Lecture 12

November 21, 2025

1. Whitney embedding

Before turning to the Whitney Embedding Theorem, let us record a few remarks about Sard's theorem in the manifold setting. Last time we proved the Sard theorem for smooth maps defined on open subsets of Euclidean space. Since all our work from now on takes place on smooth manifolds, it is useful to clarify what "measure zero" and "almost everywhere" mean in that context.

Remark 1.1 (Measure zero sets on manifolds)**.** *Although we have not formally introduced a measure on smooth manifolds, the concept of a measure-zero subset makes perfect sense. A subset $A \subset M$ is said to have measure zero if, for every coordinate chart (U, φ), the set*

$$\varphi(A \cap U) \subset \mathbf{R}^m$$

has Lebesgue measure zero. Since diffeomorphisms preserve measure-zero sets, this definition is coordinate-independent. Thus it is meaningful to speak of properties holding "almost everywhere" on a manifold.

Remark 1.2 (Measure zero vs. meagre sets)**.** *Measure-zero sets provide one notion of "smallness" of a subset. A different notion, coming from topology rather than measure theory, is that of a* first category *or* meagre *set: a subset is meagre if it is a countable union of nowhere dense sets (also known as a set of the first Baire category). Critical value sets in the Morse–Sard theorem are unions of compact measure-zero sets, hence are nowhere dense and therefore meagre. This notion extends naturally to smooth manifolds.*

In infinite-dimensional settings (for example, Banach or Hilbert manifolds), Lebesgue measure is no longer available. The corresponding version of Sard's theorem is the Sard–Smale theorem*, which states that the set of regular values of a smooth Fredholm map is residual (comeagre).*

We continue the Whitney Embedding Theorem. This is a fundamental and beautiful result: although we have defined manifolds abstractly, the manifolds we visualize in our minds are almost always realized as submanifolds of Euclidean space. Whitney's theorem guarantees that our abstract definition does not produce objects that deviate significantly from this intuition.

Recall that in the proof of the partition of unity theorem, we constructed a locally finite collection $\{W_i\}$, where each W_i is contained in the domain U_i of some coordinate chart (U_i, φ_i). For each i we also produced a compactly supported smooth function η_i, which is identically 1 on a neighborhood of W_i and whose support lies inside U_i. If you like Lee's terminology, the sets W_i are called *regular coordinate balls.*

We begin with the following lemma, which will be the starting point for the construction of embeddings.

Lemma 1.3. *Let $K \subset M$ be a compact subset of an m-dimensional manifold M, and let U be a neighborhood of K. Then there exists a natural number n and a map*

$$g \in C_c^\infty(M, \mathbf{R}^n)$$

such that g is an injective immersion on a neighborhood of K, and $g \equiv 0$ outside U.

Proof. Choose finitely many sets W_j, $j = 1, \dots, k$, from the locally finite family constructed above so that they cover K. For each j, define

$$g_j = (\eta_j \varphi_j, \eta_j) \colon M \to \mathbf{R}^{m+1}.$$

Each g_j is an injective immersion on a neighborhood of $\overline{W}_j$. Therefore the direct sum

$$g = g_1 \oplus \cdots \oplus g_k$$

is an injective immersion on some neighborhood of K. Here

$$g = g_1 \oplus \cdots \oplus g_k$$

means the map whose value at x is the concatenation of the vectors $g_1(x), \dots, g_k(x)$. Thus each $g_j : M \to \mathbf{R}^{m+1}$ yields

$$g : M \to \mathbf{R}^{k(m+1)}, \qquad g(x) = (g_1(x), \dots, g_k(x)).$$

Finally, multiply g by a bump function which is identically 1 near K and vanishes outside U. This yields a compactly supported smooth map into $\mathbf{R}^n$ that is an injective immersion near K, as required. □

The weakness of this argument is that it gives no control over the target dimension n. When M is compact, this is harmless: we only need a single embedding into some $\mathbf{R}^n$. However, when M is noncompact, one typically exhausts M by larger and larger compact subsets, and the above construction forces us to choose larger values of n at each stage. If we proceed naïvely, we are eventually led to embeddings into an infinite-dimensional Euclidean space–an outcome that is neither geometric nor desirable.

We now turn to the question of *dimension reduction*, which overcomes this issue and leads to the finite-dimensional Whitney Embedding Theorem.

We first introduce a useful family of linear projections. For $a = (a_1, \dots, a_{n-1}) \in \mathbf{R}^{n-1}$ we denote by

$$\pi_a : \mathbf{R}^n \to \mathbf{R}^{n-1}$$

the projection along the vector $(a, 1)$ onto the hyperplane $\{x_n = 0\}$, i.e.

$$\pi_a(x^1, \dots, x^{n-1}, x^n) = \left(x^1 - a_1 x^n, \dots, x^{n-1} - a_{n-1} x^n\right).$$

We now prove that, for a generic choice of a, such a projection preserves immersions on a fixed compact set.

Lemma 1.4. *Let M be an m-dimensional manifold, $K \subset M$ a compact set, and $f \in C^\infty(M, \mathbf{R}^n)$ an immersion on a neighborhood of K. Assume $n > 2m$. Then there is a closed measure-zero set $E \subset \mathbf{R}^{n-1}$ such that, for all $a \in \mathbf{R}^{n-1} \setminus E$, the map $\pi_a \circ f$ is an immersion on K.*

Proof. Since measure zero is preserved under countable unions and this is a local statement, we may assume that K is contained in a single coordinate chart and, furthermore, that M is an open subset of $\mathbf{R}^m$.

Let $E \subset \mathbf{R}^{n-1}$ be the set of all parameters a for which $\pi_a \circ f$ fails to be an immersion at some point of K. Concretely, $a \in E$ if and only if there exist $x \in K$ and $\lambda \in \mathbf{R}^m$ with $|\lambda| = 1$ such that

$$\sum_{k=1}^{m} \lambda_k \Big(\frac{\partial f_j}{\partial x_k}(x) - a_j \frac{\partial f_n}{\partial x_k}(x) \Big) = 0, \qquad j = 1, \dots, n-1.$$

These equations define a closed subset of

$$K \times S^{m-1} \times \mathbf{R}^{n-1},$$

and the projection to the a-factor is proper (since K is compact), hence E is closed.

Set

$$\mu = \sum_{k=1}^{m} \lambda_k \frac{\partial f_n}{\partial x_k}(x),$$

so that the above equations can be rewritten as

$$\sum_{k=1}^{m} \lambda_k \frac{\partial f_j}{\partial x_k}(x) = \mu\, a_j, \qquad j = 1, \dots, n-1,$$

and, with $a_n := 1$,

$$\sum_{k=1}^{m} \lambda_k \frac{\partial f_j}{\partial x_k}(x) = \mu\, a_j, \qquad j = 1, \dots, n.$$

This means that the vector $(a, 1) \in \mathbf{R}^n$ is tangent to the immersed submanifold $f(M)$ at the point $f(x)$.

Since f is an immersion, the vector

$$v = \sum_{k=1}^{m} \lambda_k \frac{\partial f}{\partial x_k}(x)$$

is nonzero, hence $\mu \neq 0$ and $(a, 1)$ lies in the range of the smooth map

$$F : \mathbf{R}^m \times K \to \mathbf{R}^n, \qquad F(\lambda, x) = \sum_{k=1}^{m} \lambda_k \frac{\partial f}{\partial x_k}(x).$$

The domain of F has dimension $2m < n$, so by a simple special case of the Morse-Sard theorem the image $F(\mathbf{R}^m \times K)$ has measure zero in $\mathbf{R}^n$. For each fixed $\mu \neq 0$, the intersection of this image with the affine hyperplane

$$H_\mu = \{(a, \mu) \in \mathbf{R}^n : a \in \mathbf{R}^{n-1}\}$$

also has measure zero (by Fubini's theorem and homogeneity). Projecting $H_\mu \cap \operatorname{im} F$ onto the first $n-1$ coordinates yields a measure-zero subset of $\mathbf{R}^{n-1}$.

Since, for $a \in E$, the direction $(a, 1)$ lies in $\operatorname{im} F$ with some $\mu \neq 0$, we conclude that E is a closed measure-zero subset of $\mathbf{R}^{n-1}$. $\square$

We next analyze when injectivity is preserved under projection.

Lemma 1.5. *Let M be an m-dimensional manifold, $K \subset M$ a compact set, and $f \in C^\infty(M, \mathbf{R}^n)$ an injective immersion on a neighborhood of K. Assume $n > 2m + 1$. Then there exists a closed measure-zero set $F \subset \mathbf{R}^{n-1}$ such that, for all $a \in \mathbf{R}^{n-1} \setminus F$, the map $\pi_a \circ f$ is an injective immersion on a neighborhood of K.*

Proof. By Lemma 1.4, there is a closed measure-zero set $E \subset \mathbf{R}^{n-1}$ such that $\pi_a \circ f$ is an immersion near K for all $a \notin E$. It remains to rule out failure of injectivity.

Let $E' \subset \mathbf{R}^{n-1}$ be the set of parameters a such that $\pi_a \circ f$ is not injective on K. First observe that $E \cup E'$ is closed. Indeed, suppose $a_j \in E'$ and $a_j \to a$. For each j there exist $x'_j, x''_j \in K$, $x'_j \neq x''_j$, with

$$\pi_{a_j} f(x'_j) = \pi_{a_j} f(x''_j).$$

Passing to a subsequence, we may assume $x'_j \to x'$ and $x''_j \to x''$ for some $x', x'' \in K$. If $a \notin E$, then for j sufficiently large, $\pi_{a_j} \circ f$ is an injective immersion on a fixed neighborhood of x', so in particular $x' \neq x''$ and

$$\pi_a f(x') = \pi_a f(x''),$$

showing that $a \in E'$. Thus $E \cup E'$ is closed.

Now describe E' more explicitly. The condition $a \in E'$ means that there exist $x', x'' \in K$, $x' \neq x''$, such that

$$f_j(x') - a_j f_n(x') = f_j(x'') - a_j f_n(x''), \qquad j = 1, \dots, n-1.$$

Setting $a_n = 1$ and

$$\mu = f_n(x') - f_n(x''),$$

we can rewrite this as

$$f(x') - f(x'') = \mu(a_1, \dots, a_{n-1}, 1).$$

Since f is injective on K, we have $f(x') \neq f(x'')$ and therefore $\mu \neq 0$. Thus the vector $(a, 1)$ lies in the range of the smooth map

$$G : \mathbf{R} \times K \times K \to \mathbf{R}^n, \qquad G(t, x', x'') = t\big(f(x') - f(x'')\big).$$

The domain of G has dimension $1 + 2m < n$ by assumption, so the image $G(\mathbf{R} \times K \times K)$ has measure zero in $\mathbf{R}^n$. As before, by homogeneity and Fubini's theorem, its intersection with each hyperplane $\{(a, \mu) : a \in \mathbf{R}^{n-1}\}$, $\mu \neq 0$, has measure zero, and hence the corresponding sets of a in $\mathbf{R}^{n-1}$ also have measure zero.

It follows that E' is a measure-zero subset of $\mathbf{R}^{n-1}$, and hence so is $F := E \cup E'$, which is closed. For $a \notin F$, the map $\pi_a \circ f$ is both an immersion and injective on a neighborhood of K, as claimed. □

In these two lemmas we have successively excluded two types of "bad" projection directions: in Lemma 1.4 we avoided projecting along directions tangent to $f(M)$, and in Lemma 1.5 we further avoided directions parallel to chords joining distinct points of $f(K)$. The sets of forbidden directions are controlled by $2m$ and $2m + 1$ parameters, respectively, which explains the dimension assumptions $n > 2m$ and $n > 2m + 1$.

References

[1] Alexander Kupers, *Lectures on Differential Topology*, 2020. Lecture notes.

[2] John M. Lee, *Introduction to Smooth Manifolds*, 2nd ed., Graduate Texts in Mathematics, vol. 218, Springer, 2012.

[3] Lars Hörmander, *Advanced Differential Calculus*, 1994. Lecture notes, Lund University.

Mathematisches Institut, Universität Freiburg
Email address: yuchen.bi@math.uni-freiburg.de