Übungsaufgaben zur Vorlesung
DG II — SS2025, Serie 03
Prof. Dr. G. Wang — 12.05.25
Dr. X. Zhang

Aufgabe 1 (4 Punkte)

Let (M, g) be a Riemmanian manifold with a Riemmannian metric g. Let $\{e_1, \cdots, e_m\}$ be an orthonormal local frame on an open set $U \subset M$, together with its dual co-frame $\{\omega_1, \cdots, \omega_m\}$ and its connection 1-forms θ. Let $\{\tilde{e}_1, \cdots, \tilde{e}_m\}$ be an orthonormal local frame on an open set $\widetilde{U} \subset M$, together with its dual co-frame $\tilde{\omega}$ and its connection 1-forms $\tilde{\theta}$. Assume $\tilde{e}_i = f_i^j e_j$ on $U \cap \widetilde{U}$. It is clear that $(f_i^j)(p) \in O(m)$ for any $p \in U \cap \widetilde{U}$. Let Ω be the curvature 2-form w.r.t. $\{\tilde{e}_1, \cdots, \tilde{e}_m\}$ and $\widetilde{\Omega}$ the curvature 2-form w.r.t. $\{\tilde{e}_1, \cdots, \tilde{e}_m\}$. Compute $\widetilde{\Omega}$ in terms of Ω and (f_i^j), which would imply that the curvature form is independent of the choice of local frame.

Aufgabe 2 (4 Punkte)

Let g be a Riemannian metric on a 2 dimensional manifold. The method of moving frames makes calculating curvature particularly easy, since there is exactly one connection form and one curvature form. Let us check this fact in this exercise: Consider the special case when the metric is diagonal, i.e. with line element

$$g = ds^2 = Edu^2 + Gdv^2.$$

Choose a local orthonormal frame by

$$e_1 = \frac{1}{\sqrt{E}}\frac{\partial}{\partial u} \qquad e_2 = \frac{1}{\sqrt{G}}\frac{\partial}{\partial v}.$$

Its dual frame is given

$$\omega_1 = \sqrt{E}du, \qquad \omega_2 = \sqrt{G}dv$$

Compute its connection forms θ, its curvature 2-form Ω and its its Gaussian curvature, which is given by $K = R^1_{212}$.

Aufgabe 3 (4 Punkte)

The standard sphere $\mathbb{S}^m$ can be viewed as $\mathbb{R}^m$ with the metric $g = \frac{4}{(1+|x|^2)^2}|dx|^2$. Please use the method of moving frame to compute its sectional curvature, Ricci curvature and its scalar curvature.

Aufgabe 4 (4 Punkte)

Let ω be the $(n-1)$-form on $\mathbb{R}^n$ given by

$$\omega = \sum_{j=1}^{2} (-1)^{j-1} x^j dx^1 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^n.$$

(a) Show that ω is invariant under the orthogonal group of $\mathbb{R}^n$.
(b) Show that the restriction of ω on $\mathbb{S}^{n-1}(r)$, $\omega_{|_{\mathbb{S}^{n-1}(r)}}$, is $\pm$ the $(n-1)$-dimensional volume form on $\mathbb{S}^{n-1}(r)$ for every $r > 0$.

Bitte schreiben Sie Ihre(n) Namen, die Matrikelnummer sowie die Nummer Ihrer Übungsgruppe auf jedes Lösungsblatt. **Abgabe ist am Montag, 19.5.25, vor der Vorlesung**.