Übungsaufgaben zur Vorlesung
DG II SS2025, Serie 08
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Aufgabe 1 (4 Punkte)

Prove that for any 2-tensor $a_{ij\tau_i \otimes \tau_j}$, the Ricci identity holds:

$$a_{ij,kl} = a_{ij,lk} + a_{im} R_{mjlk} + a_{mj} R_{milk},$$

where $a_{ij,kl}$ means $a_{ij,k,l}$-i.e., the covariant derivative with respect to τ_l of the tensor $a_{ij,k}\tau_i \otimes \tau_j \otimes \tau_k$.

Facts

Let M^n be an n-dimensional cone embedded in $\mathbb{R}^{n+1}$ (i.e., $\lambda M = M$ for any $\lambda > 0$), with zero mean curvature, with only singularity at the origin. Define the quantity

$$\mathcal{D} := \sum_{i,j,k} h_{ij,k}^2 - \sum_{k=1}^{n} |A|^{-2} (h_{ij} h_{ij,k})^2. \tag{1}$$

Let $x \in M$ and select the frame $\tau_1, \ldots, \tau_n$ so that τ_n is radial ($i.e.$, $\frac{x}{|x|}$) along the ray l_x through x, and so (as vectors in $\mathbb{R}^{n+1}$) $\tau_1, \ldots, \tau_n$ are constant along l_x. One has the following facts:

1. $h_{nj} = h_{jn=0}$ on l_x for $j = 1, \ldots, n$, and (hint: since M is a cone, $h_{ij}(\lambda x) = \lambda^{-1} h_{ij}(x)$, $\forall \lambda > 0$)
$$h_{ij,n} = -r^{-1} h_{ij} \quad on \quad l_x. \tag{2}$$

2. Rearranging the expression of $\mathcal{D}$, one can show that
$$\mathcal{D} = \frac{1}{2} \sum_{k=1}^{n} \sum_{i,j,r,s=1}^{n} |A|^{-2} (h_{rs} h_{ij,k} - h_{ij} h_{rs,k})^2,$$

3. Finally, one can prove that
$$\mathcal{D}(x) \geq 2|x|^{-2} |A(x)|^{-2}, \quad \forall 0 \neq x \in M.$$

Remark: note that
$$\sum_{i,j,r,s=1}^{n} (h_{rs} h_{ij,k} - h_{ij} h_{rs,k})^2 \geq 4 \sum_{i,j,r=1,s=n}^{n-1} (h_{rs} h_{ij,k} - h_{ij} h_{rs,k})^2,$$

which implies

$$\mathcal{D} \geq 2|A|^{-2} \sum_{k=1}^{n} \sum_{i,j,r=1}^{n-1} (h_{ij} h_{rn,k})^2.$$

Then use Codazzi equation and eqn. (2), one can complete the proof.

Aufgabe 2 (4 + 4 + 4 + 4* Punkte)

Suppose $2 \leq n \leq 6$, and M^n is an n-dimensional cone embedded in $\mathbb{R}^{n+1}$ (i.e., $\lambda M = M$ for any $\lambda > 0$), with zero mean curvature, with only singularity at the origin. Suppose that M is stable, in the sense that the stability inequality holds:

$$\int_M \left(|\nabla \xi|^2 - \xi^2 |A|^2\right) \mathrm{d}\mathcal{H}^n \geq 0$$

for every $\xi \in C_c^1(M)$ (note that $0 \notin M$, so such ξ vanish in a neighborhood of 0).

1. Prove the integral estimate

$$2 \int_M r^{-2} \xi^2 |A|^2 \leq \int_M |A|^2 |\nabla \xi|^2, \quad \forall \xi \in C_c^1(M).$$

   Here $r(x)$ denotes the Euclidean distance of x to the origin.

   Hint: Use **Facts: Item (3)** *in Simons identity to get a differential equation for* $\frac{1}{2}\Delta_M(|A|^2)$*. Then replace* ξ *by* $\xi|A|$ *in the stability inequality.*

2. Suppose that we have already shown that Item 1 (the integral estimate) is valid even for those ξ that does not have compact support in M, but just with the property that ξ is locally Lipschitz and

$$\int_M r^{-2} \xi^2 |A|^2 < \infty. \tag{3}$$

   For the cone M, we can write

$$\int_M \varphi(x) \mathrm{d}\mathcal{H}^n(x) = \int_0^\infty r^{n-1} \int_S \varphi(r\omega) \mathrm{d}\mathcal{H}^{n-1}(\omega) \mathrm{d}r, \tag{4}$$

   for any non-negative continuous φ on M, where $S = M \cap \mathbb{S}^n$ is a compact $(n-1)$-dimensional embedded submanifold. Use eqn. (4) to check that $\xi_1 := r^{1+\epsilon} r_1^{1-\frac{n}{2}-2\epsilon}$, where $r_1 = \max\{1, r\}$, is a valid choice to ensure eqn. (3).

   Hint: note that since M *is a cone, we have* $|A(x)|^2 = r^{-2}|A(\frac{x}{|x|})|^2$.

3. Use the above ξ_1 to test the integral estimate shown in Item (1), and show that

$$2 \int_M r^{2\epsilon} r_1^{2-n-4\epsilon} |A|^2$$

   can be estimated by the sum of integrals $\int_{M \cap \{r>1\}}(\cdot)$ and $\int_{M \cap \{r<1\}}(\cdot)$, and the sum of these two integrals is in particular finite.

4. For $2 \leq n \leq 6$, use Item (3) to conclude that $|A|^2 \equiv 0$ on M.

Remark: This proves J. Simons [Sim] result on non-existence in $\mathbb{R}^{n+1}$ *of* n*-dimensional stable minimal cones for* $2 \leq n \leq 6$*. The proof follows from Schoen-Simon-Yau [ScSiYa], which is slightly different from the original proof by Simons.*

Bitte schreiben Sie Ihre(n) Namen, die Matrikelnummer sowie die Nummer Ihrer Übungsgruppe auf jedes Lösungsblatt. **Abgabe ist am Montag, 30.6.25, vor der Vorlesung**.