Lecture 4

November 21, 2025

1. Examples of smooth manifolds

Having defined the concept of a differential structure, we now examine several important examples that illustrate this fundamental notion.

Example 1.1 (Euclidean Space)**.** *The Euclidean space* $\mathbf{R}^n$ *carries a natural differential structure. The* standard smooth structure *is given by the maximal atlas containing the identity chart* $(\mathbf{R}^n, \mathbf{1}_{\mathbf{R}^n})$. *This atlas consists of all charts that are smoothly compatible with the identity map, forming what we call the* standard differentiable structure *on* $\mathbf{R}^n$.

Example 1.2 (Open Submanifolds)**.** *If* M *is a smooth manifold with maximal atlas* $\mathscr{A}(M)$ *and* $U \subset M$ *is an open subset, then* U *naturally inherits a smooth manifold structure. The* induced maximal atlas *on* U *is given by:*

$$\mathscr{A}(U) = \{(V, \phi) : (V, \phi) \in \mathscr{A}(M) \text{ and } V \subset U\}.$$

That is, we simply restrict the charts of M *to the open subset* U. *One readily verifies that this collection forms a maximal atlas on* U, *making it a smooth submanifold of* M.

Example 1.3 (Distinct but Diffeomorphic Structures on $\mathbf{R}$)**.** *Consider the topological manifold* $\mathbf{R}$. *The* standard differential structure *is given by the maximal atlas containing the identity chart* $(\mathbf{R}, \mathbf{1}_{\mathbf{R}})$. *However, we can define a* different *differential structure on the same underlying topological space using the chart* $(\mathbf{R}, \phi)$ *where* $\phi(x) = x^3$.

These two charts are not smoothly compatible: the transition map $id_{\mathbf{R}} \circ \phi^{-1}(t) = t^{1/3}$ *is not differentiable at* $t = 0$. *Therefore, the maximal atlases generated by these charts are distinct, giving* $\mathbf{R}$ *(as a topological manifold) at least two different smooth structures.*

This example demonstrates that equality *of differential structures (i.e., requiring the maximal atlases to be identical) is too restrictive a notion of equivalence. A more appropriate concept, which we will introduce later, is that of* diffeomorphism - *two manifolds are considered equivalent if there exists a smooth bijection between them with smooth inverse, even if their maximal atlases differ.*

IIn our previous lectures, we began with a topological space and imposed additional structures–local Euclideanness, Hausdorff, and second countability–to define a topological manifold. We then refined this by introducing an atlas and requiring smooth transition maps to arrive at the concept of a smooth manifold.

However, we can adopt an alternative perspective. Instead of starting with a pre-existing topological space, we can take the *charts* themselves as the primitive objects. By collecting a sufficient family of compatible charts, we can *construct* the manifold from the ground up.

This alternative approach provides a different way to understand manifolds. Just as in quantum mechanics we can understand particles through their measurable interactions rather than predetermined trajectories, here we can understand manifolds through their local coordinate descriptions and how they relate to one another.

This alternative perspective is formalized in the following construction lemma, which allows us to build a manifold using only set-theoretic data.

Lemma 1.4 (Manifold Construction Lemma)**.** *Let X be a set. Suppose we are given a collection $\{U_\alpha\}_{\alpha\in I}$ of subsets of X and a collection of bijections $\{\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)\}_{\alpha\in I}$ where each $\phi_\alpha(U_\alpha)$ is an open subset of $\mathbf{R}^n$, such that:*

(1) For all $\alpha, \beta \in I$, the sets $\phi_\alpha(U_\alpha \cap U_\beta)$ and $\phi_\beta(U_\alpha \cap U_\beta)$ are open in $\mathbf{R}^n$.
(2) The transition maps $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ are smooth.
(3) The collection $\{U_\alpha\}$ covers X.
*(4) (**Hausdorff Separation**) For any two distinct points $p, q \in X$, there exist indices $\alpha, \beta \in I$ with $p \in U_\alpha$, $q \in U_\beta$, and open subsets $V_\alpha \subset U_\alpha$, $V_\beta \subset U_\beta$ such that:*
   - *$p \in V_\alpha$ and $q \in V_\beta$*
   - *$V_\alpha \cap V_\beta = \emptyset$*
   - *$\phi_\alpha(V_\alpha)$ and $\phi_\beta(V_\beta)$ are open in $\mathbf{R}^n$*

*(5) (**Second Countability**) There exists a countable subcollection $\{U_{\alpha_i}\}_{i=1}^\infty$ that still covers X.*

Then there exists a unique topology and smooth structure on X making it a smooth n-dimensional manifold for which the (U_α, ϕ_α) form a smooth atlas.

Proof. Define a topology on X by declaring $O \subseteq X$ open if and only if $\phi_\alpha(O \cap U_\alpha)$ is open in $\mathbf{R}^n$ for all $\alpha \in I$. This is the unique topology making all ϕ_α homeomorphisms.

The given conditions ensure:

- Local Euclideanness: Each (U_α, ϕ_α) is a homeomorphism onto an open subset of $\mathbf{R}^n$.
- Hausdorff: Condition (4) provides separation via charts.
- Second countability: Condition (5) gives a countable cover, and $\mathbf{R}^n$ has countable bases.
- Smooth structure: Conditions (1)-(3) ensure the charts form a compatible atlas.

Uniqueness follows because any topology making the ϕ_α homeomorphisms must coincide with our definition. □

This lemma tells us that we don't need to start with a topology on X; the topology is *induced* by the charts.

It is instructive to compare this with another gluing construction:

Theorem 1.5 (Patching Smooth Structures)**.** *Let M be a topological manifold that can be written as a union of subsets $M = \bigcup_\alpha U_\alpha$. Suppose each U_α is endowed with a smooth manifold structure such that for every α, β, the smooth structures induced on $U_\alpha \cap U_\beta$ from U_α and U_β coincide. Then there exists a unique smooth structure on M that restricts to the given smooth structure on each U_α.*

Proof. For each α, let $\mathscr{A}_\alpha$ be the maximal smooth atlas on U_α defining its smooth structure. Define an atlas on M by taking the union:

$$\mathscr{A} = \bigcup_\alpha \mathscr{A}_\alpha.$$

Then:

- $\mathscr{A}$ covers M since the U_α cover M and each $\mathscr{A}_\alpha$ covers U_α.
- Any two charts in $\mathscr{A}$ are compatible: if $(V,\phi) \in \mathscr{A}_\alpha$ and $(W,\psi) \in \mathscr{A}_\beta$, then the transition map $\psi \circ \phi^{-1}$ is smooth because the smooth structures on $U_\alpha \cap U_\beta$ from $\mathscr{A}_\alpha$ and $\mathscr{A}_\beta$ coincide.

The maximal atlas containing $\mathscr{A}$ gives the desired smooth structure on M. Uniqueness follows since any such structure must contain all $\mathscr{A}_\alpha$. □

2. Grassmann Manifolds via the Manifold Construction Lemma

We now construct the Grassmann manifold $G(k,n)$ using the Manifold Construction Lemma. Let $X = G(k,n)$ be the set of all k-dimensional linear subspaces of $\mathbf{R}^n$.

2.0. **Coordinate Charts.** There is a natural identification between k-dimensional subspaces of $\mathbf{R}^n$ and equivalence classes of full-rank $n \times k$ matrices, where two matrices are equivalent if their column spaces coincide (i.e., they differ by right multiplication by an element of $\mathrm{GL}(k)$). Under this identification, we can represent each subspace $W \in G(k,n)$ as an equivalence class $[X]$ of matrices whose columns form a basis for W.

For each k-element subset $I \subset \{1,\dots,n\}$, define:

$$U_I = \{W \in G(k,n) : \text{the projection } \pi_I : W \to \mathbf{R}^k \text{ is an isomorphism}\}$$

Equivalently, if we represent W as the column space of an $n \times k$ matrix X of full rank, then:

$$U_I = \{[X] : \det(X_I) \neq 0\}$$

where X_I is the $k \times k$ submatrix of X with rows indexed by I.

Define the chart map $\phi_I : U_I \to \mathrm{Mat}((n-k) \times k, \mathbf{R}) \cong \mathbf{R}^{k(n-k)}$ by:

$$\phi_I([X]) = X_{I^c} X_I^{-1}$$

where I^c is the complement of I.

Remark 2.1. *The chart map ϕ_I is an open map. To see this, fix X_I and let X_{I^c} vary over an open set in $\mathrm{Mat}((n-k)\times k, \mathbf{R})$. Since for fixed X_I, the map $X_{I^c} \mapsto X_{I^c} X_I^{-1}$ is a linear isomorphism. Therefore, ϕ_I sends open sets to open sets.*

Proposition 2.2. *The collection $\{(U_I, \phi_I)\}$ satisfies the conditions of the Manifold Construction Lemma.*

Proof. We verify each condition:

Covering: For any k-dimensional subspace W, there exists some I such that the projection $\pi_I : W \to \mathbf{R}^I$ is an isomorphism. Thus the U_I cover $G(k,n)$.

Openness: For any I, J, the sets $\phi_I(U_I \cap U_J)$ and $\phi_J(U_I \cap U_J)$ are open in $\mathbf{R}^{k(n-k)}$, as they are defined by the non-vanishing of certain determinants.

Smooth transitions: For $[X] \in U_I \cap U_J$, the transition map is:

$$\phi_J \circ \phi_I^{-1}(A) = \phi_J\left(\left[\begin{pmatrix} I_k \\ A \end{pmatrix}\right]\right)$$

To compute this, we find $g \in \mathrm{GL}(k)$ such that:

$$\begin{pmatrix} I_k \\ A \end{pmatrix} g = \begin{pmatrix} B \\ C \end{pmatrix}$$

where B corresponds to rows J and C to rows J^c. Then $\phi_J([X]) = CB^{-1}$. This is a rational function in the entries of A, hence smooth.

Hausdorff separation: Let $[X],[Y] \in G(k,n)$ be distinct. Consider the matrix $\begin{pmatrix} X \\ Y \end{pmatrix}$ which has rank at least $k+1$ since $[X] \neq [Y]$.

By the lower semicontinuity of matrix rank, there exists $\epsilon > 0$ such that any matrix within ϵ of $\begin{pmatrix} X \\ Y \end{pmatrix}$ has rank at least $k+1$.

Let V_X be the $\epsilon/2$-neighborhood of X in the space of $n \times k$ matrices of rank k, and V_Y the $\epsilon/2$-neighborhood of Y. Choose ϵ small enough so that $V_X \subset U_I$ and $V_Y \subset U_J$, where $X \in U_I$ and $Y \in U_J$.

Then $[V_X]$ and $[V_Y]$ are the required open subsets with $[X] \in [V_X] \subset U_I$, $[Y] \in [V_Y] \subset U_J$, $[V_X] \cap [V_Y] = \emptyset$, and $\phi_I([V_X])$, $\phi_J([V_Y])$ are open in $\mathbf{R}^{k(n-k)}$.

Second countability: There are only $\binom{n}{k}$ charts, which is finite, so the atlas is countable. $\square$

By the Manifold Construction Lemma, $G(k,n)$ is a smooth manifold of dimension $k(n-k)$.

This construction demonstrates the power of the Manifold Construction Lemma. We built the Grassmann manifold directly from local coordinate descriptions, without needing to first define it as a quotient space or prove it is Hausdorff by other means.

The Grassmann manifold $G(k,n)$ plays a fundamental role in many areas of mathematics, including algebraic geometry, representation theory, and topology. Its construction is remarkably versatile–indeed, the Grassmannian can be defined over any field (such as the complex numbers $\mathbb{C}$, giving the complex Grassmannian $G_{\mathbb{C}}(k,n)$), and even over more general rings, though the smooth manifold structure is specific to the real and complex cases. This universality makes Grassmannians fundamental objects across diverse branches of mathematics, from classical geometry to modern arithmetic geometry.

3. Smooth maps and diffeomorphisms

Let M^m and N^n be two differential manifolds. A map $f : M \to N$ is said to be a smooth map if:

(1) f is continuous.
(2) f is "locally given by smooth functions", that is, there exists atlases $\mathscr{A}$ of M and $\mathscr{B}$ of N such that if $(U,\phi) \in \mathscr{A}$ and $(V,\psi) \in \mathscr{B}$, then setting $W = U \cap f^{-1}(V)$, the composite

$$\phi(W) \xrightarrow{\phi^{-1}} W \xrightarrow{f} V \xrightarrow{\psi} \psi(V)$$

is smooth.

Remark 3.1. *(1) We describe condition 2 by saying that f is "locally given by smooth functions" since, in coordinates, composites of the form $\psi \circ f \circ \phi^{-1}$ may be written as n-tuples of smooth function of m variables.*

(2) Condition 2 is independent of the choice of atlases $\mathscr{A}$ and $\mathscr{B}$, as is seen by an argument similar to the one showing that compatibility of atlases in an equivalence relation.

The following formal properties of smooth maps are easily verified:

(1) The composition of smooth maps is a smooth map.
(2) The identity map on a manifold is a smooth map.

Definition 3.2. *Let $f\colon M \to N$ be a smooth map. We say that f is a diffeomorphism if there exists a smooth map $g\colon N \to M$ such that*

$$g \circ f = \mathbf{1}_M \quad \text{and} \quad f \circ g = \mathbf{1}_N.$$

In this case, we say that M and N are diffeomorphic.

Remark 3.3. *Bijective smooth maps is not necessary a diffeomorphism, as the example $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^3$ shows.*

After defining differentiable manifolds and smooth maps between them, we can employ the perspective of categories and functors to describe differential structures. While this abstract language may seem simple and mundane, it will greatly simplify our subsequent discussions on the relationship between immersions and differential structures. Moreover, the categorical viewpoint is widely used in mathematical branches such as algebraic topology and algebraic geometry, making familiarity with this language beneficial. Roughly speaking, the underlying philosophy is that an object is uniquely determined by its relationships with other objects.

Theorem 3.4. *Let M be a topological manifold, let $\mathscr{A}$ and $\mathscr{B}$ be full atlases on M, and let $M_{\mathscr{A}}$ (resp. $M_{\mathscr{B}}$) denote the smooth manifold whose underlying space is M that is determined by $\mathscr{A}$ (resp. $\mathscr{B}$). Then the following are equivalent:*

(1) $M_{\mathscr{A}} = M_{\mathscr{B}}$, that is, $\mathscr{A} = \mathscr{B}$.
(2) The identity map $\mathbf{1}_M : M_{\mathscr{A}} \to M_{\mathscr{B}}$ is a diffeomorphism.
(3) For all manifolds N, we have the equality $C^\infty(M_{\mathscr{A}}, N) = C^\infty(M_{\mathscr{B}}, N)$.
(4) For all manifolds N, we have the equality $C^\infty(N, M_{\mathscr{A}}) = C^\infty(N, M_{\mathscr{B}})$.

Proof. Although this theorem can be viewed as a special case of the general categorical principle that an object representing a functor is determined up to unique isomorphism, we provide a concrete proof in this context.

The implications (1) $\Rightarrow$ (2) and (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4)are straightforward.

We now prove (2) $\Rightarrow$ (1): We need to show that any chart $(U,\phi) \in \mathscr{A}$ is compatible with any chart $(V,\psi) \in \mathscr{B}$. If $U \cap V = \emptyset$, the compatibility is trivial. Assume $U \cap V \neq \emptyset$ and take any $p \in U \cap V$. Since $\mathbf{1}_M$ is a diffeomorphism, by definition there exist charts $(U',\phi') \in \mathscr{A}$ and $(V',\psi') \in \mathscr{B}$ such that $p \in U' \cap V'$ and the transition map

$$\psi' \circ \phi'^{-1}\big|_{\phi'(U'\cap V')} = \psi' \circ \mathbf{1}_M \circ \phi'^{-1}\big|_{\phi'(U'\cap V')} : \phi'(U' \cap V') \to \psi'(U' \cap V')$$

is a diffeomorphism. Since (U,ϕ) and (U',ϕ') belong to the same atlas $\mathscr{A}$, and (V,ψ) and (V',ψ') belong to $\mathscr{B}$, the transition maps

$$\phi' \circ \phi^{-1}\big|_{\phi(U\cap U')} : \phi(U \cap U') \to \phi'(U \cap U')$$

and

$$\psi \circ \psi'^{-1}\big|_{\psi'(V\cap V')} : \psi'(V \cap V') \to \psi(V \cap V')$$

are diffeomorphisms. Therefore, their composition

$$\psi \circ \phi^{-1}\big|_{\phi(U\cap V\cap U'\cap V')} : \phi(U \cap V \cap U' \cap V') \to \psi(U \cap V \cap U' \cap V')$$

is a diffeomorphism. Since p was arbitrary, (U,ϕ) and (V,ψ) are compatible. As $\mathscr{A}$ and $\mathscr{B}$ are both maximal atlases, this implies $\mathscr{A} = \mathscr{B}$. $\square$

This theorem demonstrates that the smooth structure of a manifold is completely determined by either:

- The collection of smooth maps from the manifold to arbitrary test manifolds, or
- The collection of smooth maps from arbitrary test manifolds to the manifold.

It has been (and still remains) a celebrated problem to compare the category of topological manifolds with the category of smooth manifolds. Here are some historical landmarks:

- H. Whitney (1936): For $k \geqslant 1$, every C^k structure is C^k equivalent to a smooth structure.
- E. Moise (1952): For $n < 4$, every topological manifold carries a unique smooth structure up to diffeomorphism.
- J. Milnor (1956): Exotic smooth structures exist. In particular, the topological manifold S^7 carries smooth structures that are not diffeomorphic to the standard one.
- M. Kervaire (1960): There exists a compact 10-dimensional topological manifold that carries no smooth structure at all.
- M. Freedman / S. Donaldson (1982): $\mathbf{R}^4$ carries exotic smooth structures.

Comment on spherical case: For spheres S^n with $n \neq 4$, the classification of smooth structures is intimately related to the homotopy groups of spheres. In fact, the number of distinct smooth structures on S^n for $n \neq 4$ is determined by a certain invariant connected to these homotopy groups. Currently, this classification is known for dimensions up to about 90, with the number of exotic spheres growing rapidly in higher dimensions. However, the case of S^4 remains one of the most famous open problems in differential topology–it is still unknown whether S^4 admits any exotic smooth structures (the smooth Poincaré conjecture in dimension 4).

In this course, we will provide a complete classification of smooth structures on 1-dimensional manifolds. While this result is relatively simple, it is by no means trivial and illustrates the fundamental ideas of differential topology. If time permits, we will present two different approaches to this classification: one using vector fields and their flows, and another using the technique of *handle straightening*–both methods offering distinct geometric insights into the structure of 1-manifolds.

The classification of 2-dimensional manifolds will be covered in subsequent courses on differential topology, where more sophisticated tools like Morse theory and handle decompositions are developed. As for the fascinating and complex world of 3-dimensional manifolds–including the celebrated Poincaré conjecture proved by Perelman–this typically becomes the subject of specialized research, requiring deep techniques from geometric analysis and low-dimensional topology.

References

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