Lecture 3

Tuesday, August 3, 2021 10:49 PM

$\mathcal{F}_S = \mathcal{F} / \mathcal{F}_0$ ($\mathcal{F}$ → multilocal functionals; $\mathcal{F}_0$ → functionals that vanish on $\mathcal{E}_S$)

Observation: Take a vector field $X \in \Gamma(T\mathcal{E})$, which is also multilocal

$\langle dL, X \rangle$ vanishes on $\mathcal{E}_S$ (by definition)

Assumption: Assume that all elements of $\mathcal{F}_0$ arise this way

↳ Henneaux: proof of that assumption fulfilled in examples of physical interest

Marc Henneaux. Lectures on the antifield-BRST formalism for gauge theories. *Nuclear Physics B - Proceedings Supplements*, 18(1):47–105, December 1990.

Let $\mathcal{V}_{loc}$ denote local vector fields
$\mathcal{V}$ multilocal
$\mathcal{V}_{reg}$ regular
} all compactly supported

$\delta = -\iota_{dL}$

$-\iota_{dL}(\mathcal{V}) \subset \mathcal{F}_0$ by definition

By assumption: $\iota_{dL}(\mathcal{V}) \cong \mathcal{F}_0$

$$\dots \to \underset{-2}{\Lambda^2 \mathcal{V}} \xrightarrow{\delta} \underset{-1}{\mathcal{V}} \xrightarrow{\delta} \underset{0}{\mathcal{F}} \to 0$$

Differential graded algebra

Extend δ to $\Lambda\mathcal{V}$ by using the graded Leibniz rule

$\Downarrow$

$\delta^2 = 0$

Compute $H^0 = \mathcal{F} / \delta(\mathcal{V}) \overset{\text{assumption}}{=} \mathcal{F} / \mathcal{F}_0 = \mathcal{F}_S$ functionals on the solution space

Compute H^{-1}: $\operatorname{Ker}\delta$ consists of v. fields such that

$$\delta(X) = -\langle dL, X \rangle \equiv 0$$
$$\parallel$$
$$\partial_X L(f),\ f \equiv 1 \text{ on supp } X$$

X is a symmetry!

$\operatorname{Im}\delta$: $X \wedge Y \in \Lambda^2 \mathcal{V}$

$$\delta(X \wedge Y) = -\langle dL, X \rangle Y + \langle dL, Y \rangle X$$

such symmetry vanishes on the solution space $\mathcal{E}_S$, hence is called trivial

$\Rightarrow H^{-1}$ contains non-trivial local symmetries

Examples: •) real scalar field has no non-trivial local symmetries

•) Yang-Mills theories do have symmetries

Sufficient condition for H^{-1} to be trivial: P_φ is normally hyperbolic (e.g. d'Alembertian)

↳ P_φ denotes the diff. op. induced by $L^{(2)}(\varphi)$ → linearized EOM, $\varphi \in \mathcal{E}$

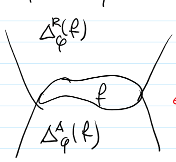
Proof: Fredenhagen, KR:

Fredenhagen, K., Rejzner, K.: Batalin-Vilkovisky formalism in the functional approach to classical field theory. Commun. Math. Phys. **314**(1), 93–127 (2012).

$P_\varphi : \mathcal{E} \to \mathcal{E}_c^* \subset \mathcal{E}'$ normally hyperbolic $\Rightarrow \exists!$ retarded and advanced Green functions:

$\Delta^R_\varphi / \Delta^A_\varphi : \mathcal{E}_c^* \to \mathcal{E}$ s.t. $\Delta^{R/A}_\varphi \circ P_\varphi\big|_{\mathcal{E}_c} = id_{\mathcal{E}_c}$ (image in $\mathcal{E}_c^*$)

$$P_\varphi \circ \Delta^{R/A}_\varphi = id_{\mathcal{E}_c^*}$$



← support properties

Define: $\Delta_\varphi \doteq \Delta^R_\varphi - \Delta^A_\varphi : \mathcal{E}^*_c \longrightarrow \mathcal{E}$ Pauli-Jordan function (commutator function)

$\hookrightarrow$ induces an integral kernel $\Pi_\varphi \in \Gamma'_c(E^{*\boxtimes 2} \longrightarrow M^2)$

$\varphi \longmapsto \Pi_\varphi$ is a generalized bivector

This is in fact a Poisson — " —. Non-trivial thing to show: Jacobi identity (use $(\Delta^R)^T = \Delta^A$ and that $L^{(2)}$ is symmetric)

Jakobs "Eichbrücken in der klassischen Feldtheorie" (Appendix B)

http://www-library.desy.de/preparch/desy/thesis/desy-thesis-09-009.pdf (AQFT Hamburg webpage)

Define: Poisson bracket (Peierls): $\lfloor F, G \rfloor(\varphi) \doteq \langle \Pi_\varphi, dF(\varphi) \otimes dG(\varphi) \rangle$

$F, G \in \mathcal{F}$

Peierls, R.E.: The commutation laws of relativistic field theory. Proc. R. Soc. Lond. Ser. A. Math. Phys. Sci. **214**(1117), 143–157 (1952)

Observation: Extends to $\Lambda\mathcal{V}$, but the space of multilocal functionals is not closed under $\lfloor\cdot,\cdot\rfloor$. Requires completion.
For example: use microcausal functionals / v. fields $\hookrightarrow$ conditions on singularity structure of derivatives $\therefore \Lambda\mathcal{V}_{\mu c}$

For a review see e.g.:
Kasia Rejzner. *Perturbative Algebraic Quantum Field Theory: An Introduction for Mathematicians*. Springer, March 2016.

Classical dg model: $\mathcal{O} \longmapsto (\Lambda\mathcal{V}_{\mu c}(\mathcal{O}), \lfloor\cdot,\cdot\rfloor, \delta)$ contains local non-linear observables

Also possible: $\mathcal{O} \longmapsto (\Lambda\mathcal{V}_{reg}(\mathcal{O}), \lfloor\cdot,\cdot\rfloor, \delta)$ well-defined but boring, only contains linear local observables and their products

More graded geometry:

Observation: $\Lambda\mathcal{V} = \Gamma(T^*[-1]\mathcal{E})$ shifted cotangent bundle

Observation: $\Lambda\mathcal{V}$ as the space of multivector fields comes with the Schouten bracket:

- $\{X, F\} := -\partial_X F$, $F \in \mathcal{F}, X \in \mathcal{V}$
- $\{X, Y\} := -[X, Y]$, $X, Y \in \mathcal{V}$
- graded Leibniz rule

$\Bigg\}$ (−1)-shifted Poisson bracket

AKA "antibracket"

Notation: For a vector field $X \in \mathcal{V}$, denote: $X = \int X(x) \frac{\delta}{\delta\varphi(x)}$ $\equiv \varphi^\ddagger(x)$ antifield

$\partial_X F = \int X(x) \frac{\delta F}{\delta\varphi(x)}$

$$\{X, Y\} = \left\langle \frac{\delta_r X}{\delta\varphi}, \frac{\delta_l Y}{\delta\varphi^\ddagger} \right\rangle - \left\langle \frac{\delta_r X}{\delta\varphi^\ddagger}, \frac{\delta_l Y}{\delta\varphi} \right\rangle$$

left derivative; right derivative

$$\{X, F\} = -\int X(x) \frac{\delta_l F}{\delta\varphi(x)} \equiv -\int X(x) \frac{\delta F}{\delta\varphi(x)} = -\partial_X F$$