

Lecture 1

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1. REVIEW OF TOPOLOGY

I want to briefly review some basics of topology that you have probably heard already.

1.0. Topological spaces. The first thing I should discuss is what is a topological space?

Definition 1.1 (Topological space). *A topological space is a pair $(X, \mathcal{O}_X)$, where X is a set, and $\mathcal{O}_X \subset 2^X$ is a set of subsets of X (equivalently, subset of power set of X) satisfies:*

- $\emptyset, X \in \mathcal{O}_X$.
- $A, B \in \mathcal{O}_X \implies A \cap B \in \mathcal{O}_X$.
- $A_i \in \mathcal{O}_X, i \in I \implies \bigcup_{i \in I} A_i \in \mathcal{O}_X$.

We say A is an open set of X if and only if $A \in \mathcal{O}_X$.

Topology defines open sets to be stable under finite intersections and arbitrary unions. It is natural to ask if this stability extends to infinite intersections. To build intuition, consider the real line. The sets $A_n = (-1/n, 1/n)$ are all open intervals. What is $\bigcap_{n=1}^{\infty} A_n$? It is the single point 0, which is not an open set. This shows that openness is not preserved under infinite intersections.

Example 1.2. *Given a set X , there are two natural topologies that can be defined on it:*

Trivial topology: $\mathcal{O}_X := \{\emptyset, X\}$.

Discrete topology: $\mathcal{O}_X := 2^X$.

We now introduce two fundamental concepts related to open sets.

- (1) A subset $A \subset X$ is said to be *closed* if and only if its complement $X - A$ is open.
- (2) A subset $U \subset X$ is called a *neighborhood* of a point $p \in X$ if there exists an open set A such that $p \in A \subset U$.

The definition of a topological space is highly general. It is built on only three axioms, making it applicable in a wide range of contexts. As a result, there exist certain highly unconventional topologies that defy geometric intuition. A notable example of such a topology can be found in **Furstenberg's proof of the infinitude of primes**.

In this course, we will focus exclusively on well-behaved topological spaces. Arguably the most important class of such spaces is that which arises from metric spaces.

Example 1.3. *Let (X, d) be a metric space. That is, X is a set, and $d : X \times X \rightarrow \mathbf{R}^+$ is a non-negative function satisfying:*

positivity: $d(x, y) = 0 \iff x = y$.

symmetry: $d(x, y) = d(y, x)$.

triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$.

Then we define

$$\mathcal{O}_X := \{A \subset X : \forall p \in A, \exists r > 0 \text{ such that } B(p, r) \subset A\}.$$

It is straightforward to verify $(X, \mathcal{O}_X)$ is a topological space.

Thus, $\mathbf{R}^n$ carries a natural topology induced by the Euclidean metric. When referring to $\mathbf{R}^n$ as a topological space, we implicitly consider this metric-induced topology, and the notation $\mathcal{O}_{\mathbf{R}^n}$ is often omitted for simplicity.

It is a good point to review another concept, which is called a *basis* for a topology.

Definition 1.4. Assume $(X, \mathcal{O}_X)$ is a topological space. Then $\mathcal{B} \subset 2^X$ is called a *basis* for this topological space if

$$\begin{aligned} A \in \mathcal{O}_X &\iff A \text{ is a union of elements in } \mathcal{B}. \\ &\iff \forall p \in A, \exists B \in \mathcal{B} \text{ such that } p \in B \subset A. \end{aligned}$$

Example 1.5. (1) $\mathcal{B} = \{B(x, r) \subset \mathbf{R}^n : x \in \mathbf{R}^n, r > 0\}$ is a basis for $\mathbf{R}^n$.

(2) $\mathcal{B} = \{B(x, r) \subset \mathbf{R}^n : x \in \mathbf{Q}^n, r \in \mathbf{Q}, r > 0\}$ is also a basis for $\mathbf{R}^n$.

The previous example shows that $\mathbf{R}^n$ has a countable basis. This give us a way to say that $\mathbf{R}^n$ is not too big. In general, we have the following definition.

Definition 1.6 (Second countable). If the topological space $(X, \mathcal{O}_X)$ has a countable basis, we say X is *second countable*.

1.6. Continuous maps. Let us now discuss maps between topological spaces. It is convenient to simplify our notation at this stage. We will often write X to represent the entire topological space $(X, \mathcal{O}_X)$, implicitly assuming that a topology $\mathcal{O}_X$ (the family of open sets) has been assigned. This abbreviation is harmless, as the relevant topology will always be clear from the context.

Now, let $X = (X, \mathcal{O}_X)$, $Y = (Y, \mathcal{O}_Y)$ be topological spaces. A central idea in topology is that specifying a topology on a space allows us to define what continuity means. In fact, this is one of the primary motivations behind the very definition of a topological space.

Definition 1.7. A map $\varphi : X \rightarrow Y$ is said to be *continuous* if for any open set $B \subset Y$, the preimage $\varphi^{-1}(B) \subset X$ is open.

It is essential to note that continuity is defined in terms of the preimage of open sets being open—not the image. For instance, a constant map $\mathbf{R} \rightarrow \mathbf{R}$ is continuous, even though the image of any non-empty open set is a single point, which is not open in $\mathbf{R}$.

Example 1.8. Let $X = Y = \mathbf{R}$, equipped with the standard topology. In this case, the topological definition of continuity given above is equivalent to the classical $\varepsilon - \delta$ definition from analysis.

The following defines when two topological spaces are considered "the same."

Definition 1.9. A continuous map $\varphi : X \rightarrow Y$ is called *homeomorphism* if φ is bijective, and both φ, φ^{-1} are continuous.

An important warning is necessary here: a continuous bijective map $\varphi : X \rightarrow Y$ is not necessarily a homeomorphism. In other words, the continuity of the inverse is not automatic. This phenomenon does occur, though in relatively special situations. One elementary example is given in Item 5 of Fact 2.8. A deeper example is provided by **Brouwer's theorem on invariance of domain**.

1.9. Subspace topology. Let X be a topological space and $Y \subset X$ a subset. We can naturally equip Y with a topology, called the *subspace topology*, by declaring the open sets in Y to be exactly those of the form

$$\mathcal{O}_Y = \{A \cap Y : A \in \mathcal{O}_X\}.$$

Whenever we refer to a map defined on Y —even without explicitly specifying its topology—we always assume that Y is endowed with this subspace topology. In particular, continuity of such a map is understood with respect to this induced topology.

It is important to understand (or at least feel comfortable with) the following two facts:

Fact 1.10. *If $f : X \rightarrow Z$ is continuous, then its restriction $f|_Y : Y \rightarrow Z$ is also continuous with respect to the subspace topology on Y .*

Fact 1.11. *The subspace topology is the coarsest topology on Y (i.e., the one with the fewest open sets) for which the inclusion map $Y \hookrightarrow X$ is continuous.*

Example 1.12. *Consider the inclusion map $\iota : \mathbf{R} \rightarrow \mathbf{R}^2$ defined by $x \mapsto (x, 0)$, which embeds the real line as the x -axis in the plane. Then the subspace topology on $\mathbf{R}$ induced by this inclusion coincides with the standard topology on $\mathbf{R}$.*

2. TOPOLOGICAL MANIFOLDS

A topic of particular interest in this course concerns topological spaces that, in some sense, "resemble" $\mathbf{R}^n$. Indeed, $\mathbf{R}^n$ serves as a key model throughout our study. In this context, it is worth mentioning a theorem that, while important to know, is not central to our development—the so-called *topological invariance of dimension*.

Theorem 2.1 (Topological invariance of dimension). *Let $\varphi : U \rightarrow V$ be a homeomorphism between nonempty open sets $U \subset \mathbf{R}^m$ and $V \subset \mathbf{R}^n$. Then $m = n$.*

This tells us that, on a topological level, $\mathbf{R}^m$ and $\mathbf{R}^n$ are fundamentally different when $m \neq n$. While this may align with our intuition, the proof is surprisingly deep and lies beyond the scope of this course—it typically makes use of tools such as homology groups.

Let us now return to the idea of spaces that look like $\mathbf{R}^n$.

Definition 2.2 (Locally Euclidean). *A topological space X is called locally Euclidean at a point $p \in X$ of dimension n if there exists an open neighborhood $p \in U \subset X$ that is homeomorphic to some open subset of $\mathbf{R}^n$. That is, there exist an open set $\tilde{U} \subset \mathbf{R}^n$ and a homeomorphism $\varphi : U \rightarrow \tilde{U}$.*

This means that, locally, the topology around p is indistinguishable from that of a Euclidean space. This is a very special property, and one we will encounter frequently. As a useful exercise, we note the following:

Exercise 2.3. *One may always take $\tilde{U} = B(0, 1) \subset \mathbf{R}^n$ in the above definition. (Hint: use a dilation to identify $\mathbf{R}^n$ with the unit ball, and restrict the original homeomorphism accordingly.)*

You might wonder why this property is so interesting. Consider, for example, the universe—or the surface of the Earth—as a topological space. We may not know its global structure (indeed, ancient people did not know the Earth was a sphere until they could observe it from outside), but we do know that locally, it appears Euclidean: our universe looks like $\mathbf{R}^3$ in small regions,

and the Earth's surface looks like $\mathbf{R}^2$. Thus, while the local topology is trivial, the global structure can be far more interesting.

Here is a lemma whose proof is worth understanding, as the underlying argument will be used repeatedly.

Lemma 2.4. *The dimension n in the definition of a locally Euclidean space is uniquely determined at each point p .*

Proof. Suppose X is locally Euclidean at p with two possible dimensions n_1 and n_2 . Then there exist open neighborhoods $p \in U_i \subset X$ and homeomorphisms $\varphi_i: U_i \rightarrow \tilde{U}_i \subset \mathbf{R}^{n_i}$ for $i = 1, 2$. Since $p \in U_1 \cap U_2$, the intersection is nonempty. Now consider the composition

$$\varphi_2 \circ \varphi_1^{-1}: \varphi_1(U_1 \cap U_2) \subset \mathbf{R}^{n_1} \rightarrow \varphi_2(U_1 \cap U_2) \subset \mathbf{R}^{n_2},$$

which is a homeomorphism between nonempty open subsets of $\mathbf{R}^{n_1}$ and $\mathbf{R}^{n_2}$. By the topological invariance of dimension, it follows that $n_1 = n_2$. $\square$

To further develop the language of manifolds, we introduce several key topological properties. The first is the following:

Definition 2.5. *A topological space X is called Hausdorff if for any two distinct points $p, q \in X$, there exist open sets $U, V \subset X$ such that $p \in U$, $q \in V$, and $U \cap V = \emptyset$.*

Example 2.6. (1) *Any metric space is Hausdorff in its induced topology.*

(2) *If X is separated by continuous functions in the sense that for every pair $p \neq q$ there exists a continuous $f: X \rightarrow \mathbf{R}$ with $f(p) \neq f(q)$, then X is Hausdorff.*

Next, we recall the important notion of compactness.

Definition 2.7. *A subset $K \subset X$ is called compact if every open cover $\{A_i\}_{i \in I}$ of K -that is, $K \subset \bigcup_{i \in I} A_i$ -admits a finite subcover $\{A_i\}_{i \in I'}$ satisfying*

$$K \subset \bigcup_{i \in I'} A_i, \quad I' \subset I, \quad |I'| < \infty.$$

If X itself is compact, we call X a compact space.

Although compactness is not always included in the definition of manifolds, it interacts fruitfully with the Hausdorff property. Moreover, since Euclidean space $\mathbf{R}^n$ is locally compact-meaning every point has a compact neighborhood within any given neighborhood-we will see later that all manifolds inherit this local compactness.

The following collection of basic topological facts will be useful throughout this course.

Fact 2.8. (1) *If X is Hausdorff and $Y \subset X$, then Y is Hausdorff under the subspace topology.*

(2) *In a Hausdorff space X , every compact subset $K \subset X$ is closed.*

(3) *If X is compact and $K \subset X$ is closed, then K is compact.*

(4) *The continuous image of a compact set is compact: if $\varphi: X \rightarrow Y$ is continuous and $K \subset X$ is compact, then $\varphi(K)$ is compact.*

(5) *If X is compact, Y is Hausdorff, and $\varphi: X \rightarrow Y$ is a continuous bijection, then φ is a homeomorphism.*

These facts complete our review of the necessary topological background. We now turn to the central object of interest in this course: manifolds.

Definition 2.9. A topological space M is called an n -dimensional topological manifold if it satisfies the following three conditions:

- (1) M is locally Euclidean of dimension n ,
- (2) M is Hausdorff,
- (3) M is second countable.

The requirement that M be locally Euclidean is fairly intuitive—without it, the space could exhibit pathological local behavior. It is natural to ask, however, why we also impose the Hausdorff and second countable properties. To illustrate their importance, we now examine what can go wrong if either of these conditions is omitted.

Non-Example 2.10 (Dropping the Hausdorff condition). Define $X := (\mathbf{R} \times \{0, 1\}) / \sim$ as the quotient of two real lines, where $(x, 0) \sim (x, 1)$ for all $x < 0$. A subset $A \subset X$ is open if and only if its preimage under the quotient map π is open in $\mathbf{R} \times \{0, 1\}$. One may verify that the sets $\{(x, 0) : x \in \mathbf{R}\} \cong \mathbf{R}$ and $\{(x, 1) : x \in \mathbf{R}\} \cong \mathbf{R}$ are open in X . Thus, X is locally Euclidean. However, X is not Hausdorff: the two distinct points $[(0, 0)]$ and $[(0, 1)]$ do not admit disjoint neighborhoods.

In essence, the Hausdorff condition prevents such pathological "branching" behavior, which is undesirable in a well-behaved geometric object.

Let us now examine the second countable condition.

Non-Example 2.11 (Dropping the second countable condition). Define $X := \mathbf{R}^2$ with the topology given by

$$\mathcal{O}_X = \{U \times \{y\} : U \in \mathcal{O}_{\mathbf{R}}, y \in \mathbf{R}\}.$$

Then $(X, \mathcal{O}_X)$ is locally Euclidean of dimension 1 at every point. However, this space fails to be second countable and has uncountably many connected components.

Here, the second countable condition serves to exclude spaces with too many components—in particular, uncountably many.

A more sophisticated example is the **long line**, which also violates second countability. Its construction is somewhat involved and relies on set-theoretic concepts such as well-ordered sets. Interested readers may refer to the linked Wikipedia entry for further details.

Now we give some examples of manifolds.

Example 2.12. If M is a 0-dimensional topological manifold, then M is a finite or countable set equipped with the discrete topology.

Example 2.13. If M^n is a topological manifold and $M' \subset M^n$ is an open subset, then M' is also an n -dimensional topological manifold. For instance, any open subset of $\mathbf{R}^n$ is a manifold.

Example 2.14.

$$\mathbf{S}^1 := \{(x_1, x_2) \in \mathbf{R}^2 : x_1^2 + x_2^2 = 1\}.$$

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