Lecture 10

November 21, 2025

1. Embedding

Recall that a subset M of a smooth manifold N is a *smooth submanifold at p* if there exist local coordinates $x^1, \dots, x^n$ defined near p in N and an integer $0 \leqslant k \leqslant n$ such that M is locally given by $x^1 = \cdots = x^k = 0$. If M is locally a smooth submanifold of N at every point $x \in M$, we say M is a *(smooth) submanifold* of N. This definition captures the "regularity" of the subset as it sits inside N.

In the previous lecture, we justified the term "submanifold" by showing that there exists a unique smooth structure on M compatible with the subspace topology, such that the inclusion map $\iota : M \to N$ is an immersion.

Now consider a smooth immersion $f : M \to N$ such that $f : M \to f(M)$ is a homeomorphism, where $f(M)$ carries the subspace topology inherited from N. In this case, $f(M)$ inherits a smooth submanifold structure from M, and we call such an immersion an *embedding*.

To see this explicitly, take any $q \in f(M) \subset N$ and choose $p \in M$ with $f(p) = q$. Since f is an immersion, there exist coordinate charts $\varphi = (x^1, \dots, x^m) : U \to \mathbf{R}^m$ around p and $\psi = (y^1, \dots, y^n) : V \to \mathbf{R}^n$ around q with $f(U) \subset V$, such that:

$$y^i \circ f = x^i \quad \text{for } 1 \leqslant i \leqslant m,$$

and

$$y^i \circ f = 0 \quad \text{for } m+1 \leqslant i \leqslant n.$$

Moreover, since f is a homeomorphism onto its image, there exists an open neighborhood $W \subset N$ of q such that $W \cap f(M) = W \cap f(U)$. Therefore, within $W \cap V$, the set $f(M)$ is precisely given by $y^{m+1} = \cdots = y^n = 0$, confirming that $f(M)$ is a smooth submanifold of N.

2. Preimage Construction

Recall condition (4) in the characterization of submersions: a smooth map $f : M \to N$ is a submersion at p if there exist local coordinates $\{x^i\}$ around p and $\{y^i\}$ around $f(p)$ such that $x^i = y^i \circ f$ for $1 \leqslant i \leqslant n$ (where $n = \dim N$).

This local form immediately implies that the fiber $f^{-1}(q)$ is locally given by $x^1 = \cdots = x^n = 0$ near p, hence is a smooth submanifold of dimension $m - n$ at p. This observation motivates the following definition.

Definition 2.1. *Let $f : M \to N$ be a smooth map. A point $q \in N$ is called a* regular value *of f if f is a submersion at every point $x \in f^{-1}(q)$.*

The discussion above leads directly to the following fundamental result:

Theorem 2.2 (Preimage Theorem)**.** *If $f : M \to N$ is a smooth map and $q \in N$ is a regular value, then $f^{-1}(q)$ is a smooth submanifold of M of dimension* $\dim M - \dim N$*. Moreover, for every $p \in f^{-1}(q)$, the tangent space satisfies*

$$T_p(f^{-1}(q)) = \ker(T_p f : T_p M \to T_q N).$$

It is often convenient to think not in terms of the absolute dimension of $f^{-1}(q)$, but rather its *codimension*—the amount by which its dimension is less than that of M. In the theorem above, $f^{-1}(q)$ has codimension $n = \dim N$.

Example 2.3 (The Sphere)**.** *The $(n-1)$-sphere $S^{n-1} \subset \mathbf{R}^n$ can be realized as $f^{-1}(1)$, where $f : \mathbf{R}^n \to \mathbf{R}$ is defined by*

$$f(x_1, \dots, x_n) = x_1^2 + \cdots + x_n^2.$$

This map is smooth, and its derivative at any point is $[2x_1, \dots, 2x_n]$. Since this is nonzero for all $(x_1, \dots, x_n) \neq (0, \dots, 0)$, every nonzero real number is a regular value of f. In particular, 1 is a regular value, so S^{n-1} is a smooth manifold of dimension $n-1$ (or codimension 1).

Example 2.4 (Orthogonal Group)**.** *The orthogonal group $O(n)$, consisting of $n \times n$ orthogonal matrices, can be described as $f^{-1}(I)$, where $f : \mathrm{GL}(n, \mathbf{R}) \to \mathrm{Sym}(n)$ is defined by $f(A) = A^T A$. Here* $\mathrm{Sym}(n)$ *denotes the space of symmetric $n \times n$ matrices. One can verify that I is a regular value of f, hence $O(n)$ is a smooth submanifold of* $\mathrm{GL}(n, \mathbf{R})$ *of dimension $\frac{1}{2}n(n-1)$.*

Definition 2.5. *Let $f : M \to N$ be a smooth map. A point $q \in N$ is called a* critical value *of f if it is not a regular value.*

Example 2.6. *If* $\dim M = m < n = \dim N$ *and $f : M \to N$ is a smooth map, then every point in the image $f(M)$ is a critical value of f. This is because at any $p \in M$, the linear map $T_p f : T_p M \to T_{f(p)} N$ cannot be surjective for dimensional reasons.*

The preimage construction provides a powerful method for producing new manifolds from old ones, often avoiding the need for explicit coordinate charts. The sphere and orthogonal group examples demonstrate how naturally occurring geometric objects can be recognized as smooth manifolds through this approach.

3. Transversality

We now address the natural question: when is the intersection of two submanifolds again a submanifold? In general, the intersection of two submanifolds can be quite pathological. For instance, when discussing partitions of unity, we showed that any closed subset $K \subset M$ of a manifold M can arise as the zero set of a non-negative smooth function $f \in C^\infty(M)$.

Let us consider the graph of this function:

$$\Gamma_f := \{(x, f(x)) \in M \times \mathbf{R} : x \in M\},$$

which is a smooth submanifold of $M \times \mathbf{R}$. On the other hand, we have the zero section:

$$\Gamma_0 := M \times \{0\},$$

which is also a smooth submanifold. Their intersection is

$$\Gamma_f \cap \Gamma_0 = \{(x, 0) \in M \times \mathbf{R} : f(x) = 0\} = K \times \{0\},$$

which, in general, may have no manifold structure at all. This shows that submanifold intersections need not themselves be submanifolds.

We will now introduce a sufficient condition that ensures the intersection of two submanifolds is again a submanifold. This condition is called *transversality*. The intuitive idea is that if two submanifolds intersect "as little as possible," their intersection will behave nicely and inherit a manifold structure.

Before giving the definition, we recall a basic lemma from linear algebra:

Lemma 3.1. *Let V be a finite-dimensional vector space and $V_1, V_2 \subset V$ be subspaces. Then the sequence*

$$0 \to V_1 \cap V_2 \xrightarrow{i} V \xrightarrow{j} V/V_1 \oplus V/V_2 \xrightarrow{k} V/(V_1 + V_2) \to 0$$

is exact, where:

- *$i : V_1 \cap V_2 \to V$ is the natural inclusion*
- *$j : V \to V/V_1 \oplus V/V_2$ is given by $j(v) = (v + V_1, v + V_2)$*
- *$k : V/V_1 \oplus V/V_2 \to V/(V_1 + V_2)$ is given by $k(v_1 + V_1, v_2 + V_2) = (v_1 - v_2) + (V_1 + V_2)$*

Proof of Lemma. We verify exactness at each term:

(1) At $V_1 \cap V_2$: The map i is injective, so $\ker i = 0$.
(2) At V: We have $\ker j = \{v \in V : v + V_1 = 0 \text{ and } v + V_2 = 0\} = V_1 \cap V_2 = \operatorname{im} i$.
(3) At $V/V_1 \oplus V/V_2$:
   - $\operatorname{im} j \subset \ker k$: For any $v \in V$, $k(j(v)) = k(v + V_1, v + V_2) = (v - v) + (V_1 + V_2) = 0$.
   - $\ker k \subset \operatorname{im} j$: Suppose $k(v_1 + V_1, v_2 + V_2) = 0$. Then $v_1 - v_2 \in V_1 + V_2$, so we can write $v_1 - v_2 = w_1 + w_2$ with $w_i \in V_i$. Let $v = v_1 - w_1 = v_2 + w_2$. Then $j(v) = (v + V_1, v + V_2) = (v_1 + V_1, v_2 + V_2)$.
(4) At $V/(V_1 + V_2)$: The map k is surjective since for any $w + (V_1 + V_2)$, we have $k(w + V_1, 0 + V_2) = w + (V_1 + V_2)$.

□

Theorem 3.2. *Let M be a manifold, N_1 and N_2 be submanifolds of M, and $p \in N_1 \cap N_2$. If $T_pM = T_pN_1 + T_pN_2$, then there exists a chart (U, φ) at p of M such that*

$$\varphi(U) = V_1 \times V_2 \times W$$
$$\varphi(U \cap N_1) = \{0\} \times V_2 \times W$$
$$\varphi(U \cap N_2) = V_1 \times \{0\} \times W.$$

Equivalently, there exists a coordinate system $x^1, \ldots, x^n$ at p and integers $r_1, r_2 \geqslant 0$ with $r_1 + r_2 \leqslant n$ such that:

$$N_1 \textit{ is given by } x^1 = \ldots = x^{r_1} = 0 \textit{ in a neighborhood of } p,$$
$$N_2 \textit{ is given by } x^{r_1+1} = \ldots = x^{r_1+r_2} = 0 \textit{ in a neighborhood of } p.$$

Proof. Since N_1 and N_2 are submanifolds of M, we can find submersions

$$f_1 : M \to \mathbf{R}^{r_1}, \quad f_2 : M \to \mathbf{R}^{r_2}$$

such that $N_i = f_i^{-1}(0)$ for $i = 1, 2$. Let $(x^1, \ldots, x^{r_1})$ and $(x^{r_1+1}, \ldots, x^{r_1+r_2})$ be the coordinate components of f_1 and f_2, respectively.

Apply Lemma 3.1 to $V = T_pM$, $V_1 = T_pN_1$, $V_2 = T_pN_2$:

$$0 \to T_p(N_1 \cap N_2) \xrightarrow{i} T_pM \xrightarrow{j} T_pM/T_pN_1 \oplus T_pM/T_pN_2 \xrightarrow{k} T_pM/(T_pN_1 + T_pN_2) \to 0.$$

The condition $T_pM = T_pN_1 + T_pN_2$ implies $T_pM/(T_pN_1 + T_pN_2) = 0$, so the sequence becomes short exact:

$$0 \to T_p(N_1 \cap N_2) \xrightarrow{i} T_pM \xrightarrow{j} T_pM/T_pN_1 \oplus T_pM/T_pN_2 \to 0.$$

In particular, j is surjective.

Now, the differential $d_p(f_1, f_2) : T_pM \to T_0\mathbf{R}^{r_1} \oplus T_0\mathbf{R}^{r_2}$ factors through j and the isomorphisms

$$T_pM/T_pN_1 \oplus T_pM/T_pN_2 \cong T_0\mathbf{R}^{r_1} \oplus T_0\mathbf{R}^{r_2}$$

induced by the submersions f_i. More precisely, we have a factorization:

$$d_p(f_1, f_2) = \Phi \circ j,$$

where $\Phi : T_pM/T_pN_1 \oplus T_pM/T_pN_2 \to T_0\mathbf{R}^{r_1} \oplus T_0\mathbf{R}^{r_2}$ is an isomorphism.

Since j is surjective and Φ is an isomorphism, it follows that $d_p(f_1, f_2)$ is also surjective. Therefore, $(f_1, f_2) : M \to \mathbf{R}^{r_1} \times \mathbf{R}^{r_2}$ is a submersion at p.

By the submersion theorem, there exists a coordinate system $(x^1, \dots, x^n)$ around p such that (f_1, f_2) corresponds to the projection onto the first $r_1 + r_2$ coordinates. In these coordinates:

- $N_1 = f_1^{-1}(0)$ is given by $x^1 = \cdots = x^{r_1} = 0$
- $N_2 = f_2^{-1}(0)$ is given by $x^{r_1+1} = \cdots = x^{r_1+r_2} = 0$

This completes the proof. □

If N_1 and N_2 satisfy the condition of the preceding theorem at p, we say that N_1 and N_2 are **transversal at** p.

Corollary 3.3. *Suppose N_1 and N_2 are transversal at p. Then:*

(1) N_1 and N_2 are transversal in a neighborhood of p.
(2) $N_1 \cap N_2$ is locally a submanifold of M at p.
(3) $T_p(N_1 \cap N_2) = T_pN_1 \cap T_pN_2$.

Proof. The first statement follows from the continuity of the transversality condition. The second and third statements are immediate consequences of the coordinate representation in the preceding theorem. □

References

[1] Alexander Kupers, *Lectures on Differential Topology*, 2020. Lecture notes.
[2] Jean-Pierre Serre, *Lie Algebras and Lie Groups*, Lecture Notes in Mathematics, vol. 1500, Springer, 1992.
[3] John M. Lee, *Introduction to Smooth Manifolds*, 2nd ed., Graduate Texts in Mathematics, vol. 218, Springer, 2012.

Mathematisches Institut, Universität Freiburg
Email address: yuchen.bi@math.uni-freiburg.de