

Lecture 2

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1. CONNECTIVITY

Let us recall some basic terminology regarding connectedness. There are essentially two fundamental ways to characterize when a topological space is connected.

Definition via Clopen Sets. The more basic definition states that a topological space X is *connected* if the only subsets that are both open and closed (clopen) are $\emptyset$ and X itself.

Why is this definition meaningful? Suppose there exists a non-trivial clopen subset $A \subset X$ with $A \neq \emptyset$ and $A \neq X$. Then its complement $X \setminus A$ is also clopen. Consequently, the topology of X decomposes into disjoint unions of open sets from A and $X \setminus A$, indicating that X can be separated into two independent components. Thus, connectedness precisely prohibits such a separation.

Path Connectedness. In practice, a more operational notion is the stronger concept of *path connectedness*. A space X is path connected if for any two points $p, q \in X$, there exists a continuous path $\gamma : [0, 1] \rightarrow X$ such that $\gamma(0) = p$ and $\gamma(1) = q$.

It is straightforward to verify that path connectedness implies connectedness. However, the converse is generally false. A classic counterexample is the **topologist's sine curve**. Notably, for topological manifolds, these two notions coincide.

Theorem 1.1. *Let M^n be a topological manifold. Then*

$$M \text{ is connected} \iff M \text{ is path connected.}$$

Proof. ($\Leftarrow$) This direction is straightforward and will be left as an exercise.

($\Rightarrow$) We now prove the non-trivial direction. Assume M is connected and we want to show it is path connected. To understand the proof strategy, it is helpful to first consider the case when M is an open subset of $\mathbf{R}^n$, a proof that may be familiar from analysis or topology courses.

Case: $M \subset \mathbf{R}^n$ open. Fix a base point $p \in M$. Define the set

$$U = \{q \in M : \exists \text{ a continuous path from } p \text{ to } q \text{ within } M\}.$$

Our goal is to show that $U = M$.

First, we show U is open. Take any $q \in U$. Since M is open, there exists an open ball $B_\epsilon(q) \subset M$. For any $r \in B_\epsilon(q)$, we can connect q to r by a straight line segment (which is continuous) contained entirely in $B_\epsilon(q) \subset M$. By composing this with the path from p to q , we obtain a path from p to r . Hence, $B_\epsilon(q) \subset U$, proving U is open.

Next, we show U is closed in M . Suppose $q \in M \setminus U$. Again, by openness of M , there exists an open ball $B_\epsilon(q) \subset M$. We claim $B_\epsilon(q) \subset M \setminus U$. If not, there would exist some $r \in B_\epsilon(q) \cap U$. But

then we could connect p to r (since $r \in U$) and then connect r to q by a straight line segment within $B_\epsilon(q)$, contradicting $q \notin U$. Thus, $M \setminus U$ is open, so U is closed.

Since M is connected and U is non-empty (as $p \in U$), we conclude $U = M$.

General case: M is a topological manifold. The proof follows the same strategy. Fix $p \in M$ and define U as above.

To show U is open: For $q \in U$, take a coordinate chart (V, φ) with $q \in V$ and $\varphi(V)$ homeomorphic to an open subset of $\mathbb{R}^n$. Since $\varphi(V)$ contains an open ball around $\varphi(q)$, we can find a neighborhood $W \subset V$ of q that is homeomorphic to an open ball. Any point in W can be connected to q by a path in W (via the straight line in coordinates), hence can be connected to p by extending the existing path. Thus, $W \subset U$.

To show U is closed: Suppose $q \in M \setminus U$. Take a coordinate chart (V, φ) around q with $\varphi(V)$ homeomorphic to an open ball. If there were some $r \in V \cap U$, we could connect p to r and then r to q within V (using the coordinate representation), contradicting $q \notin U$. Hence, $V \subset M \setminus U$, so $M \setminus U$ is open.

By connectedness of M and non-emptiness of U , we conclude $U = M$. $\square$

2. LOCAL COMPACTNESS AND PARACOMPACTNESS

We now turn to slightly more technical aspects of topological manifolds. We will see how the second countability condition translates into more practical tools for working with manifolds.

2.0. Local Compactness.

Proposition 2.1. *Let M be a topological manifold. Then for every point $p \in M$ and every open neighborhood U of p , there exists a compact neighborhood K of p such that $K \subset U$.*

Proof. Since M is a topological manifold, it is locally Euclidean. Hence, there exists a coordinate chart (V, φ) such that:

- $p \in V \subset U$,
- $\varphi(V) \subset \mathbb{R}^n$ is open.

Choose an open ball $B_r(\varphi(p)) \subset \varphi(V)$ centered at $\varphi(p)$. Now, consider the closed ball $\overline{B}_{r/2}(\varphi(p)) \subset B_r(\varphi(p))$ and define:

$$K = \varphi^{-1} \left(\overline{B}_{r/2}(\varphi(p)) \right).$$

We verify the required properties:

- (1) **K is compact:** The closed ball $\overline{B}_{r/2}(\varphi(p))$ is compact in $\mathbb{R}^n$. Since φ is a homeomorphism onto its image, K is compact in M .
- (2) **K is a neighborhood of p :** The open ball $B_{r/2}(\varphi(p))$ is contained in $\overline{B}_{r/2}(\varphi(p))$, so

$$\varphi^{-1}(B_{r/2}(\varphi(p))) \subset K.$$

This set is open in M and contains p , so K is a neighborhood of p .

- (3) **$K \subset U$:** Since $K \subset V \subset U$, the result follows.

Thus, K is a compact neighborhood of p contained in U . $\square$

This property shows that M is locally compact according to the definition:

Definition 2.2. *A topological space X is locally compact if for every point $p \in X$ and every open neighborhood U of p , there exists a compact neighborhood K of p such that $p \in K \subset U$.*

2.2. Exhaustion by Compact Sets. A crucial consequence of second countability and local compactness is the existence of exhaustion by compact sets:

Definition 2.3. A topological space X admits an exhaustion by compact sets if there exists a sequence $\{K_n\}_{n=1}^{\infty}$ of compact subsets such that:

- (1) $K_n \subset \text{int}(K_{n+1})$ for all $n \geq 1$,
- (2) $\bigcup_{n=1}^{\infty} K_n = X$.

Remark 2.4. For any exhaustion $\{K_i\}$ of X , we have:

$$X = \bigcup_{i=1}^{\infty} \text{Int}(K_i).$$

This has a useful consequence: any compact subset $K' \subset X$ is eventually contained in the interiors of the K_i 's. Specifically, since $\{\text{Int}(K_i)\}_{i=1}^{\infty}$ is an open cover of K' and K' is compact, there exists a finite subcover. Due to the nesting property $K_i \subset \text{Int}(K_{i+1})$, this implies:

$$K' \subset \text{Int}(K_i) \quad \text{for all } i \geq i_0(K').$$

In other words, any compact subset of X is eventually "absorbed" by the interiors of the exhaustion sets.

2.4. Construction of Compact Exhaustion.

Proposition 2.5. Let X be a second countable, locally compact, Hausdorff topological space. Then X admits an exhaustion by compact subsets.

This result highlights one of the key motivations for assuming second countability in the definition of topological manifolds.

Proof. We construct the exhaustion explicitly. Let $\mathcal{B}$ be a countable basis for the topology of X . Define the subcollection:

$$\mathcal{B}' = \{B \in \mathcal{B} : \overline{B} \text{ is compact}\}.$$

By local compactness and Hausdorff condition, $\mathcal{B}'$ remains a basis for X . Actually, Let $p \in X$ be an arbitrary point and $A \in \mathcal{O}_X$ be any open neighborhood of p . By local compactness, there exists an open set A' and a compact set K such that $p \in A' \subset K \subset A$.

Since $\mathcal{B}$ is a basis, there exists some $B \in \mathcal{B}$ with $p \in B \subset A'$. Now, using the Hausdorff property, we observe that K is closed in X (as compact subsets of Hausdorff spaces are closed). Therefore, we have:

$$\overline{B} \subset \overline{A'} \subset K.$$

Since K is compact and $\overline{B}$ is a closed subset of K , it follows that $\overline{B}$ is compact. This shows that $B \in \mathcal{B}'$, and we have found $B \in \mathcal{B}'$ with $p \in B \subset A$, proving that $\mathcal{B}'$ is indeed a basis for X .

Enumerate the elements of $\mathcal{B}'$ as $U_1, U_2, U_3, \dots$

We now construct the exhaustion recursively:

Base case: Let $K_1 = \overline{U_1}$.

Inductive step: Suppose K_n has been constructed. Since K_n is compact and $\{U_i\}$ is an open cover of X , there exists $m_n > n$ such that:

$$K_n \subset U_1 \cup U_2 \cup \dots \cup U_{m_n}.$$

Define:

$$K_{n+1} = \overline{U_1} \cup \overline{U_2} \cup \cdots \cup \overline{U_{m_n}}.$$

By construction, each K_n is compact (finite union of compact sets), $K_n \subset \text{int}(K_{n+1})$ (since K_n is contained in the union of the U_i 's, which are open subsets of K_{n+1}), and $\bigcup_{n=1}^{\infty} K_n = X$ (since $\{U_i\}$ covers X). $\square$

2.5. Paracompactness. We now introduce a fundamental topological property that will play a crucial role in our study of manifolds.

2.5. Basic Definitions.

Definition 2.6. Let X be a topological space.

- (1) An open cover of X is a collection $\mathcal{U} \subset \mathcal{O}_X$ of open sets such that $X = \bigcup_{U \in \mathcal{U}} U$.
- (2) An open cover $\mathcal{U}$ is called *locally finite* if every point $p \in X$ has a neighborhood W that intersects only finitely many $U \in \mathcal{U}$.
- (3) An open cover $\mathcal{V}$ is a *refinement* of an open cover $\mathcal{U}$ if for every $V \in \mathcal{V}$, there exists $U \in \mathcal{U}$ such that $V \subset U$.
- (4) X is *paracompact* if every open cover has a locally finite refinement.

Remark 2.7. Paracompactness serves to "tame" potentially complicated open covers. In the context of manifolds, we frequently work with collections of coordinate charts covering the space. Although there may be infinitely many charts, paracompactness guarantees the existence of a refinement where only finitely many charts interact at any given point. This localization property makes many local arguments feasible and is essential for various global constructions.

2.7. Paracompactness of Topological Manifolds.

Theorem 2.8. Every topological manifold is paracompact.

Remark 2.9. The primary application of paracompactness in differential geometry is the construction of partitions of unity. These are families of smooth functions that sum to 1 everywhere while maintaining local finiteness (only finitely many are nonzero in any compact neighborhood). Partitions of unity enable us to glue locally defined objects into global ones. We will explore this in detail in subsequent lectures.

Proof. The proof proceeds by constructing a locally finite refinement using the compact exhaustion of M . Let $\{K_i\}_{i=1}^{\infty}$ be an exhaustion by compact subsets with $K_i \subset \text{int}(K_{i+1})$ for all i .

Define the compact annuli:

$$N_i = K_i \setminus \text{int}(K_{i-1}), \quad \text{with } K_0 = \emptyset.$$

These sets cover M since $\bigcup_{i=1}^{\infty} K_i = M$ and $K_i \subset \text{int}(K_{i+1})$.

Now define open neighborhoods for these annuli:

$$\tilde{N}_i = \text{int}(K_{i+1}) \setminus K_{i-2},$$

where we take $K_{-1} = K_0 = \emptyset$. Note that $N_i \subset \tilde{N}_i$ for each i .

Let $\mathcal{U}$ be an arbitrary open cover of M . For each i , the compact set N_i can be covered by finitely many elements of $\mathcal{U}$. Let $\mathcal{U}_i$ be such a finite subcover for N_i .

Define the refinement:

$$\mathcal{V}_i = \{\tilde{N}_i \cap U : U \in \mathcal{U}_i\},$$

and let $\mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$.

We now verify that $\mathcal{V}$ is a locally finite refinement of $\mathcal{U}$:

(1) **Refinement:** Each $V \in \mathcal{V}$ is of the form $\tilde{N}_i \cap U$ for some $U \in \mathcal{U}_i \subset \mathcal{U}$, so $V \subset U$.

(2) **Locally finite:** For any $p \in M$, there exists a minimal i such that $p \in K_i$. Consider the neighborhood $W = \text{int}(K_{i+1}) \setminus K_{i-2}$. Then:

- If $j > i + 2$, then $W \cap \tilde{N}_j \subset K_{i+1} \cap (M \setminus K_j) = \emptyset$ since $j > i + 2$ implies $K_{i+1} \subset K_{j-2}$.
- If $j < i - 2$, then $W \cap \tilde{N}_j \subset (M \setminus K_{i-2}) \cap K_{j+1} = \emptyset$ since $j < i - 2$ implies $K_{j+1} \subset K_{i-2}$.

Thus W intersects only $\tilde{N}_{i-2}, \tilde{N}_{i-1}, \tilde{N}_i, \tilde{N}_{i+1}, \tilde{N}_{i+2}$ (with appropriate adjustments for boundary cases). Since each $\mathcal{V}_j$ is finite, W intersects only finitely many elements of $\mathcal{V}$.

Therefore, $\mathcal{V}$ is a locally finite refinement of $\mathcal{U}$, proving that M is paracompact. $\square$

REFERENCES

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