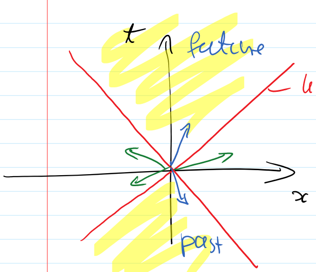
Globally hyperbolic

Sunday, August 1, 2021 9:14 PM



M_2

$g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$v \in \mathbb{R}^2$

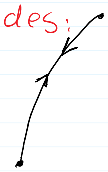
- $g(v,v) = v^T g v = 0$ v is light-like
- > 0 v is time-like
- < 0 v is spacelike

(light-like and time-like: causal)

Def: A smooth curve is a smooth map $\gamma : I \to M$ from a non-empty open interval $I \subset \mathbb{R}$, with everywhere non-vanishing derivative $\dot\gamma$

→ excludes I being just a point

→ excludes:



A smooth curve γ is causal if $\dot\gamma$ is causal everywhere along γ

spacelike	spacelike
timelike	timelike
null	null

Points p,q are joint by a (causal / timelike / ...) smooth curve means that they both lie on a smooth curve or (equivalently) are joint by a curve $\gamma : [0,1] \to M$, $\gamma(0) = p$, $\gamma(1) = q$ s.t. it can be extended to a smooth curve on an open $I \supset [0,1]$.

Fact: for distinct $p,q \in M$ the following are equivalent:

- p,q joint by a piecewise smooth future-directed timelike / causal curve
- — " — a smooth — " —

in particular, no closed causal curves

Def (BS03) M is *globally hyperbolic* if it is strongly causal and $J^+(p) \cap J^-(q)$ is compact for any $p, q \in M$.

Def: A timelike curve is future-inextendible if there is no $p \in M$ such that: $\forall\, U \subset M$ open neigh. of p, $\exists\, t'$ s.t. $\gamma(t) \in U$ $\forall\, t > t'$

Def: A Cauchy hypersurface in M is a smooth subspace of M such that every inextendible timelike curve intersects it exactly once.

From chapter 14 of reference [2]

29. Lemma. A Cauchy hypersurface S is a closed achronal topological hypersurface and is met by every inextendible causal curve.

→ at least once

A subset A of M is *achronal* provided the relation $p \ll q$ never holds for $p, q \in A$; that is, provided no timelike curve meets A more than once. For

Example:

in M_2

Theorem (after [1]) *The following definitions of global hyperbolicty of a Lorentzian manifold $\mathcal{M}$ are equivalent:*

- *$\mathcal{M}$ does not contain closed causal curves and for any two points x and y the set $J_+(x) \cap J_-(y)$ is compact.*
- *$\mathcal{M}$ contains a Cauchy surface.*
- *$\mathcal{M}$ admits a foliation by Cauchy surfaces.*

[1] Bernal, A.N., Sánchez, M.: On smooth Cauchy hypersurfaces and Geroch's splitting theorem. Commun. Math. Phys. **243**(3), 461–470 (2003)

[2] B. O'Neill, *Semi-Riemannian geometry with applications to Relativity*, Academic Press Inc., New York (1983).

[3] John K. Beem, Paul E. Ehrlich, and Kevin L. Easley. *Global Lorentzian geometry*, volume 202 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker Inc., New York, second edition, 1996.