Lecture 13

November 25, 2025

1. Whitney embedding

Last week, we prove

Lemma 1.1. *Let $K \subset M$ be a compact subset of an m-dimensional manifold M, and let U be a neighborhood of K. Then there exists a natural number n and a map*

$$g \in C_c^\infty(M, \mathbf{R}^n)$$

such that g is an injective immersion on a neighborhood of K, and $g \equiv 0$ outside U.

and

Lemma 1.2. *Let M be an m-dimensional manifold, $K \subset M$ a compact set, and $f \in C^\infty(M, \mathbf{R}^n)$ an injective immersion on a neighborhood of K. Assume $n > 2m+1$. Then there exists a closed measure-zero set $F \subset \mathbf{R}^{n-1}$ such that, for all $a \in \mathbf{R}^{n-1} \setminus F$, the map $\pi_a \circ f$ is an injective immersion on a neighborhood of K.*

Theorem 1.3 (Approximation by proper embeddings)**.** *Let M be an m-dimensional manifold and let*

$$f \in C^\infty(M, \mathbf{R}^n)$$

be a proper map, where $n \geqslant 2m+1$. Then for every positive continuous function $\varepsilon : M \to (0, \infty)$ there exists a proper embedding

$$g \in C^\infty(M, \mathbf{R}^n)$$

such that

$$|g(x) - f(x)| \leqslant \varepsilon(x), \qquad x \in M.$$

Proof. First replace ε by

$$\varepsilon'(x) := \min\{1, \varepsilon(x)\},$$

and relabel $\varepsilon' = \varepsilon$. Then the function

$$M \to \mathbf{R}, \qquad x \mapsto |f(x)| - \varepsilon(x)$$

is proper: indeed, for any $c \in \mathbf{R}$ we have

$$\{x : |f(x)| - \varepsilon(x) \leqslant c\} \subset \{x : |f(x)| \leqslant c+1\},$$

and the latter set is compact because f is proper and $\varepsilon \leqslant 1$. Consequently, if g satisfies $|g - f| \leqslant \varepsilon$ on M, then

$$|g(x)| \geqslant |f(x)| - |g(x) - f(x)| \geqslant |f(x)| - \varepsilon(x),$$

so the properness of $x \mapsto |f(x)| - \varepsilon(x)$ implies that g is also proper. Thus it suffices to construct an embedding g with $|g - f| \leqslant \varepsilon$.

Let

$$K_1 \subset K_2 \subset \cdots \subset M$$

be a compact exhaustion of M. We construct inductively a sequence

$$g_j \in C^\infty(M, \mathbf{R}^n), \qquad j = 0, 1, 2, \ldots,$$

such that:

(1) $g_0 = f$;
(2) for all $j \geqslant 1$,
$$|g_j(x) - g_{j-1}(x)| \leqslant \frac{\varepsilon(x)}{2^j}, \qquad x \in M;$$
(3) g_j is an injective immersion on a neighborhood of K_j;
(4) $g_j = g_{j-1}$ on K_{j-1}.

Assume g_{j-1} has been constructed for some $j \geqslant 1$. By Lemma 1.1, there exist an integer ℓ and a map

$$h \in C_c^\infty(M, \mathbf{R}^\ell)$$

such that $g_{j-1} \oplus h$ is an injective immersion on a neighborhood of K_j. Moreover, we may choose h so that $h \equiv 0$ near K_{j-1}: for instance, apply Lemma 1.1 with

$$K = K_j \setminus V_{j-1}, \qquad U = M \setminus K_{j-1},$$

where V_{j-1} is a neighborhood of K_{j-1} on which g_{j-1} is already an injective immersion.

Now repeatedly apply Lemma 1.2 to the map

$$M \to \mathbf{R}^{n+\ell}, \qquad x \mapsto (g_{j-1}(x), h(x)),$$

to obtain a linear map $T : \mathbf{R}^\ell \to \mathbf{R}^n$ with arbitrarily small operator norm such that

$$g_j := g_{j-1} + T \circ h$$

is an injective immersion on a neighborhood of K_j. Since $h \equiv 0$ near K_{j-1}, we have $g_j = g_{j-1}$ on K_{j-1}. By choosing $\|T\|$ sufficiently small, we can also ensure that

$$|g_j(x) - g_{j-1}(x)| \leqslant \frac{\varepsilon(x)}{2^j}, \qquad x \in M.$$

This completes the inductive step.

By construction,

$$\sum_{j=1}^\infty |g_j(x) - g_{j-1}(x)| \leqslant \sum_{j=1}^\infty \frac{\varepsilon(x)}{2^j} = \varepsilon(x),$$

so the series converges pointwise, and in fact stabilizes on each compact set: for any fixed K_N, all maps g_j with $j \geqslant N$ coincide on K_N because of property (4). We can therefore define

$$g(x) := \lim_{j \to \infty} g_j(x),$$

and obtain a smooth map $g \in C^\infty(M, \mathbf{R}^n)$ with

$$|g(x) - f(x)| \leqslant \varepsilon(x), \qquad x \in M.$$

Moreover, on each K_j we have $g = g_j$, and g_j is an injective immersion near K_j, hence g is an injective immersion on all of M.

Finally, since g is a proper injective immersion, the following lemma shows that g is an embedding. $\square$

Lemma 1.4. *Let X and Y be Hausdorff spaces, with Y locally compact, and let $f : X \to Y$ be proper (i.e. $f^{-1}(K)$ is compact for every compact $K \subset Y$). Then f is a closed map: the image of every closed subset of X is closed in Y.*

Proof. Let $C \subset X$ be closed, and let $y \in \overline{f(C)}$. We must show $y \in f(C)$.

Since Y is locally compact and Hausdorff, there exists a compact neighborhood $K \subset Y$ of y. Then

$$f(C) \cap K = f\big(C \cap f^{-1}(K)\big).$$

The set $C \cap f^{-1}(K)$ is closed in the compact set $f^{-1}(K)$, hence compact. Thus $f(C) \cap K$ is the continuous image of a compact set and therefore compact, in particular closed in K.

Because $y \in \overline{f(C)}$, we also have $y \in \overline{f(C) \cap K}$. But $f(C) \cap K$ is closed in K and $y \in K$, so $y \in f(C) \cap K$. Hence $y \in f(C)$.

We have shown that every point in $\overline{f(C)}$ lies in $f(C)$, so $f(C)$ is closed in Y. Thus f is a closed map. $\square$

Finally, let us explain why proper maps are abundant, so the approximation theorem above always applies. In Lecture 5, when we constructed partitions of unity, we also proved the existence of an *exhaustion function* on any smooth manifold M. Recall that an exhaustion function is a continuous map

$$\varphi : M \to \mathbf{R}$$

such that every sublevel set

$$\varphi^{-1}((-\infty, c])$$

is compact for all $c \in \mathbf{R}$. In particular, every exhaustion function is a *proper map.*

Therefore, given any $n \geqslant 1$, we may consider the map

$$x \longmapsto (\varphi(x), 0, \dots, 0) \in \mathbf{R}^n,$$

which is again proper. Thus proper maps $M \to \mathbf{R}^n$ always exist, and Theorem 1.3 shows that we can approximate any such proper map by a proper embedding arbitrarily well. This guarantees the existence of smooth embeddings of M into $\mathbf{R}^n$ for all sufficiently large n.

We now describe the normal bundle of an embedded submanifold and show that it is an embedded submanifold of the tangent bundle of $\mathbf{R}^n$.

Let $M \subset \mathbf{R}^n$ be an embedded m-dimensional submanifold. For each $x \in \mathbf{R}^n$, the tangent space $T_x\mathbf{R}^n$ is canonically identified with $\mathbf{R}^n$, and hence inherits the standard Euclidean inner product $\langle \cdot, \cdot \rangle$.

For $x \in M$, define the normal space

$$N_xM := \{v \in T_x\mathbf{R}^n : \langle v, w \rangle = 0 \text{ for all } w \in T_xM\}.$$

The *normal bundle* of M is then defined as

$$NM := \{(x, v) \in T\mathbf{R}^n \cong \mathbf{R}^n \times \mathbf{R}^n \mid x \in M,\ v \in N_xM\}.$$

There is a natural projection

$$\pi_{NM} : NM \longrightarrow M, \qquad \pi_{NM}(x, v) = x,$$

which is just the restriction of the standard bundle projection $\pi : T\mathbf{R}^n \to \mathbf{R}^n$.

Proposition 1.5. *The normal bundle NM is an embedded submanifold of $T\mathbf{R}^n$.*

Proof. Let $p_0 \in M$ and choose a coordinate chart

$$(U, \varphi = (x^1, \dots, x^n))$$

on $\mathbf{R}^n$ such that $p_0 \in U$ and

$$M \cap U = \{p \in U : x^{m+1}(p) = \cdots = x^n(p) = 0\}.$$

On U the coordinate vector fields

$$\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n},$$

form a basis of $T_p\mathbf{R}^n$ for each $p \in U$.

Let $(u^1, \dots, u^n)$ be the standard coordinates on $\mathbf{R}^n$. Then on U we can write

$$\frac{\partial}{\partial x^j} = \sum_{i=1}^n \frac{\partial u^i}{\partial x^j} \frac{\partial}{\partial u^i}.$$

Define a smooth map

$$\Phi : U \times \mathbf{R}^n \to \varphi(U) \times \mathbf{R}^n$$

by

$$\Phi(p, v) = \left(x^1(p), \dots, x^n(p), \langle v, \tfrac{\partial}{\partial x^1}\rangle, \dots, \langle v, \tfrac{\partial}{\partial x^n}\rangle\right).$$

In product coordinates (p, v) the differential has block form

$$D\Phi_{(p,v)} = \begin{pmatrix} A & 0 \\ * & A^{-1} \end{pmatrix},$$

where $A = (\partial x^i/\partial u^j)_{i,j}$ is the Jacobian of $u \mapsto x(u)$. Since φ is a local diffeomorphism, A is invertible, hence $D\Phi_{(p,v)}$ is invertible. So Φ is a local diffeomorphism.

We claim that Φ is injective. If $\Phi(p, v) = \Phi(p', v')$, then

$$(x^1(p), \dots, x^n(p)) = (x^1(p'), \dots, x^n(p')),$$

so $p = p'$ by injectivity of φ. For this p,

$$\langle v, \tfrac{\partial}{\partial x^i}\rangle = \langle v', \tfrac{\partial}{\partial x^i}\rangle, \quad i = 1, \dots, n,$$

so $v = v'$ because the $\partial/\partial x^i$ form a basis of $T_p\mathbf{R}^n$. Thus Φ is injective and defines a smooth coordinate chart on $U \times \mathbf{R}^n$.

Now $(p, v) \in NM$ iff:

$$x^{m+1}(p) = \cdots = x^n(p) = 0, \qquad \langle v, \tfrac{\partial}{\partial x^i}\rangle = 0 \ (i = 1, \dots, m).$$

Hence, writing

$$\Phi(p, v) = (x^1, \dots, x^n, w^1, \dots, w^n), \qquad w^i = \langle v, \tfrac{\partial}{\partial x^i}\rangle,$$

we have

$$(p, v) \in NM \iff x^{m+1} = \cdots = x^n = 0, \quad w^1 = \cdots = w^m = 0.$$

Thus

$$\Phi\big(NM \cap (U \times \mathbf{R}^n)\big) = \{(x, w) : x^{m+1} = \cdots = x^n = 0,\ w^1 = \dots w^m = 0\},$$

a linear subspace of $\varphi(U) \times \mathbf{R}^n$, hence an embedded submanifold.

Therefore NM is an embedded submanifold of $T\mathbf{R}^n$. $\square$

Tubular Neighborhoods. Let $M \subset \mathbf{R}^n$ be an embedded m–dimensional submanifold. Its normal bundle

$$NM = \{(x,v) \in \mathbf{R}^n \times \mathbf{R}^n : x \in M,\ v \perp T_x M\}$$

is an embedded submanifold of $T\mathbf{R}^n \cong \mathbf{R}^n \times \mathbf{R}^n$. Define the (normal) exponential map

$$E : NM \to \mathbf{R}^n, \qquad E(x,v) = x + v.$$

This is smooth because it is the restriction of the addition map on $\mathbf{R}^n \times \mathbf{R}^n$.

Definition 1.6. *A* tubular neighborhood *of M is an open set $U \subset \mathbf{R}^n$ for which there exists an open subset*

$$V = \{(x,v) \in NM : |v| < \delta(x)\}$$

with $\delta : M \to (0,\infty)$ continuous, such that $E|_V : V \to U$ is a diffeomorphism.

Theorem 1.7 (Tubular Neighborhood Theorem)**.** *Every embedded submanifold $M \subset \mathbf{R}^n$ admits a tubular neighborhood.*

Proof. Let $M_0 = \{(x,0) : x \in M\}$ be the zero section of NM. We first show that E is a local diffeomorphism near M_0.

Fix $x \in M$. On M_0, the restriction

$$E|_{M_0} : M_0 \to M, \qquad (x,0) \mapsto x$$

is a diffeomorphism, so its differential

$$dE_{(x,0)} : T_{(x,0)} M_0 \to T_x M$$

is an isomorphism. On the other hand, the restriction of E to the fiber $N_x M$ is the affine map

$$N_x M \to \mathbf{R}^n, \qquad w \mapsto x + w,$$

whose differential at $w = 0$ is the identity on $N_x M$. Thus

$$dE_{(x,0)} : T_{(x,0)}(N_x M) \to N_x M$$

is also an isomorphism.

Since E restricts to a diffeomorphism $M_0 \to M$ along the zero section $M_0 = \{(x,0) : x \in M\}$, its differential

$$dE_{(x,0)} : T_{(x,0)} M_0 \longrightarrow T_x M$$

is an isomorphism. Similarly, the restriction of E to the fiber $N_x M$ is the affine map $w \mapsto x + w$, whose differential at 0 is the identity on $N_x M$. Hence

$$dE_{(x,0)} : T_{(x,0)}(N_x M) \longrightarrow N_x M$$

is also an isomorphism.

Now set

$$V_1 := T_{(x,0)} M_0, \qquad V_2 := T_{(x,0)}(N_x M) \subset T_{(x,0)} NM.$$

Since M_0 and the fiber directions meet transversely in NM at $(x,0)$, we have

$$V_1 \cap V_2 = \{0\}, \qquad \dim V_1 = \dim T_x M = m, \qquad \dim V_2 = \dim N_x M = n - m.$$

Because $\dim T_{(x,0)} NM = n = \dim V_1 + \dim V_2$, it follows that

$$T_{(x,0)} NM = V_1 \oplus V_2.$$

Choose a basis of $T_{(x,0)}NM$ consisting of a basis of V_1 followed by a basis of V_2, and choose a basis of $T_x\mathbf{R}^n$ consisting of a basis of T_xM followed by a basis of N_xM. With respect to these adapted bases, the matrix of

$$dE_{(x,0)} : T_{(x,0)}NM \longrightarrow T_x\mathbf{R}^n$$

has block diagonal form

$$dE_{(x,0)} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

where

$$A = dE_{(x,0)}|_{V_1} : V_1 \to T_xM, \qquad B = dE_{(x,0)}|_{V_2} : V_2 \to N_xM.$$

Both A and B are linear isomorphisms by the discussion above, hence the entire block diagonal matrix is invertible. Thus $dE_{(x,0)}$ is a linear isomorphism.

By the inverse function theorem, E is a local diffeomorphism at $(x,0)$.

Since NM is a vector bundle, we have a direct sum decomposition

$$T_{(x,0)}NM = T_{(x,0)}M_0 \oplus T_{(x,0)}(N_xM),$$

and similarly

$$T_x\mathbf{R}^n = T_xM \oplus N_xM.$$

By the preceding discussion, $dE_{(x,0)}$ maps each summand isomorphically onto the corresponding summand, hence is an isomorphism $T_{(x,0)}NM \to T_x\mathbf{R}^n$. By the inverse function theorem, E is a local diffeomorphism at $(x,0)$.

Therefore, for each $x \in M$ there exists $\delta > 0$ such that

$$E : V_\delta(x) \to E\big(V_\delta(x)\big)$$

is a diffeomorphism, where we may take

$$V_\delta(x) := \{(x', v') \in NM : |x - x'| < \delta,\ |v'| < \delta\}.$$

For each $x \in M$, define

$$\rho(x) := \sup\big\{\delta \in (0,1] : E|_{V_\delta(x)} \text{ is a diffeomorphism onto its image}\big\}.$$

The local argument above shows $\rho(x) > 0$ for all x. To see that ρ is 1–Lipschitz, fix $x, x' \in M$ and assume $|x - x'| < \rho(x)$. Set $\delta := \rho(x) - |x - x'| > 0$. Then $V_\delta(x') \subset V_{\rho(x)}(x)$, so E is a diffeomorphism on $V_\delta(x')$, hence $\rho(x') \geqslant \delta$ and

$$\rho(x) - \rho(x') \leqslant |x - x'|.$$

If $|x - x'| \geqslant \rho(x)$ this inequality is trivial. Exchanging x and x' gives $\rho(x') - \rho(x) \leqslant |x - x'|$, so

$$|\rho(x) - \rho(x')| \leqslant |x - x'|.$$

Now set

$$V := \{(x, v) \in NM : |v| < \tfrac{1}{2}\rho(x)\}.$$

We claim that E is injective on V. Suppose $(x, v), (x', v') \in V$ satisfy $E(x, v) = E(x', v')$, i.e. $x + v = x' + v'$. Then

$$|x - x'| = |v - v'| \leqslant |v| + |v'| < \tfrac{1}{2}\rho(x) + \tfrac{1}{2}\rho(x').$$

Without loss of generality assume $\rho(x') \leqslant \rho(x)$. Then

$$|x - x'| < \rho(x),$$

so both (x,v) and (x',v') lie in $V_{\rho(x)}(x)$. By definition of $\rho(x)$, E is injective on $V_{\rho(x)}(x)$, hence $(x,v)=(x',v')$. Thus E is injective on V.

Finally, let $U := E(V) \subset \mathbf{R}^n$. Since E is a local diffeomorphism and V is open, U is open and

$$E|_V : V \to U$$

is a bijective local diffeomorphism, hence a global diffeomorphism. By construction, V is of the form $\{(x,v) : |v| < \delta(x)\}$ with $\delta(x) = \frac{1}{2}\rho(x)$, so U is a tubular neighborhood of M. □

Proposition 1.8 (Existence of a smooth retraction)**.** *If U is any tubular neighborhood of M, then there exists a smooth map*

$$r : U \to M$$

which is a retraction ($r|_M = \mathrm{id}_M$) and a smooth submersion.

Proof. Write $U = E(V)$ with

$$V = \{(x,v) \in NM : |v| < \delta(x)\},$$

and $E|_V : V \to U$ a diffeomorphism. Let $\pi_{NM} : NM \to M$ be the bundle projection $(x,v) \mapsto x$. Define

$$r := \pi_{NM} \circ (E|_V)^{-1} : U \to M.$$

Then r is smooth and $r(x) = x$ for all $x \in M$, so r is a retraction. Moreover, π_{NM} is a submersion, and $(E|_V)^{-1}$ is a diffeomorphism, so r is a submersion as well. □

References

[1] John M. Lee, *Introduction to Smooth Manifolds*, 2nd ed., Graduate Texts in Mathematics, vol. 218, Springer, 2012.

[2] Lars Hörmander, *Advanced Differential Calculus*, 1994. Lecture notes, Lund University.

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