Restrict to regular functionals (all derivatives are smooth sections)

Free theory: $\mathcal{O} \longmapsto \big(BV_{reg}(\mathcal{O})[[\hbar]],\ s_0 = \{\cdot, L_0\},\ \star \big)$

$|||$ regular functionals (not necessarily local) on $T^*[-1]\mathcal{E}$

Def: Time-ordering operator $\mathcal{T}$ is defined on $BV_{reg}[[\hbar]]$ by:

$$\mathcal{T} = e^{\frac{\hbar}{2}\left\langle \Delta^F, \frac{\delta^2}{\delta\varphi^2} \right\rangle}$$

with appropriate signs

$\Delta^F = \frac{i}{2}(\Delta^R + \Delta^A) + H$ Feynman propagator, symmetric

NB: $\Pi = \Delta^R - \Delta^A$

Def: the time-ordered product $\cdot_T$ is defined by:

$$F \cdot_T G = \mathcal{T}(\mathcal{T}^{-1}F \cdot \mathcal{T}^{-1}G)$$

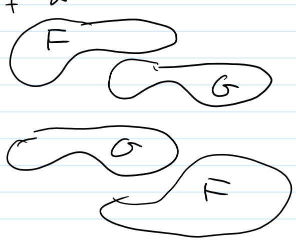
(equiv. to the pointwise product of the classical theory)

Formally: "$(\mathcal{T}F)(\varphi) = \int F(\varphi - \psi)\, d\mu_{i\hbar\Delta^F}(\psi)$" (path integral)

is the oscillating Gaussian measure with cov. Δ^F

Remark: $\cdot_T$ is the "time-ordered" version of $\star$

$$F \cdot_T G = \begin{cases} F \star G \\ G \star F \end{cases}$$



Def: The formal S-matrix is defined by:

$$\mathcal{S}(\lambda V) = e_T^{\frac{i}{\hbar}\lambda \mathcal{T}V}$$

Laurent series in $\hbar$ with coeff. in formal power series in λ

interaction

coupling constant
another formal parameter

"normally-ordered" V

Total Lagrangian: $L = L_0 + V$

Def: Interacting observables are defined as follows:

given $F \in BV_{reg}$, define $R_V(F) \doteq \mathcal{S}(V)^{-1} \star (\mathcal{S}(V) \cdot_T \mathcal{T}F)$ (Bogoliubov)

retarded Møller operator
(starts at order zero in $\hbar$)

Def: The interacting $\star$ product is defined by:

$$F \star_{\lambda V} G \doteq R_{\lambda V}^{-1}\left(R_{\lambda V} F \star R_{\lambda V} G\right)$$

$\hookrightarrow$ this is convergent (can be resummed in λ)
(Hawkins, KR)

E. Hawkins and K. Rejzner, *The star product in interacting quantum field theory*, Letters in Mathematical Physics (2020), [arXiv:math-ph/1612.09157].

Note: $s_0(X \cdot_T Y) = (s_0 X)\cdot_T Y + (-1)^{|X|} X \cdot_T (s_0 Y) - i\hbar \underbrace{\{X, Y\}_T}_{= J^{-1}\{JX, JY\}}$

(BD algebra)

Def: $\hat{s}_0 = J^{-1} \circ s_0 \circ J = s_0 - i\hbar \Delta$ (free quantum BV differential)

Δ → BV Laplacian (graded Laplacian)

$$\Delta = \int \frac{\delta^2}{\delta\varphi(x)\,\delta\varphi^\ddagger(x)}$$

Note: $\hat{s}_0(X \cdot Y) = (s_0 X)\cdot Y + (-1)^{|X|} X \cdot (s_0 Y) - i\hbar \{X, Y\}$

$$(\mathcal{BV}_{reg}(\mathcal{O})[[\hbar]], s_0, \cdot_T) \xrightarrow{J^{-1}} (\mathcal{BV}_{reg}(\mathcal{O})[[\hbar]], \hat{s}_0, \cdot)$$

pAQFT — Costello–Gwilliam

Condition: assume $S(V)$ to be invariant under s_0

$$s_0\left(e_T^{iJV/\hbar}\right) = 0 \qquad\Big|\qquad \hat{s}_0\left(e^{iV/\hbar}\right) = 0$$

$$\parallel \qquad\qquad\qquad \parallel$$

$$s_0 \circ J\left(e^{iV/\hbar}\right) \qquad\Big|\qquad \# \; e^{iV/\hbar}\left(\{V, L_0\} + \tfrac{1}{2}\{V, V\} - i\hbar \Delta V\right)$$

Assume: $\{L_0, L_0\} - i\hbar \Delta L_0 = 0$

$$\tfrac{1}{2}\{\underbrace{L_0 + V}_{L}, \underbrace{L_0 + V}_{L}\} - i\hbar \Delta (\underbrace{L_0 + V}_{L}) = 0$$

(Quantum master equation)

Def: Quantum BV operator in the interacting theory:

$$\hat{s} = R_V^{-1} \circ s_0 \circ R_V \underset{\substack{\text{assuming}\\ \text{QME}}}{=} \underbrace{\{\cdot, L_0 + V\}}_{s} - i\hbar \Delta$$

$(\mathcal{BV}_{reg}[[\hbar]], \hat{s})$ computes the space of quantum observables

Where are the ∞?

Note: Regular functionals are non-local if non-linear

Problem: How to extend $\cdot_T$ to local non-linear functionals?

↳ renormalisation problem

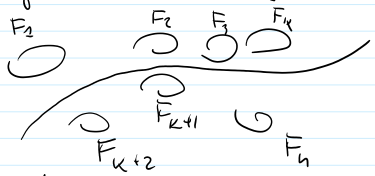
Observation: $J_n(F_1 \otimes \dots \otimes F_n) \doteq F_1 \cdot_T \dots \cdot_T F_n$
is well-defined for $F_i \in BV_{loc}$ as long as the supports are disjoint

↳ want to extend this to functionals with coinciding supports

↳ Epstein-Glaser renormalisation

$$J_n : \mathcal{F}_{loc}^{\otimes n} \to \mathcal{F}_{\mu c} \qquad n \in \mathbb{N}$$

Causal factorisation property: $J_n(F_1 \otimes \dots \otimes F_k \otimes \dots \otimes F_n) = J_k(F_1 \otimes \dots \otimes F_k) \star J_{n-k}(F_{k+1} \otimes \dots \otimes F_n)$



The renormalisation problem now reduces to the problem of extension of certain distributions defined outside the diagonal to the full diagonal.

H. Epstein and V. Glaser, *The role of locality in perturbation theory*, AHP **19** (1973), no. 3, 211–295.