Aufgabe 1 (4 + 2 Punkte)

i) Let $\delta : \Lambda^p(M) \to \Lambda^{p-1}(M)$ be defined by

$$\delta\omega = (-1)^{m(p+1)+1} * d * \omega$$

where $*$ is the Hodge star operator sending

$$\omega_1 \wedge \cdots \wedge \omega_p \mapsto \omega_{p+1} \wedge \cdots \wedge \omega_m,$$

if $\{\omega_1, \cdots, \omega_m\}$ is a positively oriented frame. Prove that δ is a “formal adjoin” of d, i.e.,

$$\langle \delta\omega, \eta \rangle = \langle \omega, d\eta \rangle, \qquad \forall \omega \in \Lambda^p(M), \eta \in \Lambda^{p-1}(M),$$

where ω or η has compact support. Let $-\Delta = \delta d + d\delta$ be the Hodge Laplacian. Prove that Δ is self-adjoint, i.e.,

$$\langle \Delta\omega, \eta \rangle = \langle \omega, \Delta\eta \rangle, \qquad \forall \omega, \eta \in \Lambda^p(M),$$

where ω or η has compact support.

ii) If $\omega = a_j \omega_j$ is a 1-form, calculate $\delta\omega$.

Aufgabe 2 (4 + 2 + 4 Punkte)

1) $X = a_i e_i$ is a conformal Killing vector field, there exists a function f such that

$$a_{i,j} + a_{j,i} = f\delta_{ij}, \quad \forall 1 \le i, j \le m.$$

Prove the infinitesimal generator of a 1-parameter of conformal transformations of M is a conformal Killing vector field. By conformal transformation $\phi : (M, g) \to (M, g)$ it means that $\phi^*(g) = u^2 g$ for some function $u : M \to \mathbb{R}$.

2) Let us consider the following transformation of $\mathbb{S}^n$. For any $b \in \mathbb{R}^{n+1}$ with $|b| < 1$

$$\xi(x) = \frac{x + (\mu\langle x, b\rangle + \nu)b}{\nu(1 + \langle x, b\rangle)},$$

where $\nu = (1 - |b|^2)^{-\frac{1}{2}}$ and $\mu = (\nu - 1)|b|^{-2}$.
2a) Prove $\xi : \mathbb{S}^n \to \mathbb{S}^n$.

2b) Prove that the differential map ξ_* of Ξ is

$$\xi_*(v) = \nu^{-2}(1 + \langle x, b\rangle)^{-2}\{\nu(1 + \langle x, b\rangle)v - \nu\langle v, b\rangle x + \langle v, b\rangle(1 - \nu)|b|^{-2}b\},$$

where v is a tangent vector to $\mathbb{S}^n$ at x. It follows

$$\langle \xi_*(v), \xi_*(w)\rangle = \frac{1 - |b|^2}{(1 + \langle x, b\rangle)^2}\langle v, w\rangle.$$

This means that ξ is a conformal transformation of $\mathbb{S}^n$

Bitte schreiben Sie Ihre(n) Namen, die Matrikelnummer sowie die Nummer Ihrer Übungsgruppe auf jedes Lösungsblatt. **Abgabe ist am Montag, 26.5.25, vor der Vorlesung**.