Lecture 3

November 21, 2025

1. Smooth (Differential) Manifolds

In the previous lecture, we introduced the concept and properties of topological manifolds. In this lecture, we will equip a topological manifold with an additional structure that allows us to perform calculus on it, such as computing derivatives of various orders. This additional structure is called a *smooth (differential) structure*. To begin, let us first recall the concept of a smooth differentiable function.

Definition 1.1. *Let $U \subset \mathbf{R}^m$ be an open set and $f : U \to \mathbf{R}^n$ be a function. We say that f is* smooth *(or C^∞) on U if:*

(1) f is continuous on U
(2) All partial derivatives of f of all orders exist and are continuous on U

That is, for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbf{Z}^m_{\geqslant 0}$, the partial derivative

$$\partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}}$$

exists and is continuous on U, where $|\alpha| = \alpha_1 + \cdots + \alpha_m$.

Remark 1.2. *This definition can be equivalently stated as:*

- *f is C^k for every $k \geqslant 0$*
- *f is infinitely differentiable (C^∞)*
- *The function f and all its derivatives vary continuously throughout U.*

Let M^n be an n-dimensional topological manifold. Recall that a manifold is a locally Euclidean, Hausdorff, and second-countable topological space. In the previous lecture, we used these properties to deduce that manifolds are paracompact, meaning they are, in a certain sense, not too "large". However, the most fundamental property for our purposes is still the *local Euclidean* structure. In subsequent lectures, we will frequently rely on this property. To facilitate this, we now introduce the terminology of a *chart*, which provides a convenient way to describe the local Euclidean structure.

Definition 1.3. *A* chart *on M is a pair (U, ϕ) where:*

(1) $U \subset M$ is an open set.
(2) $\phi : U \to \phi(U) \subset \mathbf{R}^n$ is a homeomorphism onto an open subset of $\mathbf{R}^n$.

We can now compare how different charts relate to each other. This leads to the concept of *transition maps*. By imposing certain regularity conditions on these transition maps, we can define special types of topological manifolds. In this course, we typically require the transition maps to be smooth, which gives rise to the concept of a *smooth (or differentiable) manifold*âthe primary object of our study.

Definition 1.4. *Let (U,ϕ) and (V,ψ) be two charts on M. Let $W = U \cap V$ be their intersection. The maps*

$$\psi \circ \phi^{-1}|_{\phi(W)} : \phi(W) \to \psi(W) \quad \textit{and} \quad \phi \circ \psi^{-1}|_{\psi(W)} : \psi(W) \to \phi(W)$$

are called the transition maps *between the charts. The two charts are called* smoothly compatible *if both transition maps are smooth (i.e., C^∞ maps between open subsets of $\mathbf{R}^n$).*

Example 1.5 (Polar Coordinates)**.** *Let $M = \mathbf{R}^2$ be the Euclidean plane. Consider two charts:*

- *The* Cartesian chart*: (U,ϕ) where $U = \mathbf{R}^2$ and $\phi = \mathbf{1}_{\mathbf{R}^2}$ is the identity map.*
- *The* polar chart*: (V,ψ) where $V = \mathbf{R}^2 \setminus \mathbf{R}^{\geqslant 0} \times \{0\}$ (the plane with the non-negative x-axis removed), and $\psi : V \to \mathbf{R}^2$ is defined by*

$$\psi(x, y) = \left(\sqrt{x^2 + y^2}, \arctan\left(\frac{y}{x}\right)\right).$$

Denoting $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$, we obtain the classical polar coordinates. The transition map from polar to Cartesian coordinates is given by:

$$\phi \circ \psi^{-1} : \psi(V) \to \mathbf{R}^2, \quad (r,\theta) \mapsto (r\cos\theta, r\sin\theta).$$

This is the familiar coordinate transformation between Cartesian and polar coordinates. Note that both transition maps $\psi \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$ are smooth on their respective domains, making these charts smoothly compatible. The exclusion of the non-negative x-axis in the polar chart is necessary to ensure that ψ is a homeomorphism, as the angular coordinate θ requires a consistent branch choice.

Now we attempt to give the precise definition of a differentiable manifold. Intuitively, we need a "complete" family of charts whose domains cover the entire manifold, and the transition maps between any two charts should be smooth. This is analogous to how an atlas of maps is used in geography to describe the Earth.

Definition 1.6. *An* atlas *$\mathscr{A}$ on M is a collection of charts on M such that:*

(1) The charts cover M: $\bigcup_{(U,\phi)\in\mathscr{A}} U = M$
(2) Any two charts in $\mathscr{A}$ are mutually compatible.

Example 1.7 (Standard Charts on S^1)**.** *The circle $S^1 = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 = 1\}$ can be endowed with a smooth structure using different atlases.*

Atlas 1: Four Charts via Projections

Consider the following four open subsets of S^1:

- *$U_1 = \{(x, y) \in S^1 : y > 0\}$ (upper semicircle)*
- *$U_2 = \{(x, y) \in S^1 : y < 0\}$ (lower semicircle)*
- *$U_3 = \{(x, y) \in S^1 : x > 0\}$ (right semicircle)*
- *$U_4 = \{(x, y) \in S^1 : x < 0\}$ (left semicircle)*

Define the chart maps as projections onto coordinates:

- *$\phi_1 : U_1 \to \mathbf{R}, \quad (x, y) \mapsto x$ (projection to x-axis)*
- *$\phi_2 : U_2 \to \mathbf{R}, \quad (x, y) \mapsto x$*

- $\phi_3 : U_3 \to \mathbf{R}, \quad (x, y) \mapsto y$
- $\phi_4 : U_4 \to \mathbf{R}, \quad (x, y) \mapsto y$

Each ϕ_i is a homeomorphism onto $(-1,1) \subset \mathbf{R}$. The transition maps between these charts are smooth. For example, for $U_1 \cap U_3 = \{(x, y) \in S^1 : x > 0, y > 0\}$, we have:

$$\phi_3 \circ \phi_1^{-1}(x) = \phi_3(x, \sqrt{1-x^2}) = \sqrt{1-x^2}, \quad x \in (0,1)$$

which is smooth on $(0,1)$. Similar calculations show all transition maps are smooth.

Atlas 2: Two Charts via Stereographic Projection

The stereographic projection provides a more elegant atlas with only two charts. Let:

- $V_1 = S^1 \setminus \{(0,1)\}$ *(circle without north pole)*
- $V_2 = S^1 \setminus \{(0,-1)\}$ *(circle without south pole)*

Define the stereographic projections:

- $\psi_1 : V_1 \to \mathbf{R}$ *(projection from north pole $(0,1)$):*
$$\psi_1(x, y) = \frac{x}{1-y}$$
- $\psi_2 : V_2 \to \mathbf{R}$ *(projection from south pole $(0,-1)$):*
$$\psi_2(x, y) = \frac{x}{1+y}$$

Both ψ_1 and ψ_2 are homeomorphisms onto $\mathbf{R}$. The transition map on $V_1 \cap V_2 = S^1 \setminus \{(0,1), (0,-1)\}$ is:

$$\psi_2 \circ \psi_1^{-1}(t) = \psi_2\left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right) = \frac{1}{t}, \quad t \in \mathbf{R} \setminus \{0\}$$

which is smooth on $\mathbf{R} \setminus \{0\}$. Similarly, $\psi_1 \circ \psi_2^{-1}(t) = 1/t$ is smooth.

Both atlases define the same smooth structure on S^1, demonstrating that different collections of charts can yield equivalent differentiable manifolds.

As the examples illustrate, a manifold can admit many different atlases. Analogous to the variety of physical atlases one might find for the Earth, we require a method to determine when two such atlases describe the same underlying smooth structure.

Definition 1.8. *Two atlases $\mathscr{A}$ and $\mathscr{A}'$ on M are* compatible *if any of the following equivalent conditions holds:*

(1) *$\mathscr{A} \cup \mathscr{A}'$ is an atlas.*
(2) *Every chart in $\mathscr{A}$ is compatible with every chart in $\mathscr{A}'$.*

Example 1.9 (Compatible Atlases on S^1)**.** *The two atlases on S^1 described above—the four-chart atlas via projections and the two-chart atlas via stereographic projection—are in fact* compatible. *This means that they both belong to the same equivalence class of atlases and thus define the* same *smooth structure on the circle.*

While one could verify this directly by checking the smoothness of all transition maps between charts of the first atlas and charts of the second, such a verification, though straightforward, is computationally tedious.

We will later establish this fact in a more conceptual and elegant way. Once we develop the theory of immersions and their relationship with differential structures, the compatibility of these atlases will follow naturally as a consequence of a more general principle. This approach highlights *the power of the categorical viewpoint in differential geometry, where the fundamental properties of a manifold can be understood through its smooth maps to and from other manifolds.*

Remark 1.10. *Compatibility of atlases is an equivalence relation. Indeed, the reflexive and symmetric properties are obvious. To check transitivity, let $\mathscr{A}_1$, $\mathscr{A}_2$ and $\mathscr{A}_3$ be atlases such that $\mathscr{A}_1$ is compatible with $\mathscr{A}_2$, and $\mathscr{A}_2$ is compatible with $\mathscr{A}_3$. Take any charts $(U_1, \phi_1) \in \mathscr{A}_1$ and $(U_3, \phi_3) \in \mathscr{A}_3$. We must show that c_1 and c_3 are compatible. Let $V = U_1 \cap U_3$. If $V = \emptyset$, then c_1 and c_3 are trivially compatible. Suppose $V \neq \emptyset$. It suffices to check that $\phi_3 \circ \phi_1^{-1}$ is smooth on $\phi_1(V)$. We verify this by checking differentiability at $\phi_1(x)$ for each $x \in V$. Choose $(U_2, \phi_2) \in \mathscr{A}_2$ such that $x \in U_2$. Then:*

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2) \text{ is smooth at } \phi_1(x),$$
$$\phi_3 \circ \phi_2^{-1} : \phi_2(U_2 \cap U_3) \to \phi_3(U_2 \cap U_3) \text{ is smooth at } \phi_2(x).$$

Hence, the composition $\phi_3 \circ \phi_1^{-1} = (\phi_3 \circ \phi_2^{-1}) \circ (\phi_2 \circ \phi_1^{-1})$ is smooth at $\phi_1(x)$ as desired.

This gives us an appropriate way to say when two atlases are essentially the same. We can now define:

A *differential manifold structure* on M is an equivalence class of compatible atlases on M.

An alternative definition can be formulated as follows. We say that an atlas $\mathscr{A}$ on M is *full* (or *maximal*) if whenever (U, ϕ) is a chart on M that is compatible with every chart $(V, \psi) \in \mathscr{A}$, then (U, ϕ) necessarily belongs to $\mathscr{A}$. It is straightforward to verify that each equivalence class of atlases on M contains exactly one full atlas.

We may therefore equivalently define:

A *differential manifold structure* on M is the choice of a full atlas on M.

To establish the equivalence of this definition with the previous one, we must show that for any atlas $\mathscr{A}$ on M, there exists a unique full atlas $\overline{\mathscr{A}}$ that is compatible with $\mathscr{A}$.

Theorem 1.11. *Let M be a topological manifold and $\mathscr{A}$ a smooth atlas on M. Then there exists a unique full atlas $\overline{\mathscr{A}}$ compatible with $\mathscr{A}$.*

Proof. The uniqueness is immediate. For existence, define $\overline{\mathscr{A}}$ to be the collection of all charts on M that are compatible with every chart in $\mathscr{A}$:

$$\overline{\mathscr{A}} := \{(U, \phi) : (U, \phi) \text{ is a chart on } M \text{ and is compatible with all } (V, \psi) \in \mathscr{A}\}.$$

We must verify that any two charts $(U_1, \phi_1), (U_2, \phi_2) \in \overline{\mathscr{A}}$ are smoothly compatible. Let $q \in \phi_1(U_1 \cap U_2)$ be arbitrary, and choose $p \in U_1 \cap U_2$ such that $\phi_1(p) = q$. Since $\mathscr{A}$ covers M, there exists a chart $(V, \psi) \in \mathscr{A}$ with $p \in V$.

Consider the transition map $\phi_2 \circ \phi_1^{-1}$ restricted to $\phi_1(U_1 \cap U_2 \cap V)$. This can be expressed as the composition:

$$\phi_2 \circ \phi_1^{-1}\big|_{\phi_1(U_1 \cap U_2 \cap V)} = \left(\phi_2 \circ \psi^{-1}\big|_{\psi(U_2 \cap V)}\right) \circ \left(\psi \circ \phi_1^{-1}\big|_{\phi_1(U_1 \cap V)}\right).$$

By the definition of $\overline{\mathscr{A}}$, both $\phi_2 \circ \psi^{-1}$ and $\psi \circ \phi_1^{-1}$ are smooth on their respective domains. Therefore, their composition $\phi_2 \circ \phi_1^{-1}$ is smooth in a neighborhood of q. Since q was arbitrary, the charts (U_1, ϕ_1) and (U_2, ϕ_2) are smoothly compatible.

Finally, $\overline{\mathscr{A}}$ is maximal by construction, which completes the proof of existence. $\square$

Henceforth, we will assume that M is a topological space equipped with a fixed differential manifold structure. We denote by $\mathscr{A}(M)$ the full atlas corresponding to this structure. By a *chart* (U,ϕ) on M, we will always mean a chart belonging to $\mathscr{A}(M)$; similarly, an *atlas* $\mathscr{A}$ of M will always refer to a collection of charts that forms a subset of $\mathscr{A}(M)$ and covers M.

References

[1] Jean-Pierre Serre, *Lie Algebras and Lie Groups*, Lecture Notes in Mathematics, vol. 1500, Springer, 1992.
[2] Alex Waldron, *Math 761: Differentiable Manifolds*. University of Wisconsin–Madison, Lecture notes, Fall 2024.
[3] John M. Lee, *Introduction to Smooth Manifolds*, 2nd ed., Graduate Texts in Mathematics, vol. 218, Springer, 2012.

Mathematisches Institut, Universität Freiburg
Email address: yuchen.bi@math.uni-freiburg.de