Lecture 11

November 21, 2025

1. Fibre Products, Pullbacks, and Cartesian Squares

We now present a unified framework that generalizes both the preimage construction and the transversal intersection of submanifolds: the concept of *fibre products* or *pullbacks*.

1.0. Definition and Basic Properties. Consider a pair of smooth maps $f_i : N_i \to M$, $i = 1, 2$. Their **fibre product** (also called **pullback**) is defined as:

$$N_1 \times_M N_2 = \{(y_1, y_2) \in N_1 \times N_2 : f_1(y_1) = f_2(y_2)\}.$$

Let $p_i : N_1 \times_M N_2 \to N_i$ be the natural projections, and define $f = f_1 \circ p_1 = f_2 \circ p_2$. This yields a commutative diagram:

$$\begin{tikzcd}
N_1 \times_M N_2 \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, "f"] & N_2 \arrow[d, "f_2"] \\
N_1 \arrow[r, "f_1"'] & M
\end{tikzcd}$$

In category theory, such a square is called a **cartesian square** or **pullback square** because it satisfies the following universal property:

> For any manifold X with maps $g_1 : X \to N_1$ and $g_2 : X \to N_2$ such that $f_1 \circ g_1 = f_2 \circ g_2$, there exists a unique map $h : X \to N_1 \times_M N_2$ making the following diagram commute:

$$\begin{tikzcd}
X \arrow[dr, "h"] \arrow[drr, bend left, "g_2"] \arrow[ddr, bend right, "g_1"'] & & \\
& N_1 \times_M N_2 \arrow[r, "p_2"'] \arrow[d, "p_1"] & N_2 \arrow[d, "f_2"] \\
& N_1 \arrow[r, "f_1"'] & M
\end{tikzcd}$$

1.0. Transversality and Smoothness. In the smooth category, the fibre product $N_1 \times_M N_2$ may not be a manifold in general. To ensure smoothness, we need a transversality condition.

Let $(y_1, y_2) \in N_1 \times_M N_2$ and let $p = f(y_1, y_2)$. We say that f_1 and f_2 are **transverse at** (y_1, y_2) if

$$T_p M = \mathrm{Im}(T_{y_1} f_1) + \mathrm{Im}(T_{y_2} f_2).$$

we write $f_1 \pitchfork_p f_2$ indicating that the transversality condition holds at the point p, that is , f_1 and $_2$ are transverse at $\forall (y_1, y_2) \in f^{-1}(p)$. A commutative square of the above form is called **transversal cartesian at** p if this transversality condition holds.

As usual, if this condition holds for every $(y_1, y_2) \in N_1 \times_M \times N_2$, we simply say that f_1 and f_2 are **transverse** and write

$$f \pitchfork g.$$

Theorem 1.1 (Smoothness of Transverse Fibre Products)**.** *Suppose f_1 and f_2 are transverse at (y_1, y_2). Then:*

(1) f_1 and f_2 are transverse at all points in a neighborhood of (y_1, y_2) in $N_1 \times_M N_2$.
(2) $N_1 \times_M N_2$ is locally a submanifold of $N_1 \times N_2$ at (y_1, y_2).
(3) The tangent space is given by the fibre product:

$$T_{(y_1,y_2)}(N_1 \times_M N_2) = T_{y_1}N_1 \times_{T_pM} T_{y_2}N_2 = \{(v_1, v_2) \in T_{y_1}N_1 \times T_{y_2}N_2 : T_{y_1}f_1(v_1) = T_{y_2}f_2(v_2)\}.$$

Proof. Set $N = N_1 \times N_2$ and $P = N_1 \times_M N_2$. Define maps $g_i : N \to N \times M$ by:

$$g_1(y_1, y_2) = ((y_1, y_2), f_1(y_1)), \quad g_2(y_1, y_2) = ((y_1, y_2), f_2(y_2)).$$

Let $g = g_i|_P$. Then:

(1) $g_1(N)$ and $g_2(N)$ are transverse at $g(y_1, y_2)$.
(2) $g(P) = g_1(N) \cap g_2(N)$.

The result follows by applying the transverse intersection theorem to the submanifolds $g_1(N)$ and $g_2(N)$ of $N \times M$. □

1.1. Special Cases and Examples.

The fibre product construction unifies several important concepts:

Example 1.2 (Preimage as Fibre Product)**.** *Given $f : N \to M$ and a point $q \in M$, consider f and the inclusion $i : \{q\} \hookrightarrow M$. Their fibre product is:*

$$N \times_M \{q\} = \{(y, q) \in N \times \{q\} : f(y) = q\} \cong f^{-1}(q).$$

The transversality condition becomes the requirement that q is a regular value of f.

Example 1.3 (Intersection as Fibre Product)**.** *For submanifolds $N_1, N_2 \subset M$ with inclusions $\iota_i : N_i \hookrightarrow M$, their fibre product is:*

$$N_1 \times_M N_2 = \{(y_1, y_2) \in N_1 \times N_2 : \iota_1(y_1) = \iota_2(y_2)\} \cong N_1 \cap N_2.$$

The transversality condition is exactly

$$T_pM = T_pN_1 + T_pN_2, \qquad p \in N_1 \cap N_2.$$

When this condition holds, we say that N_1 and N_2 intersect transversely and write

$$N_1 \pitchfork N_2.$$

Example 1.4 (Product as Fibre Product)**.** *When M is a single point, the fibre product reduces to the ordinary product:*

$$N_1 \times_{\{*\}} N_2 \cong N_1 \times N_2.$$

Example 1.5 (Graph of a Map)**.** *Given $f : N_1 \to N_2$, consider the maps $f : N_1 \to N_2$ and $\mathrm{id} : N_2 \to N_2$. Their fibre product is the graph of f:*

$$N_1 \times_{N_2} N_2 = \{(y_1, y_2) \in N_1 \times N_2 : f(y_1) = y_2\} =: \Gamma_f.$$

2. The Morse-Sard Theorem

The usefulness of the preimage construction and of the transversality condition lies in the fact that the hypotheses are, in a very strong sense, "typically" satisfied. The Morse-Sard theorem guarantees that the set of non-regular values has measure zero, so for a fixed smooth map, almost every value is regular.

Theorem 2.1 (Morse–Sard)**.** *Let $f \in C^\infty(U, \mathbf{R}^n)$, where $U \subset \mathbf{R}^m$ is open. Then the set of critical values of f has Lebesgue measure 0 in $\mathbf{R}^n$.*

Proof. The proof proceeds by induction on m. Assume the theorem is already known for dimension $m-1$ whenever $m > 1$.

For $j \geqslant 1$, define

$$C_j = \{x \in U : f'(x) = 0,\ f''(x) = 0,\ \dots, f^{(j)}(x) = 0\}.$$

We first show:

$$\mathscr{L}^n(f(C_j)) = 0 \qquad \text{whenever } (j+1)n > m. \tag{2.1}$$

It suffices to prove $\mathscr{L}^n(f(K \cap C_j)) = 0$ for a compact cube $K \subset U$ of side length ℓ. Divide K into k^m subcubes of side $\varepsilon = \ell/k$. Let $I_1, \dots, I_N$ be the subcubes intersecting C_j, and choose $x_t \in I_t \cap C_j$.

Taylor's theorem and $x_t \in C_j$ give

$$|f(x) - f(x_t)| \leqslant A|x - x_t|^{j+1} \leqslant A\varepsilon^{j+1}, \qquad x \in I_t.$$

If $(j+1)n > m$, then

$$\mathscr{L}^n(f(I_t)) \leqslant A^n \varepsilon^{(j+1)n} \leqslant A^n \varepsilon^{m+1} = A^n \varepsilon \mathscr{L}^m(I_t).$$

Summing over t,

$$\mathscr{L}^n(f(K \cap C_j)) \leqslant \sum_{t=1}^{N} \mathscr{L}^n(f(I_t)) \leqslant A^n \varepsilon \sum_{t=1}^{N} \mathscr{L}^m(I_t) \leqslant A^n \ell^m \varepsilon.$$

Letting $\varepsilon \to 0$ proves (2.1).

Next, note that the set

$$E_j := C_j \setminus C_{j+1}$$

is contained in a smooth submanifold of codimension 1 near each of its points. Indeed, at $x_0 \in E_j$ there exists a component g of $f^{(j)}$ with $dg(x_0) \neq 0$. Thus, E_j lies locally in the level set $g^{-1}(g(x_0))$, which is a smooth $(m-1)$-dimensional submanifold S.

If f has a critical point on S, then this point is also a critical point of $f|_S$. By the induction hypothesis (applied to the dimension $m-1$ domain), the set of critical values of $f|_S$ has measure zero. Since E_j is covered by countably many such neighborhoods, we obtain

$$\mathscr{L}^n(f(C_j \setminus C_{j+1})) = 0.$$

Finally, we handle the set $C \setminus C_1$, where C is the full critical set. Since C is invariant under precomposition with local diffeomorphisms, we may work in coordinates. At a point of $C \setminus C_1$ where $\frac{\partial f_1}{\partial x_1} \neq 0$, let ψ be the inverse of the map

$$x \mapsto (f_1(x), x_2, \dots, x_m).$$

Then

$$f \circ \psi(y) = (y_1, g(y)), \qquad g : \mathbf{R}^m \to \mathbf{R}^{n-1}.$$

A point $y = (y_1, y')$ is critical for $f \circ \psi$ iff y' is a critical point of the map $y' \mapsto g(y_1, y')$. For each fixed y_1, the set of critical values of $f \circ \psi$ in the slice $\{y_1\} \times \mathbf{R}^{n-1}$ has measure zero. Since the critical set in a compact K is compact, the image under f is compact and hence measurable. By Fubini's theorem, for each compact K

$$\mathscr{L}^n(f(C \setminus C_1) \cap f(K)) = 0.$$

Thus $f(C \setminus C_1)$ has measure zero, completing the proof. $\square$

Appendix: Categorical Properties of Cartesian Squares

Proposition 2.2. *The following properties hold for transversal cartesian squares:*

Consider a commutative diagram of smooth manifolds:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[r, "h"] \arrow[d, "k"'] & C \arrow[d, "l"] \\
D \arrow[r, "m"'] & E \arrow[r, "n"'] & F
\end{tikzcd}$$

Then:

(1) If both squares are transversal cartesian, then the outer rectangle is transversal cartesian.
(2) If the right square and the outer rectangle are transversal cartesian, then the left square is transversal cartesian.

Proof. By the Pasting Lemma for pullbacks below, the corresponding statements for the underlying commutative squares as pullbacks in the category of smooth manifolds hold. It remains to verify the transversality conditions.

(1) Let $a \in A$, $b = f(a)$, $c = h(b)$, $d = g(a)$, $e = m(d) = k(b)$, $f = n(e) = l(c)$. Given the transversality conditions:

$$T_e E = \mathrm{Im}(T_b k) + \mathrm{Im}(T_d m)$$
$$T_f F = \mathrm{Im}(T_c l) + \mathrm{Im}(T_e n)$$

For any $v \in T_f F$, write $v = T_c l(u) + T_e n(w)$ with $u \in T_c C$, $w \in T_e E$. Then write $w = T_b k(x) + T_d m(y)$ with $x \in T_b B$, $y \in T_d D$. Thus $v = T_c l(u) + T_e n(T_b k(x)) + T_e n(T_d m(y)) = T_b(l \circ h)(u' + x) + T_d(n \circ m)(y)$ for some $u' \in T_b B$, showing $v \in \mathrm{Im}(T_a(l \circ h \circ f)) + \mathrm{Im}(T_d(n \circ m))$.

(2) Let $e = k(b) = m(d)$, $w \in T_e E$. Then $T_e n(w) \in T_f F$. By outer rectangle transversality: $T_e n(w) = T_c(l)(u) + T_d(n \circ m)(v)$ for some $u \in T_c C$, $v \in T_d D$. Thus $T_e n(w) = T_c(l)(u) + T_e n(T_d m(v))$. Since the right square is transversal cartesian, there is $x \in T_b B$ such that $T_b k(x) = w - T_d m(v)$ and $T_b h(x) = u$, giving $w = T_b k(x) + T_d m(v)$. $\square$

Note: The properties in Proposition 2.2 are standard for cartesian squares (pullbacks) in category theory. However, for transversal cartesian squares we must additionally verify that the transversality condition is preserved under composition and decomposition of squares. The above proofs establish precisely that the transversality condition holds in these situations.

Lemma 2.3 (Pasting Lemma for Cartesian Squares)**.** *In any category with pullbacks, consider a commutative diagram:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[r, "h"] \arrow[d, "k"'] & C \arrow[d, "l"] \\ D \arrow[r, "m"'] & E \arrow[r, "n"'] & F \end{tikzcd}$$

Then:

(1) *If both squares are cartesian, then the outer rectangle is cartesian.*
(2) *If the right square and the outer rectangle are cartesian, then the left square is cartesian.*

Proof. We prove these statements using the universal property of pullbacks.

(1) Assume both inner squares are cartesian. We need to show the outer rectangle is cartesian, i.e., that A is the pullback of $D \to F$ and $C \to F$.

Let X be any object with maps $\alpha : X \to D$ and $\gamma : X \to C$ such that $n \circ m \circ \alpha = l \circ \gamma$. Since the right square is cartesian, there exists a unique map $\beta : X \to B$ such that $k \circ \beta = m \circ \alpha$ and $h \circ \beta = \gamma$.

Now, since the left square is cartesian, there exists a unique map $\delta : X \to A$ such that $g \circ \delta = \alpha$ and $f \circ \delta = \beta$.

This δ satisfies $g \circ \delta = \alpha$ and $(h \circ f) \circ \delta = \gamma$, showing that A is indeed the pullback.

(2) Assume the right square and outer rectangle are cartesian. We need to show the left square is cartesian.

Let X be any object with maps $\alpha : X \to D$ and $\beta : X \to B$ such that $m \circ \alpha = k \circ \beta$.

Consider the map $l \circ h \circ \beta : X \to F$. Note that $n \circ m \circ \alpha = n \circ k \circ \beta = l \circ h \circ \beta$, where the last equality follows from commutativity of the right square.

Since the outer rectangle is cartesian, there exists a unique map $\delta : X \to A$ such that $g \circ \delta = \alpha$ and $(h \circ f) \circ \delta = h \circ \beta$.

Now, both $f \circ \delta$ and β are maps from X to B that satisfy:

- $k \circ (f \circ \delta) = k \circ f \circ \delta = m \circ g \circ \delta = m \circ \alpha = k \circ \beta$
- $h \circ (f \circ \delta) = h \circ f \circ \delta = h \circ \beta$

Since the right square is cartesian, the map $(k, h) : B \to E \times_F C$ is a monomorphism (in fact, it's the pullback of l along n). Therefore, $f \circ \delta = \beta$.

Thus δ satisfies $g \circ \delta = \alpha$ and $f \circ \delta = \beta$, showing that the left square is cartesian. □

References

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