Lecture 2

Monday, August 2, 2021 12:26 PM

Explicit construction

Configuration space: $\mathcal{E} = \Gamma(E \to M)$ (v. bundle), e.g. scalar field: $\mathcal{E} = \mathcal{C}^\infty(M, \mathbb{R})$

$\mathcal{E}_c = \Gamma_c(E \to M)$ sections w/ comp. supp

$\mathcal{D} = \mathcal{C}^\infty_c(M)$

A functional $F : \mathcal{E} \to \mathbb{C}$ is differentiable at $\varphi \in \mathcal{E}$ if

$$\langle F^{(1)}(\varphi), \psi \rangle \doteq \lim_{t \to 0} \frac{1}{t}\left(F(\varphi + t\psi) - F(\varphi)\right)$$

exists for all $\psi \in \mathcal{E}$.

It is cont. differentiable in a neigh. $\mathcal{U}$ of φ in the sense of Bastiani if

the map $(\varphi, \psi) \longmapsto \langle F^{(1)}(\varphi), \psi \rangle$ is continuous w.r.t

$\mathcal{U} \times \mathcal{E} \longrightarrow \mathbb{C}$ the product topology

$\rightsquigarrow$ smoothness

We take $\mathcal{C}^\infty(\mathcal{E}, \mathbb{C})$ smooth functionals

In particular: $F^{(n)}(\varphi) \in \Gamma'(E^{\boxtimes n} \longrightarrow M^n)$

distributional sections, so the dual of $\Gamma(E^{\boxtimes n} \to M^n)$

Def: Spacetime support of a functional $F \in \mathcal{C}^\infty(\mathcal{E}, \mathbb{C})$ is:

$$\operatorname{supp} F = \{ x \in M \mid \forall \underset{\text{open}}{\mathcal{U}} \ni x, \ \exists\, \varphi, \psi, \ \operatorname{supp}\psi \subset \mathcal{U} \text{ such that } F(\varphi + \psi) \neq F(\varphi) \}$$

Fact: $\operatorname{supp} F = \overline{\bigcup_{\varphi \in \mathcal{E}} \operatorname{supp}(F^{(1)}(\varphi))}$

Important classes of functionals:

(assume to be compactly supported)

- $\to$ $\mathcal{F}_{loc}$ local functionals $\Leftrightarrow F(\varphi) = \int_M f(j^k(\varphi))\, d\mu$ (j^k: k-th jet; $d\mu$: the volume form on M)

  NB: $\mathcal{O} \mapsto \mathcal{F}_{loc}(\mathcal{O})$ is a sheaf

- $\to$ $\mathcal{F}$ multilocal functionals, algebraic closure of $\mathcal{F}_{loc}$ under the pointwise product: $(F \cdot G)(\varphi) \doteq F(\varphi)\, G(\varphi)$

  NB: $\mathcal{O} \mapsto \mathcal{F}(\mathcal{O})$ is a sheaf in Weiss top.

- $\to$ $\mathcal{F}_{reg}$ regular functionals: $F^{(n)}(\varphi) \in \Gamma_c(E^{*\boxtimes n} \longrightarrow M^n)$ $\forall\, n \in \mathbb{N}, \varphi \in \mathcal{E}$

  $\cap$ $\Gamma'(E^{\boxtimes n} \longrightarrow M^n)$

II. Classical dynamics

Def: A generalized Lagrangian is a map: $L : \mathcal{D} \to \mathcal{F}_{loc}$ such that:

- $\operatorname{supp} L(f) \subset \operatorname{supp} f$ $\quad f \in \mathcal{D}$
- $L(f + g + h) = L(f+g) + L(g+h) - L(g)$, for all $f, g, h \in \mathcal{D}$ such that $\operatorname{supp} f \cap \operatorname{supp} h = \emptyset$

Example: $L_0(f)[\varphi] = \frac{1}{2}\int_M f\left(-\nabla_\mu\varphi\nabla^\mu\varphi + m^2\varphi^2\right)d\mu$

NB: It turns out that $T_\varphi\mathcal{E} = \mathcal{E}_c$, so this is a good choice: $T^*_\varphi\mathcal{E} = \mathcal{E}_c'$

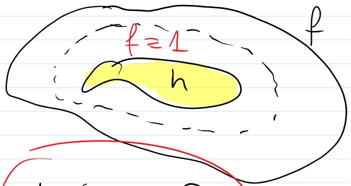
Def: The Euler-Lagrange derivative $dL \in \Gamma(T^*\mathcal{E})$ (1-form) is defined:

$$\langle \underbrace{dL(\varphi)}_{\in\, \mathcal{E}_c'}, \underbrace{h}_{\in\, \mathcal{E}_c} \rangle \doteq \langle L(f)^{(1)}, h \rangle, \text{ where } f \equiv 1 \text{ on supp } h$$

locality of L

$\Downarrow$

this does not depend on the cutoff



Def: The equation of motion is $dL(\varphi) = 0$

the space of solutions $\mathcal{E}_s$ is the zero locus of dL

Functional description: $\mathcal{F}_s = \mathcal{F}/\mathcal{F}_0$ → functionals that vanish on $\mathcal{E}_s$

↳ functionals on the space of solutions

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