Übungsaufgaben zur Vorlesung
DG II — SS2025, Serie 06
Prof. Dr. G. Wang — 2.6.25
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Aufgabe 1 (4 Punkte)

Given two vector fields X and Y on (M, g) such that ∇X and ∇Y are symmetric, develop Bochner formulas for $Hess\frac{1}{2}g(X,Y)$ and $\Delta\frac{1}{2}g(X,Y)$. (Here ∇X is symmetric, if $g(\nabla_{e_i}X, e_j)$ is symmetric in i, j)

Aufgabe 2 (4 Punkte)

Let be $M^n \subset \mathbb{R}^{n+1}$ be an isometric immersion.

(1) Show that the second fundamental form II is a Codazzi tensor, i.e., $h_{ij,k} = h_{ik,j}$, where h_{ij} is the second fundamental form and $h_{ij,k}$ its covariant derivative.
(2) Show Liebmann's theorem: If (M, g) has constant mean curvature and nonnegative second fundamental from, then (M, g) is a constant curvature sphere.
(*Hint. Compute* $\Delta(|h|^2)$ *and use (1), and certainly also the commutation formula* $h_{ij,kl} = h_{ij,lk} + R_{mjkl}h_{im} + \cdots$)

Aufgabe 3 (4 Punkte)

Let f be a smooth function on M. Prove the following p-Bochner formula for $p > 2$

$$\begin{aligned}\frac{1}{p}\Delta(|\nabla f|^p) &= (p-2)|\nabla f|^{p-2}|\nabla|\nabla f||^2 \\ &\quad + \frac{1}{2}|\nabla f|^{p-2}\{|Hess\, f|^2 + \langle \nabla f, \nabla \Delta f\rangle + Ric(\nabla f, \nabla f)\}\end{aligned}$$

Aufgabe 4 (4 Punkte)

Let $\tilde{g} = e^u g$ be two (conformal) metrics. Let Δ_g and $\Delta_{\tilde{g}}$ be the Laplacian with respect to g and $\tilde{g}$ resp. You prove that these two Laplacians are related by

$$\Delta_{\tilde{g}} f = e^{-u}\Delta_g f + (1 - \frac{n}{2})e^{-2u}\nabla_g u \nabla_g f.$$

Bitte schreiben Sie Ihre(n) Namen, die Matrikelnummer sowie die Nummer Ihrer Übungsgruppe auf jedes Lösungsblatt. **Abgabe ist am Montag, 16.6.25, vor der Vorlesung**.