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Convergence & Continuity of Star Products

Today: formal star products

Quantization

classical mechanics $\rightsquigarrow$ quantum mechanics

observables — functions on phase space — operators on Hilbert space

associative *-algebra $\mathcal{A}$

states — point in phase space M — 1-dim. subspace

positive functional — $\psi \in \mathcal{H} \setminus \{0\}$

$p \rightsquigarrow \delta_p \quad p \in M$ — $\omega(a^* a) \geq 0$ — $A \mapsto \dfrac{\langle \psi, A\psi \rangle}{\langle \psi, \psi \rangle}$

$\omega : \mathcal{A} \longrightarrow \mathbb{C}$

focus on observable algebra in quantization.

M phase space — $\mathfrak{B}(\mathcal{H})$ bounded operators

Poisson manifold — non-commutative

$\hat{=}$ $\mathscr{C}^\infty(M)$ are Poisson algebra — $\rightarrow$ have intrinsic bracket

$[A, B] = AB - BA$

$\{f, g\} = \pi(df, dg)$

Poisson structure $\rightarrow$ $\pi \in \Gamma^\infty(\Lambda^2 TM)$

bracket?

→ time evolution $\quad f \in C^\infty(M) \qquad\qquad A \in \mathcal{B}(\mathcal{H})$

$$\frac{d}{dt} f(t) = \{H, f(t)\}, \quad f(0) = f \qquad\qquad \frac{d}{dt} A(t) = \frac{1}{i\hbar}[H, A(t)]$$

Hamiltonian function $\in C^\infty(M)$ $\qquad$ Hamiltonian operator

Planck's constant ! $\hbar$

Quantization: preserve as much algebraic features as possible

$$a \rightsquigarrow \hat{a}$$

„1 : 1"

$$\widehat{za + wb} \Longleftrightarrow z\hat{a} + w\hat{b}$$

$$\widehat{\bar{a}} \rightsquigarrow \hat{a}^*$$

$$\widehat{ab} \rightsquigarrow \hat{a}\,\hat{b}$$

$$\widehat{\{a, b\}} \rightsquigarrow \frac{1}{i\hbar}[\hat{a}, \hat{b}]$$

Way out? relax all of them (not the first...)

$\rightsquigarrow$ means correspondence up to higher orders in $\hbar$

idea of deformation quantization $\{\mathcal{A}_\hbar\}_{\hbar \geq 0}$

$\mathcal{A}_0$ = classical observable algebra

$\mathcal{A}_{\hbar > 0}$ = quantum observable algebra

+ some continuous dependence on $\hbar$ (smooth, analytic ...)

such that „classical limit" $\hbar \to 0$ makes sense.

$\longmapsto$

Example: $M = T^*\mathbb{R}^n = \mathbb{R}^{2n}$ classical phase space

canonical quantization:

observable: polynomials $q^1, \dots, q^n, p_1, \dots, p_n$

$q^k \longmapsto Q^k$ = position operator

$$(Q^k\psi)(x) = x^k\psi(x)$$

$\psi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$

wave functions on config. space $\mathbb{R}^n$

$p_\ell \longmapsto P_\ell = -i\hbar \dfrac{\partial}{\partial x^\ell}$

momentum operator

polynomial in q's & p's ? $\rightsquigarrow$?

$\rightsquigarrow$ choose an ordering !

very simple one is standard-ordering

$$\varrho_{std}(q^{i_1}\cdots q^{i_k} p_{j_1}\cdots p_{j_\ell}) = (-i\hbar)^\ell x^{i_1}\cdots x^{i_k} \frac{\partial^\ell}{\partial x^{j_1}\cdots\partial x^{j_\ell}}$$

+ linear extension

$$\rho_{std} : \mathbb{C}[q's, p's] \longrightarrow \mathrm{Diffops}(\mathbb{R}^n)$$

extend to $C^\infty(\mathbb{R}^n)[p_1, \dots, p_n] = \mathrm{Pol}(T^*\mathbb{R}^n)$

$\to$ possible by

$$\rho_{std}(f) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\hbar}{i}\right)^k \frac{\partial^k f}{\partial p_{i_1} \cdots \partial p_{i_k}} \Bigg|_{p=0} \frac{\partial^k}{\partial x^{i_1} \cdots \partial x^{i_k}}$$

gives :

$$\rho_{std} : \mathrm{Pol}(T^*\mathbb{R}^n) \xrightarrow[\text{bijection}]{\text{lin}} \mathrm{Diffop}(\mathbb{R}^n)$$

pull back the operator product

$$f *_{std} g = \rho_{std}^{-1}\left(\rho_{std}(f)\, \rho_{std}(g)\right)$$

do it yourself !

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\hbar}{i}\right)^k \frac{\partial^k f}{\partial p_{i_1} \cdots \partial p_{i_k}} \frac{\partial^k g}{\partial q^{i_1} \cdots \partial q^{i_k}}$$

terminates since $f, g \in \mathrm{Pol}(T^*\mathbb{R}^n)$

features :

$$f *_{std} g = fg + \text{higher orders in } \hbar$$

$$f *_{std} g - g *_{std} f = i\hbar \{f, g\} + \text{higher orders in } \hbar$$

canonical Poisson bracket of $T^*\mathbb{R}^n$

$$f *_{std} g = \sum \hbar^k C_k(f, g)$$

bidifferential operators

$$f *_{std} 1 = 1 *_{std} f = f$$

$*_{std}$ is associative !

Definition (Star product, BFFLS '78) Let (M,π) be a Poisson manifold. Then a formal star product is an associative $\mathbb{C}[[\hbar]]$-bilinear product $*$ for $\mathcal{C}^\infty(M)[[\hbar]]$,

$$f * g = \sum_{k=0}^{\infty} \hbar^k C_k(f,g)$$

such that

i) $C_0(f,g) = fg$

ii) $C_1(f,g) - C_1(g,f) = i\{f,g\}$

iii) $1 * f = f * 1 = f$

iv) C_k bidifferential operators

+ some more feature ...

Convergence problem: make the formal power series converge ?!?

$\longmapsto$

some results :

- Existence : ✓

symplectic case: De Wilde & Lecomte, Fedosov, ...

Poisson linear case: Gutt, Drinfel'd

general case: Kontsevich

- Classification: ✓

  Symplectic case: Nest-Tsygan, Bertelson-Cahen-Gutt, Fedosov, Deligne, ...

  Poisson case: Kontsevich

- States & Representations ✓
- Symmetries & Reduction ✓

But convergence in $\hbar$?

NOT perturbation theory since $\hbar$ has dimension of action

Proposal / plan:

1.) Identify a small & easy subalgebra of $C^\infty(M)[[\hbar]]$ where $\star$ trivially converges — Hope

2.) Find topology (l.c.) such that $\star$ becomes continuous — Hope

3.) Complete! — Easy

4.) Use it: what are states? representations on (pre-) Hilbert spaces ... — difficult

Examples: constant Poisson structures on vector space

V locally convex space, e.g.

$\mathbb{R}^n$ V^* is the phasespace

$\mathcal{C}_0^\infty(M)$, $\mathcal{C}_0^\infty(\Sigma) \oplus \mathcal{C}_0^\infty(\Sigma)$, ...

consider Polynomials on dual: $\mathrm{Pol}(V^*)$ complicated

better: $S(V) \subseteq \mathrm{Pol}(V^*)$

($=$ in finite dimension)

constant Poisson structure modelled by

$$\Lambda : V \times V \longrightarrow \mathbb{C} \quad \text{bilinear}$$

extend to $S(V) \underline{\otimes} S(V) \longrightarrow S(V) \otimes S(V)$

$$P_\Lambda (v_1 \cdots v_k \otimes w_1 \cdots w_l) =$$

$$\sum_{i,j=1}^{\cdot\cdot} \Lambda(v_i, w_j) \; v_1 \cdots \overset{i}{\wedge} \cdots v_k \otimes w_1 \cdots \overset{j}{\wedge} \cdots w_l$$

is a biderivation, i.e.

$$P_\Lambda (ab \otimes c) = a \otimes 1 \cdot P_\Lambda(b \otimes c)$$

$$+ \; b \otimes 1 \cdot P_\Lambda(a \otimes c)$$

...

in finite dimensions $\Lambda(e_i, e_j) = \Lambda_{ij}$

$$P_\Lambda = \Lambda_{ij} \; i_s(e^i) \otimes i_s(e^j)$$

$e_1, \dots, e_d \in V$ basis,

$e^1, \dots, e^d \in V^*$ dual basis

$i_s(\alpha) : S(V) \longrightarrow S(V)$ symmetric insertion of $\alpha \in V^*$

(derivation)

symmetric tensor product $\mu : S(V) \otimes S(V) \longrightarrow S(V)$

$$\mu(a \otimes b) = ab$$

$$\{a, b\}_\pi = \mu\big(P_\wedge(a \otimes b) - P_\wedge(b \otimes a)\big)$$

$\Rightarrow$ is a Poisson bracket (check this!)

have star product

$$a \star b = \mu \circ e^{i\hbar P_\wedge}(a \otimes b)$$

is associative ! (Gerstenhaber III) ~ 60's

1.) Hope: subalgebra with trivial convergence ✓

2.) Hope: need a lc topology to make $\star$ continuous ...

need a lc topology on $S(V) = \bigoplus_{k=0}^{\infty} S^k(V)$

comment $\quad S^k(V) \subseteq \underline{\underline{T^k(V)}} = \underbrace{V \otimes \cdots \otimes V}_{k\text{-times}}$

$\nearrow$ symmetric tensors

which topology on $T^k(V)$?

take most easy one: projective topology

$z \in V \otimes W \qquad V, W$ lc spaces

$p, q \qquad \begin{matrix} p \\ q \end{matrix}$ cont. seminorm on $\begin{matrix} V \\ W \end{matrix}$

$$(p \otimes q)(z) = \inf \left\{ \sum_i p(v_i) q(w_i) \;\middle|\; z = \sum_i v_i \otimes w_i \right\}$$

is a seminorm on $V \otimes W$

all of them together (for p, q as above) define the π-topology (= projective) on $V \otimes W$

for us: $\quad V^k = V \otimes \cdots \otimes V$ with seminorms

$$\left\{ p^k = p \otimes \cdots \otimes p \;\middle|\; p \text{ cont. seminorm on } V \right\}$$

gives $\quad V^{\otimes_\pi k}$

but now: $\quad T(V) = \bigoplus_{k=0}^{\infty} V^{\otimes_\pi k}$ $\leftarrow$ each with π-topology

$\quad\;\; \cup\!\!\!\!\; /$

$S(V) \quad ?$ $\nearrow$

Options:

1.) Cartesian product topology from

$$\bigoplus_{k=0}^{\infty} V^{\otimes k} \subseteq \prod_{k=0}^{\infty} V^{\otimes k}$$

$\rightsquigarrow$ bad idea! too coarse

2.) take (lc) direct sum topology

$\rightsquigarrow$ bad idea, too fine!

3.) Something in between?

$R \in \mathbb{R}$ fixed, p seminorm on V

$$p_R(v) = \sum_{k=0}^{\infty} k!^R \, p^k(v_k)$$

$= \sum_{k=0}^{\infty} v_k \in T(V)$

$v_k \in T^k(V)$

only finitely many $\neq 0$

take all p_R's for p a continuous seminorm on V to define

$\mathcal{T}_R$ - topology on $T(V)$

$\cup\!\mid$

$\mathcal{S}_R$ - topology on $S(V)$

$=$ subspace topology

Some features:

i.) $p \leq q \implies p_R \leq q_R$

ii.) $$p_R(v) \le (2p)_R(v) = \sum_{k=0}^{\infty} k!^R \, (2p)^k(v_k)$$
$$= \sum_{k=0}^{\infty} k!^R \, 2^k \, p^k(v_k)$$

can not be estimated by p_R anymore

iii.) $$p_{R,\infty}(v) = \sup_k \left\{ k!^R \, p^k(v_k) \right\}$$

$$p_{R,\infty} \le p_R \le 2 \cdot (2p)_{R,\infty}$$

$\Rightarrow$ leads to same $\mathcal{T}_R$-topology

iv.) $$p_R \Big|_{T^k(V) \subseteq T(V)} = \underbrace{k!^R}_{\text{const.}} \, p^k$$

$\Rightarrow$ subspace topology inherited by $T^k(V) \subseteq T(V)$ is original π-topology

$$T(V) \xrightarrow[\text{continuous}]{pr_k} T^k(V) \xrightarrow[\text{embedding}]{} T(V)$$

v.) $R \ge 0$ $$p_R(v \otimes w) \le (2^R p)_R(v) \; (2^R p)_R(w)$$
$v, w \in T(V)$

so: $\otimes : T(V) \times T(V) \longrightarrow T(V)$

is continuous for T_R-topology

and for $R=0$

$T(V)_{R=0}$ is $\underbrace{\text{l m c}}_{\underline{\underline{\text{free}}}}$ algebra generated by V

<u>Remark:</u> for quantization: lmc algebras are not at all useful!

Since: $\mathcal{A} \ni Q, P$ with

$$[Q,P] = i\hbar \mathbb{1} \qquad \hbar \neq 0$$

then there is <u>no</u> submultiplicative seminorm on $\mathcal{A}$ beside 0

vii) Completion of $T_R(V)$ and $S_R(V)$:

$$\hat{T}_R(V) = \left\{ v = \sum_{k=0}^{\infty} v_k \;\middle|\; v_k \in V^{\hat{\otimes}_\pi k} \text{ s.t. } p_R(v) < \infty \text{ for all } p \text{ cont. seminorms on } V \right\} \subseteq \prod_{k=0}^{\infty} V^{\hat{\otimes} k}$$

in particular:

$\sum_{k=0}^{\infty} v_k$ really converges in T_R

topology (absolutely)

$\hat{S}_R(V)$ same with symmetric tensors

vii.) symmetric tensor product is S_R - continuous

$$p_R(v \vee w) \leq 2\,(2p)_R(v)\,(2p)_R(w)$$

Exercise: describe $\hat{S}_R(V)$ for $V = \mathbb{R}^n$

viii.) $R \geq 0$ $\quad \varphi \in V'$ (top. dual of V)

$$\delta_\varphi : S(V) \ni v \longmapsto \delta_\varphi(v) \in \mathbb{C}$$

character by evaluating at φ

$$V \ni v \longmapsto \varphi(v) \in \mathbb{C}$$

is continuous in S_R - topology

$\Rightarrow$ extends to $\hat{S}_R(V)$

$\Rightarrow$ elements of $\hat{S}_R(V)$ can be viewed as functions on V'

$$\underset{v \in \hat{S}(V)}{v(\varphi)} = \delta_\varphi(v)$$

now: continuity of $\star$?

warming up: continuity of $\{\cdot,\cdot\}$

$$u \in S^n(V) \otimes S^m(V) \rightsquigarrow P_\Lambda(u) \in S^{n-1}(V) \otimes S^{m-1}(V)$$

$$\left(p^{n-1} \otimes p^{m-1}\right)\left(P_\Lambda(u)\right) \leq \underline{\underline{n}}\,\underline{\underline{m}}\; p^{n+m}(u) \qquad (*)$$

from Leibniz rule

provided Λ is <u>continuous</u> w.r.t. p, i.e.

$$|\Lambda(v,w)| \leq p(v)\,p(w) \qquad ☺$$

consequence: $R \geq 0$

$$p_R\left(\{v,w\}_\Lambda\right) \leq \left(2^{R+1}p\right)_R(v)\,\left(2^{R+1}p\right)_R(w)$$

$\Rightarrow$ $\{\cdot,\cdot\}_\Lambda$ <u>is</u> continuous if Λ is continuous.

<u>Theorem</u> (Continuity of $\star$) $\quad R \geq 1/2$ (sharp)

Suppose Λ is continuous and p is a cont seminorm satisfying ☺ then

$$p_R(a \star b) \leq c' \,(cp)_R(a)\,(cp)_R(b)$$

$\forall a,b \in S(V)$. (Exercise)

more precisely: $\quad a, b \in \hat{S}_R(V)$

$$a \star_\hbar b = \sum_{k=0}^{\infty} \frac{(i\hbar)^k}{k!} \mu \circ P_\Lambda^k (a \otimes b)$$

is absolutely convergent!

so: $\quad \mathbb{C} \ni \hbar \mapsto a \star_\hbar b \in \hat{S}_R(V)$

is entire with Taylor series given as above

Some more features depending on V:

- Thm: equivalent:

  i) V is nuclear

  ii) $\hat{S}_R(V)$ is nuclear

  iii) $\hat{T}_R(V)$ is nuclear

  in particular for $\dim V < \infty$

- good functorial properties for continuous linear maps preserving Λ.

  (important for symmetries!)

Example ($\nearrow$ Kasia's talk):

$$\underset{\text{LF space}}{V} = \mathcal{C}_0^\infty(M)$$ test functions on glob. hyp. (M,g)

$$\Lambda : \mathcal{C}_0^\infty(M) \times \mathcal{C}_0^\infty(M) \longrightarrow \mathbb{C}$$

$$= \underbrace{\tfrac{i}{2}\pi}_{\text{Peierls}} + \underbrace{H}_{\text{symmetric}}$$ continuous

is a distribution on $M \times M$
i.e. cont. linear functional on $\mathcal{C}_0^\infty(M\times M)$

$\rightsquigarrow \hat{S}_R(\mathcal{C}_0^\infty(M))$, $\star$ as above, $R \geq \frac{1}{2}$

completion in direction of tensor degree

$$S^{\otimes k}(\mathcal{C}_0^\infty(M)) = \mathcal{C}_0^\infty(M) \hat{\otimes}_s \cdots \hat{\otimes}_s \mathcal{C}_0^\infty(M)$$
$$= \mathcal{C}_0^\infty(M\times \cdots \times M)_{sym}$$

would be nice to extend

$\mathcal{C}_0^\infty(M) \longrightarrow$ much more singular things

$\mathcal{C}_{0,\Gamma}^{-\infty}(M)$ — Γ: wave front condition

but then Λ becomes discontinuous ...

Example: linear Poisson structures

$\mathfrak{g}^*$ with Poisson structure on $\mathrm{Pol}(\mathfrak{g}^*)$ $\mathrm{Fun}(\mathfrak{g}^*)$
with *linear* coefficients

$\Longleftrightarrow$ Lie algebra structure on $\mathfrak{g}$

$$\{f,g\}(x) = x\left(\left[df|_x , dg|_x\right]_{\mathfrak{g}}\right)$$

so: $\mathfrak{g}$ Lie algebra

$\cancel{\mathrm{Pol}(\mathfrak{g}^*)}$ $\rightsquigarrow$ better to use $S(\mathfrak{g})$

$\cancel{\{\cdot,\cdot\}}$ $\rightsquigarrow$?

$\cancel{\star\,?}$ $\rightsquigarrow$?

don't use that

star product for $S(\mathfrak{g})$ quantize *the* Poisson bracket of $S(\mathfrak{g})$

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

& extend by Leibniz rule

Gutt: ('83) $\quad S(\mathfrak{g}) \xrightarrow{\text{PBW}} U(\mathfrak{g})$

$$\sigma_k : \quad S^k(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{g}) \quad , \quad \sigma = \sum_{k=0}^{\infty} \sigma_k$$

$$x = \xi_1 \cdots \xi_k \longmapsto \frac{1}{k!} \sum_{\sigma \in S_k} \xi_{\sigma(1)} \bullet \cdots \bullet \xi_{\sigma(k)}$$

$$\underset{S^k(\mathfrak{g})}{\underset{\cap}{x}} \star \underset{S^\ell(\mathfrak{g})}{\underset{\cap}{y}} = \sum_{n=0}^{k+\ell} (i\hbar)^n \, \pi_{k+\ell-n} \, \sigma^{-1}\big(\sigma(x) \bullet \sigma(y)\big)$$

„Gutt star product" for $S(\mathfrak{g})$

Fact: $\xi, \eta \in \mathfrak{g}$ then

$$\exp(\xi) \star \exp(\eta) = \exp\left(\frac{1}{i\hbar} BCH(i\hbar\xi, i\hbar\eta)\right)$$

formal exp series

Baker-Campbell-Hausdorff

$$BCH(\xi,\eta) = \xi + \eta + \frac{1}{2}[\xi,\eta] + \cdots$$

allows to reconstruct $\star$ for $S(\mathfrak{g})$

$\xi, \eta \in \mathfrak{g}$

$$\xi^k = \frac{d^k}{dt^k}\bigg|_{t=0} \exp(t\xi)$$

$$\xi^k \star \eta^\ell = \frac{d^k}{dt^k}\bigg|_{t=0} \frac{d^\ell}{ds^\ell}\bigg|_{s=0} \exp(t\xi) \star \exp(s\eta)$$

$$= \frac{d^k}{dt^k}\bigg|_{t=0} \frac{d^\ell}{ds^\ell}\bigg|_{s=0} \exp\left(\frac{1}{i\hbar} \mathrm{BCH}(i\hbar t \xi, i\hbar s \eta) \right)$$

$\xi_1 \cdots \xi_k$ from polarization of $(t_1 \xi_1 + \cdots + t_k \xi_k)^k$

$\rightsquigarrow$ gives $\xi_1 \cdots \xi_k \star \eta_1 \cdots \eta_\ell = \ldots$

more explicitly

$$\mathrm{BCH}(\xi, \eta) = \sum_{n=1}^{\infty} \mathrm{BCH}_n(\xi, \eta)$$

have n letters in total

$$= \sum_{a,b=1}^{\infty} \mathrm{BCH}_{a,b}(\xi, \eta)$$

have a letters ξ
b letters η

finite number of bracket expressions

$$\xi^k * \eta^l = \sum (i\hbar)^n \, C_n(\xi^k, \eta^l)$$

$$C_n(\xi^k, \eta^l) = \frac{k!\, l!}{(k+l-n)!} \sum_{\substack{a_1, b_1, \dots, a_r, b_r \geq 0 \\ a_i + b_i \geq 1 \\ a_1 + \dots + a_r = k \\ b_1 + \dots + b_r = l}} \mathrm{BCH}_{a_1, b_1}(\xi, \eta) \cdot \dots \cdot \mathrm{BCH}_{a_r, b_r}(\xi, \eta)$$

$n \geq 1$, $\quad r = k + l - n$.

$$\mathrm{BCH}_n(\xi, \eta) = \sum_{|w| = n} \frac{g_w}{n} [w]$$

w = word in ξ's and η's with n letters in total

$[w]$ = the word we get by taking the brackets from the left

$$w = \xi\eta\xi \longmapsto [\xi\eta\xi] = [\dots[\xi, \eta], \xi]$$

then $g_w \in \mathbb{Q}$ are coefficients with

$$\sum_{|w| = n} \frac{|g_w|}{n} \leq \frac{2}{n}$$

Goldberg
Thompson
~ 60's

Assumption on $\mathfrak{g}$:

$\mathfrak{g}$ lc space and $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ should be continuous

NOT enough:

need to find defining system of seminorms $\{p\}$ on $\mathfrak{g}$ such that

$$p([\xi,\eta]) \leq p(\xi)\, p(\eta)$$

„lmc Lie algebra"

(generalization to AE Lie algebras)

Thm: $R \geq 1$, $\mathfrak{g}$ a lmc/AE Lie algebra. Then $\star_{Gutt}$ is continuous on $S_R(\mathfrak{g})$

$\rightarrow$ extends to $\widehat{S}_R(\mathfrak{g})$ by continuity and for elements in $\widehat{S}_R(\mathfrak{g})$ the (formal) Gutt star product S_R-converges & is entire in $\hbar \in \mathbb{C}$.

Remark: $\Delta : S(\mathfrak{g}) \longrightarrow S(\mathfrak{g}) \,\hat{\otimes}_\pi\, S(\mathfrak{g})$

usual coproduct turning $S(\mathfrak{g})$ into Hopf algebra

is continuous w.r.t $\hat{\otimes}_\pi$
& S_R-topologies

$\Rightarrow$ $(\hat{S}_R(\mathfrak{g}), *, \Delta)$ becomes a lc
Hopf algebra.

<u>Example</u> Cotangent bundle of a Lie group

general construction of a symbol calculus/
star product on T^*Q, Q configuration
space.

$$\begin{tikzcd} T^*Q \arrow[d, shift left, "\pi"] \\ Q \arrow[u, shift left, hook, "\iota"] \end{tikzcd}$$

$$\bigoplus_{k=0}^{\infty} \Gamma^\infty(S^k_{\mathbb{C}} TQ) \xrightarrow[\sim]{\mathcal{J}} \mathrm{Pol}^\bullet(T^*Q)$$

canonical $\mathcal{C}^\infty(Q)$-linear algebra iso

$$u \in \mathcal{C}^\infty(Q) \longmapsto \mathcal{J}(u) = \pi^* u$$
$$X \in \Gamma^\infty(TQ) \longmapsto \mathcal{J}(X) \text{ defined by}$$
$$\mathcal{J}(X)(\alpha_q) = \alpha_q(X(q))$$
$$\alpha_q \in T^*_q Q$$

need covariant torsion-free connection ∇
on Q

$\rightsquigarrow$ symmetrized covariant derivative

$$D : \Gamma^\infty(S^k T^*Q) \longrightarrow \Gamma^\infty(S^{k+1} T^*Q)$$

via Leibniz rule for $\vee$

locally in local frame $e_1, \dots, e_n \in \Gamma^\infty(TU)$
$U \subseteq Q$ open with dual frame
$e^1, \dots, e^n \in \Gamma^\infty(T^*U)$

$$D\big|_U = e^\alpha \vee \nabla_{e_\alpha}$$

multi insertion

$$i_s : \Gamma^\infty(S^k TQ) \times \Gamma^\infty(S^l T^*Q) \longrightarrow \Gamma^\infty(S^{l-k} T^*Q)$$

$$i_s(X_1 \vee \dots \vee X_k) = i_s(X_1) \cdots i_s(X_k)$$

$$i_s(X)\alpha = \alpha(X) \qquad \alpha \in \Gamma^\infty(T^*Q)$$

+ Leibniz for $\vee$

Def: Standard ordering:

$$\mathcal{J}(X) \in \mathrm{Pol}^\bullet(T^*Q) \rightsquigarrow \varrho_{\mathrm{std}}(\mathcal{J}(X)) \in \mathrm{DiffOp}(Q)$$

$$\varrho_{\mathrm{std}}(\mathcal{J}(X))\psi = \iota^*\left(i_s(X)\, e^{-i\hbar D} \psi\right)$$

restriction to function part

Remark:
- finite sum
- reproduces the ϱ_{std} from beginning
- $\varrho_{std} : \mathrm{Pol}^\bullet(T^*Q) \xrightarrow{\cong} \mathrm{DiffOp}^\bullet(Q)$ as $C^\infty(Q)$-module

pull-back operator product

$$f \star_{std} g = \varrho_{std}^{-1}\left(\varrho_{std}(f)\,\varrho_{std}(g)\right)$$

is a star product on T^*Q such that $\mathrm{Pol}^\bullet(T^*Q)[\hbar]$ is a $\mathbb{C}[\hbar]$-subalgebra

($\to$ step 1.) ✓)

Weyl-ordered version ?

choose smooth volume density μ on Q

compute $\varrho_{std}(J(X))^*$ with respect to

$$\langle \psi, \phi \rangle_\mu = \int_Q \overline{\psi}\,\phi\;\mu$$

$\phi, \psi \in C_0^\infty(Q)$

$$\nabla_X \mu = \alpha(X)\,\mu \quad \text{defines}$$

$$\alpha \in \Gamma^\infty(T^*Q)$$

$$\rightsquigarrow \quad \alpha^{\mathrm{ver}} \in \Gamma^\infty(T(T^*Q))$$

vertical lift.

construct $g_0 \in \Gamma^\infty(S^2 T^*(T^*Q))$

Riem metric of split signature

$$\alpha_q \in T_q^* Q$$

$$T_{\alpha_q}(T^*Q) \supseteq \mathrm{Ver}_{\alpha_q} = \ker T\pi$$

$$\simeq \underline{\underline{T_q^* Q}}$$

$\nabla \rightsquigarrow$ horizontal lift

$\rightsquigarrow$ horizontal subspace $\mathrm{Hor}_{\alpha_q} \subseteq T_{\alpha_q}(T^*Q)$

$$\mathrm{Hor}_{\alpha_q} \simeq \underline{\underline{T_q Q}}$$

g_0 is natural pairing between hor. & ver. vectors.

$\rightsquigarrow \quad \Delta_0$ Laplacian.

$$N = \exp\left(-\frac{i\hbar}{2}\left(\Delta_0 + \mathcal{L}_{\alpha^{\mathrm{ver}}}\right)\right)$$

$$\left(\Delta_0 = \frac{\partial^2}{\partial q^i \partial p_i} + \cdots\right)$$

$\rightsquigarrow \quad N : \mathrm{Pol}(T^*Q) \longrightarrow \mathrm{Pol}^\bullet(T^*Q)$

is well-defined bijection

<u>Theorem</u>:

$$\langle \phi, \varrho_{std}(\mathcal{J}(X))\psi \rangle_\mu$$

$$= \langle \varrho_{std}(N^2 \overline{\mathcal{J}(X)})\phi, \psi \rangle_\mu$$

$$\rightsquigarrow \quad \varrho_{Weyl}(f) = \varrho_{std}(Nf)$$

$$\Rightarrow \quad \varrho_{Weyl}(f)^* = \varrho_{Weyl}(\bar{f})$$

$$f *_{Weyl} g = N^{-1}(Nf *_{std} Ng)$$

$\vdash\!\!\!-\!\!/$

Now $\quad Q = G \quad$ connected Lie group

$$T^*G \cong G \times \mathfrak{g}^*$$

∇ half-commutative connection

$$\mathcal{C}^\infty(T^*G)^G \cong \mathcal{C}^\infty(\mathfrak{g}^*)$$

$*_{Gutt}$ from before!

$*_{Weyl} / *_{std}$ restrict to $*_{Gutt}$ on invariant functions estimates for

$$\pi^* u \ *_{std}\ \pi^* v = \pi^*(uv) \quad ✓$$

$$\pi^* u \ *_{std}\ f = \pi^* u \cdot f \quad ✓$$

$$J(X) \ *_{std}\ \pi^* u = \text{non trivial} \quad !$$

$$J(X) \ *_{std}\ J(Y) = J(X *_{Gutt} Y) \quad \text{yesterday } ✓$$

for invariant $X, Y \in S(\mathfrak{g})$

need to specify nice class of functions on G such that third part can be estimated

$\phi \in C^\omega(G)$ $\qquad R \in \mathbb{R}, R \geq 0$

$$q_{R,c}(\phi) = \sum_{k=0}^{\infty} \frac{c^k}{k!} k!^R \sum_{\alpha \in \{1,\dots,n\}^k} \left| (\mathcal{L}_{X_\alpha} \phi)(e) \right|$$

$c > 0$

$X_1, \dots, X_n \in \Gamma^\infty(TG)^G$ frame of left-invariant vector fields

$$\mathcal{L}_{X_\alpha} = \mathcal{L}_{X_{\alpha_1}} \cdots \mathcal{L}_{X_{\alpha_k}}$$

$$\mathcal{E}_R(G) = \left\{ \phi \in C^\omega(G) \;\middle|\; q_{R,c}(\phi) < \infty \ \ \forall c > 0 \right\}$$

Thm: $\mathcal{E}_R(G) \otimes_\pi \mathcal{S}_{R'}(\mathfrak{g}^*)$ $\quad R \geq 0$, $R' \geq 1$

then $\star_{std} / \star_{Weyl}$ are continuous

Thm: If $\phi \in \mathcal{C}^\omega(G)$ is a representative function, i.e. has finite-dim. G-orbit, then $\phi \in \mathcal{E}_0(G)$.

Thm: $f, g \in \mathcal{E}_R(G) \widehat{\otimes_\pi} S(\mathfrak{g}^*)$

then

$$\hbar \longmapsto f \star_{std/Weyl} g$$

is entire & Taylor expansion of it is the formal star product, converges in the above topology

Some references:

S.W.: Convergence of star products: From examples to a general framework

EMS Surveys in Mathematical Sciences 6 (2019), 1–31.

Here you find also many general references to DQ & (formal) star product + alternative approaches using integral formulas

A survey on the recent progress including a longer motivational section and many many references.

S.W.: A nuclear Weyl algebra
J. Geom. Phys. 81 (2014), 10 - 46.

Where it all started with the flat case, including fermionic cases + applications to Peierls bracket

M. Schötz, S.W.: Convergent star products for projective limits of Hilbert spaces
J. Funct. Anal. 274 (2018), 1381 - 1423

Flat case, but refined analysis in case the underlying space is proj. Hilbert + representation theory ...

D. Kraus, O. Roth, M. Schötz, S.W.:

A convergent star product on the Poincaré disc
J. Funct. Anal. 277 (2019), 2734 - 2771

Yet another example, now with curvature

S. Beiser, S.W.: Fréchet algebraic deformation quantization of the Poincaré disk
J. Reine Angew. Math. 688 (2014), 147 - 207

A first example including some representation theory of the resulting algebras.

C. Esposito, P. Stapor, S.W.:

Convergence of the Gutt star product
J. Lie Theory 27 (2017), 579 - 622

The case $\mathfrak{g}^*$ with linear Poisson bracket

M. Heins, O. Roth, S. W:

Convergent Star Products on Cotangent Bundles of Lie Groups

arXiv: 2107.14624