Lecture 5

November 21, 2025

1. Cutoff Functions

All constructions in this section rely on the existence of smooth functions that are positive on specified regions of a manifold and identically zero elsewhere. We begin by constructing a fundamental smooth function on the real line that transitions from zero to positive.

Lemma 1.1 (Smooth Transition Function)**.** *The function $f : \mathbf{R} \to \mathbf{R}$ defined by*

$$f(t) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \leqslant 0 \end{cases}$$

is smooth.

Proof. We proceed by induction on n to show that for $t > 0$,

$$f^{(n)}(t) = \frac{d^n f}{dt^n} = P_n(t^{-1})e^{-1/t}$$

where P_n is a polynomial. The base case $n = 0$ holds trivially. Assuming the formula holds for n, we differentiate to obtain:

$$f^{(n+1)}(t) = \left[P_n'(t^{-1})(-t^{-2}) + P_n(t^{-1})t^{-2}\right]e^{-1/t} = P_{n+1}(t^{-1})e^{-1/t}.$$

To prove smoothness at $t = 0$, we note that $\lim_{t\to 0^+} t^{-1} f^{(n)}(t) = 0$ for every $n \geqslant 0$. This ensures that all derivatives of f exist and are continuous at $t = 0$, and in fact $f^{(n)}(0) = 0$. □

Lemma 1.2 (One-Dimensional Cutoff Function)**.** *Given real numbers $r_1 < r_2$, there exists a smooth function $h : \mathbf{R} \to \mathbf{R}$ such that:*

- $h(t) \equiv 1$ *for* $t \leqslant r_1$
- $0 < h(t) < 1$ *for* $r_1 < t < r_2$
- $h(t) \equiv 0$ *for* $t \geqslant r_2$

Proof. Let f be the function from the previous lemma, and define

$$h(t) = \frac{f(r_2 - t)}{f(r_2 - t) + f(t - r_1)}.$$

The denominator is always positive since for any $t \in \mathbf{R}$, at least one of $r_2 - t > 0$ or $t - r_1 > 0$ holds. The function h inherits smoothness from f, and direct verification shows it satisfies all the stated properties. □

A function with these properties is called a *cutoff function.*

Lemma 1.3 (Smooth Bump Function)**.** *Given positive real numbers $r_1 < r_2$, there exists a smooth function $H : \mathbf{R}^n \to \mathbf{R}$ such that:*

- $H \equiv 1$ *on* $\overline{B(0, r_1)}$
- $0 < H(x) < 1$ *for* $x \in B(0, r_2) \setminus \overline{B(0, r_1)}$
- $H \equiv 0$ *on* $\mathbf{R}^n \setminus B(0, r_2)$

Proof. Define $H(x) = h(|x|)$, where h is the one-dimensional cutoff function from the previous lemma with the same radii $r_1 < r_2$. The function H is smooth on $\mathbf{R}^n \setminus \{0\}$ as a composition of smooth functions. At the origin, H is identically 1 in a neighborhood, hence smooth there as well. □

2. Partitions of unity

It will be convenient in various contexts to express smooth functions (as well as differential forms and vector fields that we will encounter in later lectures) as sums of locally supported functions. This leads us to the concept of a partition of unity, whose existence fundamentally relies on the assumptions that M is second-countable and Hausdorff. Recall that the *support* $\text{supp}(\eta) \subset M$ of a continuous function $\eta : M \to \mathbf{R}$ is defined as the closure of the open subset $\{x : \eta(x) \neq 0\}$.

Definition 2.1. *Let* $\mathscr{W} = \{W_i\}_{i \in I}$ *be an open cover of* M*. A partition of unity subordinate to* $\mathscr{W}$ *is a collection of smooth functions* $\eta_i : M \to [0, 1]$ *satisfying:*

(1) $\text{supp}(\eta_i) \subset W_i$ *for each* $i \in I$*,*
(2) The family $\{\text{supp}(\eta_i) : i \in I\}$ *is locally finite, meaning every point* $p \in M$ *has a neighborhood intersecting only finitely many* $\text{supp}(\eta_i)$*,*
(3) $\sum_{i \in I} \eta_i(p) = 1$ *for all* $p \in M$*.*

Theorem 2.2 (Existence of Partitions of Unity)**.** *Every open cover* $\mathscr{W} = \{W_i\}_{i \in I}$ *of* M *admits a subordinate partition of unity.*

We now proceed to prove the existence of partitions of unity. First, we recall a key result established in Lecture 2:

Proposition 2.3 (Exhaustion by Compact Sets)**.** *Every topological manifold* M *admits an* exhaustion by compact sets*, i.e., there exists a sequence* $\{K_n\}_{n=1}^{\infty}$ *of compact subsets such that:*

(1) $K_n \subset \text{int}(K_{n+1})$ *for all* $n \geqslant 1$*,*
(2) $\bigcup_{n=1}^{\infty} K_n = M$*.*

Recall also that if $\mathscr{U}$ is an open cover of M, an open cover $\mathscr{V}$ is called a *refinement* of $\mathscr{U}$ if every $V \in \mathscr{V}$ is contained in some $U \in \mathscr{U}$. The previous proposition implies that M is *paracompact*–that is, every open cover admits a locally finite refinement. We will first establish a slightly weaker version of Theorem 2.2, and in the process provide a detailed proof of the paracompactness of M (which was sketched in Lecture 2).

Proposition 2.4. *Every open cover* $\mathscr{W} = \{W_i\}_{i \in I}$ *of* M *has a refinement that admits a subordinate partition of unity.*

Proof. Let $K_1 \subset \text{int}(K_2) \subset K_2 \subset \text{int}(K_3) \subset \cdots$ be an exhaustion of M by compact sets as established previously, and let $\mathscr{W} = \{W_i\}_{i \in I}$ be an open cover. For any point $p \in M$, there exists a unique n such that $p \in K_n \setminus K_{n-1}$ (where we take $K_0 = \emptyset$). Then $\text{int}(K_{n+1}) \setminus K_{n-1}$ is an open neighborhood of p. We choose a chart (U_α, ϕ_α) around p such that $\phi_\alpha(U_\alpha) = B(z, r) \subset \mathbf{R}^n$ and

$$U_\alpha \subset W_i \cap (\text{int}(K_{n+1}) \setminus K_{n-1})$$

for some $i \in I$.

Varying p over M (and thus over all $n \geqslant 1$), the collection of open sets $\phi_\alpha^{-1}(B(z, r/3))$ covers M. In particular, for each n, the compact set $K_n \setminus \mathrm{int}(K_{n-1})$ is covered by finitely many such sets, say $\phi_{\alpha_i^n}^{-1}(B(z_i^n, r_i^n/3))$ for $1 \leqslant i \leqslant j_n$. Taking the union over n of these finite families, we obtain a countable open cover of M, since

$$\bigcup_{n\geqslant 1}(K_n \setminus \mathrm{int}(K_{n-1})) \supset \bigcup_{n\geqslant 1}(K_n \setminus K_{n-1}) = M.$$

By construction, each $\phi_{\alpha_i^n}^{-1}(B(z_i^n, r_i^n/3))$ is contained in some W_i, so this cover is a refinement of $\mathscr{W}$.

We now verify local finiteness. Let $p \in M$. Then p lies in some $K_n \setminus K_{n-1}$, and hence in the open set $\mathrm{int}(K_{n+1}) \setminus K_{n-1}$. By construction, p is contained in one of the sets $U_{\alpha_i^n} = \phi_{\alpha_i^n}^{-1}(B(z_i^n, r_i^n))$, which is contained in $\mathrm{int}(K_{n+1}) \setminus K_{n-1}$. Note that if $U_{\alpha_i^n}$ intersects $U_{\alpha_j^m}$, then we must have

$$(\mathrm{int}(K_{n+1}) \setminus K_{n-1}) \cap (\mathrm{int}(K_{m+1}) \setminus K_{m-1}) \neq \emptyset,$$

which implies that m can only be $n-1$, n, or $n+1$. Therefore, each $U_{\alpha_i^n}$ can intersect only finitely many $U_{\alpha_j^m}$ (specifically, at most $j_{n-1} + j_n + j_{n+1}$ of them). This establishes local finiteness, proving that M is paracompact.

Claim 2.5. *For each i and n, there exists a smooth function $\tilde{\rho}_i^n : B(z_i^n, r_i^n) \to [0,1]$ that vanishes outside $B(z_i^n, r_i^n/2)$ and is identically 1 on $B(z_i^n, r_i^n/3)$.*

This follows directly from the existence of smooth bump functions (Lemma 1.3) by appropriate scaling and translation.

We now define smooth functions $\tilde{\eta}_i^n : M \to [0,1]$ by

$$\tilde{\eta}_i^n(p) = \begin{cases} \tilde{\rho}_i^n(\phi_{\alpha_i^n}(p)) & \text{if } p \in \phi_{\alpha_i^n}^{-1}(B(z_i^n, r_i^n)), \\ 0 & \text{otherwise.} \end{cases}$$

Since the collection $\{\phi_{\alpha_i^n}^{-1}(B(z_i^n, r_i^n/3))\}$ covers M and the collection $\{\phi_{\alpha_i^n}^{-1}(B(z_i^n, r_i^n))\}$ is locally finite, the sum

$$p \mapsto \sum_{i,n} \tilde{\eta}_i^n(p)$$

is locally a finite sum of non-zero terms and hence defines a smooth function $M \to \mathbf{R}_{>0}$. We then define

$$\eta_i^n(p) = \frac{\tilde{\eta}_i^n(p)}{\sum_{i,n} \tilde{\eta}_i^n(p)}.$$

This yields the desired partition of unity subordinate to the refinement $\{U_{\alpha_i^n}\}$ of $\mathscr{W}$. □

Proof of Theorem 2.2. By the previous proposition we can find a refinement $\mathscr{W}' = \{W_j'\}_{j\in J}$ of $\mathscr{W} = \{W_i\}_{i\in I}$ and partition of unity $\{\eta_j' : M \to [0,1]\}$ subordinate to it.

For $j \in J$, fix a W_i such that $W_j' \subset W_i$. This gives a function $\lambda : J \to I$. We claim that

$$\eta_i := \sum_{j\in\lambda^{-1}(i)} \eta_j'$$

gives the desired partition of unity. This is a locally finite sum and hence a smooth function and the sum of η_i is 1 everywhere. Also since that $\mathrm{supp}(\eta_j') \subset W_j'$ and hence is also contained in W_i.

Now observe that

$$\mathrm{supp}(\eta_i) = \overline{\eta_i^{-1}((0,1])} = \overline{\bigcup_{j\in\lambda^{-1}(i)} (\eta'_j)^{-1}((0,1])}.$$

In general, the closure of the union of subsets is larger the the union of closures of these subset. But Note that the latter is a closure of a locally finite union of open subsets. The local finiteness ensures that this is the union of the closures, by directly verify the definition. So we conclude that

$$\mathrm{supp}(\eta_i) = \bigcap_{j\in\lambda^{-1}(i)} \overline{(\eta'_j)^{-1}((0,1])} = \bigcup_{j\in\lambda^{-1}} \mathrm{supp}(\eta'_j) \subset W_j.$$

□

3. Applications of Partitions of Unity

Proposition 3.1 (Smooth Urysohn Lemma)**.** *Let M be a smooth manifold, and let $A, B \subset M$ be disjoint closed subsets. Then there exists a smooth function $\lambda : M \to [0,1]$ such that $\lambda|_A \equiv 0$ and $\lambda|_B \equiv 1$.*

Proof. Consider the open cover of M given by $U_1 = M \setminus A$ and $U_2 = M \setminus B$. Let $\{\eta_1, \eta_2\}$ be a smooth partition of unity subordinate to this cover. Since $\mathrm{supp}(\eta_1) \subset U_1$, we have $\eta_1|_A \equiv 0$. Similarly, as $\mathrm{supp}(\eta_2) \subset U_2$, we have $\eta_2|_B \equiv 0$. Now define $\lambda = \eta_1$. Then $\lambda|_A = 0$, and on B we have $\lambda = \eta_1 = 1 - \eta_2 \equiv 1$, which completes the proof. □

The Smooth Urysohn Lemma has an equivalent formulation that is often more convenient for applications:

Proposition 3.2 (Smooth Bump Function)**.** *Let M be a smooth manifold, $A \subset M$ a closed subset, and $U \subset M$ an open subset containing A. Then there exists a smooth function $\psi : M \to [0,1]$ such that:*

- $\psi|_A \equiv 1$
- $\mathrm{supp}(\psi) \subset U$

Proof. Apply the Smooth Urysohn Lemma to the disjoint closed sets A and $M \setminus U$. This yields a smooth function $\lambda : M \to [0,1]$ with $\lambda|_A \equiv 0$ and $\lambda|_{M\setminus U} \equiv 1$. Define $\psi = 1 - \lambda$. Then $\psi|_A \equiv 1$, and since $\lambda \equiv 1$ on $M \setminus U$, we have $\psi \equiv 0$ on $M \setminus U$, hence $\mathrm{supp}(\psi) \subset U$. □

Remark 3.3. *These two propositions are equivalent: each can be derived from the other. The Smooth Bump Function version is particularly useful for constructing local extensions and cutoff functions, while the original Urysohn formulation provides a clear separation property for disjoint closed sets.*

Our second application concerns the extension of smooth functions from closed subsets. Let M and N be smooth manifolds, and $A \subset M$ an arbitrary subset. We say that a map $F : A \to N$ is *smooth on A* if for every point $p \in A$, there exists an open neighborhood $W \subset M$ containing p and a smooth map $\widetilde{F} : W \to N$ that agrees with F on $W \cap A$.

Lemma 3.4 (Extension Lemma for Smooth Functions)**.** *Let M be a smooth manifold, $A \subset M$ a closed subset, and $f : A \to \mathbf{R}^k$ a smooth function. For any open subset $U \subset M$ containing A, there exists a smooth function $\tilde{f} : M \to \mathbf{R}^k$ such that:*

- $\tilde{f}|_A = f$
- $\operatorname{supp}(\tilde{f}) \subset U$

Proof. For each point $p \in A$, choose a neighborhood $W_p \subset U$ and a smooth function $\tilde{f}_p : W_p \to \mathbf{R}^k$ that agrees with f on $W_p \cap A$. The collection $\{W_p : p \in A\} \cup \{M \setminus A\}$ forms an open cover of M. Let $\{\eta_p : p \in A\} \cup \{\eta_0\}$ be a smooth partition of unity subordinate to this cover, with $\operatorname{supp}(\eta_p) \subset W_p$ and $\operatorname{supp}(\eta_0) \subset M \setminus A$.

Define the global function $\tilde{f} : M \to \mathbf{R}^k$ by

$$\tilde{f}(x) = \sum_{p \in A} \eta_p(x) \tilde{f}_p(x).$$

This sum is well-defined since the partition of unity is locally finite. The function $\tilde{f}$ is smooth, agrees with f on A, and has support contained in U. □

Remark 3.5. *The classical Whitney Extension Theorem provides a more refined result, requiring only compatibility conditions on the partial derivatives rather than assuming smoothness in the sense defined above.*

For example, consider a function defined on the union of linear subspaces of Euclidean space in general position. If the restriction of the function to each subspace is smooth, the Whitney Extension Theorem guarantees the existence of a smooth extension to the whole Euclidean space. This situation frequently arises in geometry or analysis and cannot be handled by the Extension Lemma above, which requires the function to already have local smooth extensions near every point of the closed set.

Next, we use partitions of unity to construct a special type of smooth function. If M is a topological space, an *exhaustion function for M* is a continuous function $f : M \to \mathbf{R}$ such that for every $c \in \mathbf{R}$, the *sublevel set* $f^{-1}((-\infty, c])$ is compact. The terminology arises from the fact that as n ranges over the positive integers, the sublevel sets $f^{-1}((-\infty, n])$ form an exhaustion of M by compact sets.

Proposition 3.6 (Existence of Smooth Exhaustion Functions)**.** *Every smooth manifold admits a smooth positive exhaustion function.*

Proof. Let $\{V_j\}_{j=1}^{\infty}$ be a countable open cover of M by precompact open subsets, and let $\{\eta_j\}$ be a smooth partition of unity subordinate to this cover. Define $f \in C^\infty(M)$ by

$$f(p) = \sum_{j=1}^{\infty} j \eta_j(p).$$

To verify that f is an exhaustion function, fix $c \in \mathbf{R}$ and choose a positive integer $N > c$. If $p \notin \bigcup_{j=1}^{N} \overline{V}_j$, then $\eta_j(p) = 0$ for $1 \leqslant j \leqslant N$, and thus

$$f(p) = \sum_{j=N+1}^{\infty} j \eta_j(p) \geqslant \sum_{j=N+1}^{\infty} N \eta_j(p) = N \sum_{j=N+1}^{\infty} \eta_j(p).$$

Since $\{\eta_j\}$ is a partition of unity and $\eta_j(p) = 0$ for $j \leqslant N$, we have $\sum_{j=N+1}^{\infty} \eta_j(p) = 1$, which implies $f(p) \geqslant N > c$. Equivalently, if $f(p) \leqslant c$, then $p \in \bigcup_{j=1}^{N} \overline{V}_j$. Therefore, $f^{-1}((-\infty, c])$ is a closed subset of the compact set $\bigcup_{j=1}^{N} \overline{V}_j$ and is consequently compact. □

As our final application of partitions of unity, we prove a remarkable result in differential geometry: every closed subset of a smooth manifold can be realized as the zero set of a smooth real-valued function. While we will not use this theorem directly in subsequent lectures, it provides an interesting contrast to Sard's theorem, which we will study later.

Sard's theorem states that for any smooth map between manifolds, the set of critical values has measure zero. This implies that for a generic smooth function, almost every level set is a smooth submanifold. The present theorem, however, shows that by carefully constructing our smooth function, we can force a specific closed set (which may be highly irregular) to appear as a level set. This contrast highlights the flexibility of smooth functions compared to the generic behavior described by Sard's theorem.

Theorem 3.7 (Closed Sets as Zero Sets of Smooth Functions)**.** *Let M be a smooth manifold. For any closed subset $K \subset M$, there exists a smooth nonnegative function $f : M \to \mathbf{R}$ such that $f^{-1}(0) = K$.*

Proof. Since M is a smooth manifold, it is paracompact. Consider the open set $U = M \setminus K$. We construct a locally finite open cover $\{V_j\}_{j\in J}$ of U such that each V_j is precompact and $\overline{V_j} \subset U$.

For each $j \in J$, we construct a smooth nonnegative function $\psi_j : M \to \mathbf{R}$ with the following properties:

- $\psi_j > 0$ on V_j
- $\operatorname{supp}(\psi_j) \subset U$

Such functions can be constructed using bump functions supported in coordinate charts.

Now, let $\{\eta_j\}_{j\in J}$ be a smooth partition of unity subordinate to the cover $\{V_j\}_{j\in J}$. Define the function $f : M \to \mathbf{R}$ by

$$f(p) = \sum_{j\in J} \eta_j(p)\psi_j(p).$$

We verify that f has the desired properties:

If $p \in K$, then $p \notin U$ and thus $p \notin \operatorname{supp}(\eta_j)$ for all j, so $f(p) = 0$. Conversely, if $p \notin K$, then $p \in U$ and there exists some j with $\eta_j(p) > 0$ and $\psi_j(p) > 0$, so $f(p) > 0$. Therefore, $f^{-1}(0) = K$. □

References

[1] Alexander Kupers, *Lectures on Differential Topology*, 2020. Lecture notes.

[2] John M. Lee, *Introduction to Smooth Manifolds*, 2nd ed., Graduate Texts in Mathematics, vol. 218, Springer, 2012.

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