Übungsaufgaben zur Vorlesung
DG II — SS2025, Serie 01
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Aufgabe 1 (4 Punkte)

Let $N \subset M$ be an n dimensional submanifold in an m dimensional manifold M. Let g or $\langle \cdot, \cdot \rangle$ be a Riemannian metric on M with its Levi-Civita connection $\bar\nabla$. We consider on N the induced metric and identify $T_pN \subset T_pM$ as subspace of T_pM. Let $T_p^\perp N$ the complement of T_pN in T_pM, i.e., it normal space. Any element $Z \in TM$ we have an orthogonal decomposition $Z = Z^\top + Z^\perp \in TM \oplus T^\perp N$. Define a linear operator

$$\nabla_X Y = (\bar\nabla_X Y)^\top, \quad \forall X, Y \in \Gamma(TN).$$

1) Prove that ∇ is the Levi-Civita connection of the induced metric on N.
2) Using the decomposition

$$\bar\nabla_X Y = \nabla_X Y + II(X,Y), \quad \forall X, Y \in \Gamma(TN),$$

where $II(X,Y) = (\bar\nabla_X Y)^\perp$, compute $\bar\nabla X \bar\nabla_Y Z$ for any $X, Y, Z \in \Gamma(TN)$, which yields several geometric formulas.

Aufgabe 2 (4 Punkte)

Compute the second fundamental forms of $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ and $\mathbb{S}^n \subset \mathbb{S}^{n+1}$.

Aufgabe 3 (8 Punkte)

We consider hypersurfaces M in $\mathbb{R}^{n+1}_+ := \{x = (x_1, \cdots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} \,|\, x_{n+1} \geq 0\}$ with boundary $\partial M \subset \partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n$ and their variations, smooth families of such hypersurfaces. Let $|M|$ be the area of M. Since $\partial M \subset \mathbb{R}^n$ is a $n-1$ dimensional closed hypersurface in $\mathbb{R}^n$, it encloses a domain in $\mathbb{R}^n$, which we denote by $\widehat{\partial M}$. We also denote its area by $|\widehat{\partial M}|$. For any such a hypersurface we define a functional F by

$$F(M) := |M| + a|\widehat{\partial M}|$$

for a given constant $a \in (-1, 1)$.
Compute its first and second variational formulas

Bitte schreiben Sie Ihre(n) Namen, die Matrikelnummer sowie die Nummer Ihrer Übungsgruppe auf jedes Lösungsblatt. **Abgabe ist am Montag, 5.5.25, vor der Vorlesung**.