Quantization via integrable systems

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GEOQUANT 2021

Quantization seeks to associate a quantum system to a classical Hamiltonian system replacing functions by operators and Poisson brackets of functions by brackets of operators. Several paths have been traced for this passionate journey from geometry and analysis into Physics: geometric quantization, formal quantization, BRST quantization and semi-classical quantization to cite a few. All of them supply taylor-made master formulas to the day dreamer mathematicians who are looking into the quantum world through their quantization mirror.

In this course we focus on the Geometric Quantization approach and we provide several a model that brings us closer to the role of quantization as a mathematical tamer of quantum physics. The almost metaphysical questions still waft in the air: Can this be achieved? Do these methods depend on additional data? Can we find a universal model?

Geometric quantization and integrable systems are common mathematical objects on the interface of Geometry and Physics. Integrable systems represent a class of Hamiltonian systems which can be associated to an extra number of functions (first integrals). And geometric quantization meets integrable systems when these systems are used as data attached to the Geometric quantization process (a polarization). In this minicourse we will examine the quantization problem considering as polarizations the real polarizations associated to an integrable system and analyze the role of singularities in the quantization picture suggested by Kostant.

Lecture 1. **Introduction to Symplectic Geometry, integrable systems and action-angle systems. Non-degenerate singularities of integrable systems.**
We will provide the basics in Symplectic Geometry. Including basic concepts of integrable systems, moment maps, actions and finishing with the statement and sketch of the proof for the action-angle coordinate theorem (Arnold–Liouville–Mineur theorem). We will also present a cotangent lift action. An integrable system on a compact manifold will have singularities so we can simply not escape them. At the end of this lecture we will discuss the existence of certain "action-angle" with singularities such as normal forms for integrable systems with non-degenerate singularities.

Problem session 1 (by Pau Mir). **Integrable systems in physics.**
This problem session will have two objectives: First one revision of some theoretical aspects mentioned in Lecture 1 such as cotangent lift. We will provide a short list of problems most of them describing singularities of integrable systems

naturally appearing in some physical systems. Some of these systems will be presented as cotangent lifted actions. The list of problems will be provided in advance.

Lecture 2. **Basics on Geometric Quantization. Polarizations. Integrable systems as polarizations. Bohr–Sommerfeld leaves. Some computations.**
In this second lecture we will associate a prequantum line bundle endowed with a connection to a symplectic manifold with integral symplectic class. I will describe the flat sections equations. Determine the role of a polarization and end up the class with a computation of flat sections taking as polarization an integrable system in action-angle coordinates.
This last computation will be unveiled as the proof of a hidden theorem that computes the Geometric Quantization in terms of the Bohr–Sommerfeld leaves and identifies it with a sheaf cohomology computation. We will show our computation kit for this sheaf-theoretical definition of Geometric Quantization (Kostant's approach). The manipulations of sheaves and proof of this theorem will be done in the problem session.

Problem session 2 (by Pau Mir). **Sheaf cohomology computations.**
In this session two key examples to compute the main "bricks" of the sheaf theoretical approach to quantization will be provided. By means of an application of a Künneth formula we will provide a new proof of Snyaticky's theorem on Geometric Quantization. We end this problem session with an exercise computing Bohr–Sommerfeld leaves on a compact sphere which is symplectic away from a curve. This exercise will be an invitation to the last session.

Lecture 3. **Singularities everywhere! How to handle with singularities of integrable systems. How to handle singularities of the symplectic structures.**
In this last lecture we will deal with non-degenerate singularities and analyze their contribution in Geometric Quantization. The cotangent models provided in the first problem session will be useful now to reformulate the Geometric Quantization of singularities using these cotangent models. We end up this course discussing the extension of our techniques to a class of Poisson manifolds: b-Poisson manifolds. The example discussed in the second problem session will be essential.

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