Übungsaufgaben zur Vorlesung
DG II — SS2025, Serie 10
Prof. Dr. G. Wang — 7.7.25
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Aufgabe 1 Coarea formula (4 Punkte)

Let $\Omega \subset \mathbb{R}^n$ be an open set and $F : \Omega \to \mathbb{R}^k$ a C^1-submersion, that is, a C^1-smooth map with surjective differential at each point. As a consequence, we have that $F(\Omega)$ is open in $\mathbb{R}^k$ and that, for every $y \in F(\Omega)$, the set $F^{-1}(y) \subset \Omega$ is an immersed submanifold of dimension $n-k$. Then, for every $u \in L^1(\Omega)$ with compact support,

$$\int_\Omega u(x)J(DF(x)dx = \int_{F(\Omega)} \int_{F^{-1}(y)} u(x)dS^{n-k}dy,$$

where

$$J(DF(x)) = \sqrt{\det(DF(x)DF(x)^T)}$$

Compute $J(DF)$ when $k=1$ and $k=n-1$

Aufgabe 2 (4 Punkte)

Using the coarea formula to prove the following formulas. Let $\mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$ centered at 0. If $f \in C^1(\mathbb{R}^n)$, then, for every $u \in C^0(\mathbb{S}^{n-1})$

$$\int_{\mathbb{S}^{n-1}} u(x)|\nabla f(x) - (\nabla f(x)\cdot x)x|dS^{n-1}(x) = \int_\mathbb{R} \int_{\mathbb{S}^{n-1}\cap\{f=t\}} u(x)dS^{n-2}(x)dt.$$

If $f(x) = x_n$, then for every $u \in C^0(\mathbb{S}^{n-1})$,

$$\int_{\mathbb{S}^{n-1}} u(x)\sqrt{1-x_n^2}dS^{n-1}(x) = \int_\mathbb{R} \int_{\mathbb{S}^{n-1}\cap\{x_n=t\}} u(x)dS^{n-2}(x)dt.$$

Aufgabe 3 (4 Punkte)

Show that, if $u \in C^0(\partial B_r(0))$ for some $r > 0$, then

$$\int_{\partial B_r(0)} u(x)dS^{n-1}(x) = \int_{-r}^{r} \int_{\partial B_r(0)\cap\{x_n=t\}} u(x)dS^{n-2}(x)\frac{1}{\sqrt{r^2-t^2}}dt.$$

Aufgabe 4 (4 Punkte)

Let $u : \Omega \to \mathbb{R}$ be a smooth function without critical points in Ω, i.e, $\nabla u(x) \neq 0$ for any $x \in \Omega$, which implies that $\Gamma_t = u^{-1}(t)$ is a hypersurface. Let $\Omega_t = \{x \in \Omega | u(x) \leq t\}$, and assume that $Vol(\Omega_t) < \infty$ for any t, and that the function $t \mapsto Vol(\Omega_t)$ is smooth. Prove

$$\frac{d}{dt}Vol(\Omega_t) = \int_{\Gamma_t} \frac{1}{|\nabla u|}dS_t.$$

Bitte schreiben Sie Ihre(n) Namen, die Matrikelnummer sowie die Nummer Ihrer Übungsgruppe auf jedes Lösungsblatt. **Abgabe ist am Montag, 14.7.25, vor der Vorlesung**.