Übungsaufgaben zur Vorlesung
DG II — SS2025, Serie 05
Prof. Dr. G. Wang — 26.05.25
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Aufgabe 1 (4 Punkte)

Let us consider $S^n_K = S^n(\frac{1}{\sqrt{K}})$. Fix one point p on S^n_K and consider the (modified) distance function $u_K = \cos(\sqrt{K}r)$, where r is the distance function to p. Show that u_K satisfies $\Delta u_K = -(nK)u_K$ and $\int u_K dvol = 0$.

Aufgabe 2 (4 Punkte)

Show that any function on an m-dimensional Riemannian manifold satisfies

$$|\operatorname{Hess} u|^2 \geq \frac{1}{n}|\Delta u|^2$$

with equality holding only when $\operatorname{Hess} u = \frac{\Delta u}{m} g$. What can you say about M when $\operatorname{Hess} u = \frac{\Delta u}{m} g$?

Aufgabe 3 (4 Punkte)

Let (M, g) be an n-dimensional Riemannian manifold such that for some compact subset $K \subset M$, $M \setminus K$ is isometric to $\mathbb{R}^n \backslash C$ for some compact set $C \subset \mathbb{R}^n$. If $Ric_g \geq 0$, show that $M = \mathbb{R}^n$.

Aufgabe 4 (4 Punkte)

Let $\gamma_+ : [0, \infty) \to M$ be a ray and r be the distance function. For any $x \in M$, define $b_x(t) = t - r(x, \gamma_+(t))$. Show that

1) For fixed x, the function $t \mapsto b_x(t)$ is increasing and bounded in absolute value by $r(x, \gamma_+(0))$. It follows that the Busemann function $\beta_+(x) = \lim_{\to\infty} b_x(t)$ is well-defined

2) $|b_x(t) - b_y(t)| \leq r(x, y)$. And show that β_+ is a Lipschitz function with Lipschitz constant 1.

Bitte schreiben Sie Ihre(n) Namen, die Matrikelnummer sowie die Nummer Ihrer Übungsgruppe auf jedes Lösungsblatt. **Abgabe ist am Montag, 2.6.25, vor der Vorlesung**.