Lecture 6

November 21, 2025

1. Applications of Partitions of Unity

Proposition 1.1 (Smooth Urysohn Lemma)**.** *Let M be a smooth manifold, and let $A, B \subset M$ be disjoint closed subsets. Then there exists a smooth function $\lambda : M \to [0,1]$ such that $\lambda|_A \equiv 0$ and $\lambda|_B \equiv 1$.*

Proof. Consider the open cover of M given by $U_1 = M \setminus A$ and $U_2 = M \setminus B$. Let $\{\eta_1, \eta_2\}$ be a smooth partition of unity subordinate to this cover. Since $\mathrm{supp}(\eta_1) \subset U_1$, we have $\eta_1|_A \equiv 0$. Similarly, as $\mathrm{supp}(\eta_2) \subset U_2$, we have $\eta_2|_B \equiv 0$. Now define $\lambda = \eta_1$. Then $\lambda|_A = 0$, and on B we have $\lambda = \eta_1 = 1 - \eta_2 \equiv 1$, which completes the proof. □

The Smooth Urysohn Lemma has an equivalent formulation that is often more convenient for applications:

Proposition 1.2 (Smooth Bump Function)**.** *Let M be a smooth manifold, $A \subset M$ a closed subset, and $U \subset M$ an open subset containing A. Then there exists a smooth function $\psi : M \to [0,1]$ such that:*

- $\psi|_A \equiv 1$
- $\mathrm{supp}(\psi) \subset U$

Proof. Apply the Smooth Urysohn Lemma to the disjoint closed sets A and $M \setminus U$. This yields a smooth function $\lambda : M \to [0,1]$ with $\lambda|_A \equiv 0$ and $\lambda|_{M\setminus U} \equiv 1$. Define $\psi = 1 - \lambda$. Then $\psi|_A \equiv 1$, and since $\lambda \equiv 1$ on $M \setminus U$, we have $\psi \equiv 0$ on $M \setminus U$, hence $\mathrm{supp}(\psi) \subset U$. □

Remark 1.3. *These two propositions are equivalent: each can be derived from the other. The Smooth Bump Function version is particularly useful for constructing local extensions and cutoff functions, while the original Urysohn formulation provides a clear separation property for disjoint closed sets.*

Our second application concerns the extension of smooth functions from closed subsets. Let M and N be smooth manifolds, and $A \subset M$ an arbitrary subset. We say that a map $F : A \to N$ is *smooth on A* if for every point $p \in A$, there exists an open neighborhood $W \subset M$ containing p and a smooth map $\widetilde{F} : W \to N$ that agrees with F on $W \cap A$.

Lemma 1.4 (Extension Lemma for Smooth Functions)**.** *Let M be a smooth manifold, $A \subset M$ a closed subset, and $f : A \to \mathbf{R}^k$ a smooth function. For any open subset $U \subset M$ containing A, there exists a smooth function $\tilde{f} : M \to \mathbf{R}^k$ such that:*

- $\tilde{f}|_A = f$
- $\mathrm{supp}(\tilde{f}) \subset U$

Proof. For each point $p \in A$, choose a neighborhood $W_p \subset U$ and a smooth function $\tilde{f}_p : W_p \to \mathbf{R}^k$ that agrees with f on $W_p \cap A$. The collection $\{W_p : p \in A\} \cup \{M \setminus A\}$ forms an open cover of M. Let $\{\eta_p : p \in A\} \cup \{\eta_0\}$ be a smooth partition of unity subordinate to this cover, with $\mathrm{supp}(\eta_p) \subset W_p$ and $\mathrm{supp}(\eta_0) \subset M \setminus A$.

Define the global function $\tilde{f} : M \to \mathbf{R}^k$ by

$$\tilde{f}(x) = \sum_{p \in A} \eta_p(x) \tilde{f}_p(x).$$

This sum is well-defined since the partition of unity is locally finite. The function $\tilde{f}$ is smooth, agrees with f on A, and has support contained in U. □

Remark 1.5. *The classical Whitney Extension Theorem provides a more refined approach, requiring only compatibility conditions on partial derivatives.*

The definition of smoothness on closed subsets is not entirely canonical. For example, on the union of the coordinate planes xy, yz, and xz in $\mathbf{R}^3$, one might define smoothness as having smooth restrictions to each plane separately, which appears weaker than requiring local smooth extensions.

Remarkably, the Whitney Extension Theorem establishes the equivalence of these two definitions: a function on this union admits local smooth extensions if and only if its restrictions to each coordinate plane are smooth.

Next, we use partitions of unity to construct a special type of smooth function. If M is a topological space, an *exhaustion function for M* is a continuous function $f : M \to \mathbf{R}$ such that for every $c \in \mathbf{R}$, the *sublevel set* $f^{-1}((-\infty, c])$ is compact. The terminology arises from the fact that as n ranges over the positive integers, the sublevel sets $f^{-1}((-\infty, n])$ form an exhaustion of M by compact sets.

Proposition 1.6 (Existence of Smooth Exhaustion Functions)**.** *Every smooth manifold admits a smooth positive exhaustion function.*

Proof. Let $\{V_j\}_{j=1}^{\infty}$ be a countable open cover of M by precompact open subsets, and let $\{\eta_j\}$ be a smooth partition of unity subordinate to this cover. Define $f \in C^{\infty}(M)$ by

$$f(p) = \sum_{j=1}^{\infty} j \eta_j(p).$$

To verify that f is an exhaustion function, fix $c \in \mathbf{R}$ and choose a positive integer $N > c$. If $p \notin \bigcup_{j=1}^{N} \overline{V}_j$, then $\eta_j(p) = 0$ for $1 \leqslant j \leqslant N$, and thus

$$f(p) = \sum_{j=N+1}^{\infty} j \eta_j(p) \geqslant \sum_{j=N+1}^{\infty} N \eta_j(p) = N \sum_{j=N+1}^{\infty} \eta_j(p).$$

Since $\{\eta_j\}$ is a partition of unity and $\eta_j(p) = 0$ for $j \leqslant N$, we have $\sum_{j=N+1}^{\infty} \eta_j(p) = 1$, which implies $f(p) \geqslant N > c$. Equivalently, if $f(p) \leqslant c$, then $p \in \bigcup_{j=1}^{N} \overline{V}_j$. Therefore, $f^{-1}((-\infty, c])$ is a closed subset of the compact set $\bigcup_{j=1}^{N} \overline{V}_j$ and is consequently compact. □

Next, we prove a remarkable result in differential geometry: every closed subset of a smooth manifold can be realized as the zero set of a smooth real-valued function.

Theorem 1.7 (Closed Sets as Zero Sets of Smooth Functions)**.** *Let M be a smooth manifold. For any closed subset $K \subset M$, there exists a smooth nonnegative function $f : M \to \mathbf{R}$ such that $f^{-1}(0) = K$.*

Proof. Since M is a smooth manifold, it is paracompact. Consider the open set $U = M \setminus K$. We construct a locally finite open cover $\{V_j\}_{j\in J}$ of U such that each V_j is precompact and $\overline{V_j} \subset U$.

For each $j \in J$, we construct a smooth nonnegative function $\psi_j : M \to \mathbf{R}$ with the following properties:

- $\psi_j > 0$ on V_j
- $\operatorname{supp}(\psi_j) \subset U$

Such functions can be constructed using bump functions supported in coordinate charts.

Now, let $\{\eta_j\}_{j\in J}$ be a smooth partition of unity subordinate to the cover $\{V_j\}_{j\in J}$. Define the function $f : M \to \mathbf{R}$ by

$$f(p) = \sum_{j\in J} \eta_j(p)\psi_j(p).$$

We verify that f has the desired properties:

If $p \in K$, then $p \notin U$ and thus $p \notin \operatorname{supp}(\eta_j)$ for all j, so $f(p) = 0$. Conversely, if $p \notin K$, then $p \in U$ and there exists some j with $\eta_j(p) > 0$ and $\psi_j(p) > 0$, so $f(p) > 0$. Therefore, $f^{-1}(0) = K$. □

Remark 1.8. *This theorem reveals that smooth functions can produce highly irregular zero sets, contrasting sharply with the generic regularity guaranteed by Sard's Theorem (to be studied in detail when discussing transversality). Specifically, Sard's Theorem ensures that for a smooth function,* almost every *level set is a smooth submanifold. Such* generic properties–*where "typical" behavior differs radically from worst-case scenarios–are ubiquitous in differential manifold theory, yet offer a novel perspective from the standpoint of classical calculus.*

2. Density of Smooth Functions

We now establish a fundamental approximation theorem: smooth functions are dense in the space of continuous functions on manifolds with respect to uniform convergence. This result underpins many constructions in geometric analysis, allowing us to approximate continuous geometric structures by smooth ones.

2.0. Convolution in Euclidean Space.

The key tool for our approximation is convolution, which provides a method to smooth out functions while preserving their essential features.

Definition 2.1 (Convolution)**.** *Let $f, g : \mathbf{R}^n \to \mathbf{R}$ be measurable functions. The* convolution *of f and g is defined as*

$$(f * g)(x) = \int_{\mathbf{R}^n} f(y)g(x-y)\,dy,$$

whenever this integral exists.

Convolution provides a powerful smoothing technique when we take g to be a bump function. This construction exemplifies the utility of the bump functions we developed earlier.

Theorem 2.2 (Convolution Approximation in $\mathbf{R}^n$)**.** *Let $f : \mathbf{R}^n \to \mathbf{R}$ be a continuous function with compact support. For any $\epsilon > 0$, there exists a smooth function $g : \mathbf{R}^n \to \mathbf{R}$ with compact support* *such that:*

$$\sup_{x\in\mathbf{R}^n} |f(x) - g(x)| < \epsilon.$$

Moreover, if f is supported in a compact set K, then g can be chosen with support contained in any given open neighborhood of K.

Proof. Let $\rho : \mathbf{R}^n \to \mathbf{R}$ be a smooth bump function with $\rho \geqslant 0$, $\mathrm{supp}(\rho) \subset B(0,1)$, and $\int_{\mathbf{R}^n} \rho(x)\,dx = 1$. For $\delta > 0$, define the mollifier $\rho_\delta(x) = \delta^{-n}\rho(x/\delta)$ and consider the convolution

$$f_\delta(x) = (f * \rho_\delta)(x) = \int_{\mathbf{R}^n} f(y)\rho_\delta(x-y)\,dy.$$

This function is smooth (by differentiation under the integral sign) and has support contained in the δ-neighborhood of $\mathrm{supp}(f)$.

The key observation is that convolution averages the values of f near each point. Since f is uniformly continuous (by compact support), for any $\epsilon > 0$ we can choose $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$. Then

$$|f(x) - f_\delta(x)| = \left|\int_{\mathbf{R}^n} [f(x) - f(y)]\rho_\delta(x-y)\,dy\right| \leqslant \int_{\mathbf{R}^n} |f(x) - f(y)|\rho_\delta(x-y)\,dy < \epsilon,$$

establishing uniform convergence. □

2.2. **Global Approximation on Manifolds.**

We now extend this result to smooth manifolds using partitions of unity. The strategy is to work locally in coordinate charts and then glue the approximations together.

Theorem 2.3 (Global Uniform Approximation on Manifolds)**.** *Let M be a smooth manifold and $f : M \to \mathbf{R}$ a continuous function. For any $\epsilon > 0$, there exists a smooth function $g : M \to \mathbf{R}$ such that:*

$$\sup_{x\in M} |f(x) - g(x)| < \epsilon.$$

That is, $C^\infty(M)$ is dense in $C^0(M)$ with respect to uniform convergence.

Proof. Since M is a smooth manifold, it is paracompact and second countable. Let $\{U_i\}_{i=1}^\infty$ be a locally finite open cover of M by precompact coordinate charts, with each $\phi_i : U_i \to \mathbf{R}^n$ a diffeomorphism onto its image. Let $\{\eta_i\}_{i=1}^\infty$ be a smooth partition of unity subordinate to this cover, with $\mathrm{supp}(\eta_i) \subset U_i$.

The precompactness condition ensures that each $\overline{U_i}$ is compact, which will be crucial for our local approximations.

For each i, define $f_i = \eta_i f$. This function is continuous and supported in U_i. Consider the pushforward $\tilde{f}_i = f_i \circ \phi_i^{-1}$ defined on $\phi_i(U_i) \subset \mathbf{R}^n$. Since U_i is precompact, $\tilde{f}_i$ has compact support in $\phi_i(U_i)$.

By the Euclidean approximation theorem, there exists a smooth function $\tilde{g}_i : \phi_i(U_i) \to \mathbf{R}$ with compact support such that

$$\sup_{y\in\phi_i(U_i)} |\tilde{f}_i(y) - \tilde{g}_i(y)| < \frac{\epsilon}{2^i}.$$

Define $g_i : U_i \to \mathbf{R}$ by $g_i = \tilde{g}_i \circ \phi_i$, and extend it to all of M by setting it to zero outside U_i.

Now define the global approximation by

$$g(x) = \sum_{i=1}^{\infty} g_i(x).$$

This sum is well-defined and smooth because the cover is locally finite–each point has a neighborhood intersecting only finitely many U_i, so the sum reduces to a finite sum locally.

For the error estimate, observe that for any $x \in M$:

$$\begin{aligned} |f(x) - g(x)| &= \left| \sum_{i=1}^{\infty} \eta_i(x) f(x) - \sum_{i=1}^{\infty} g_i(x) \right| \\ &\leq \sum_{i=1}^{\infty} \left| \eta_i(x) f(x) - g_i(x) \right|. \end{aligned}$$

Note that

$$|\eta_i(x) f(x) - g_i(x)| = |\tilde{f}_i(\phi_i(x)) - \tilde{g}_i(\phi_i(x))| < \frac{\epsilon}{2^i}.$$

Therefore,

$$|f(x) - g(x)| < \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon,$$

which holds uniformly on M. □

Remark 2.4. *Even in the familiar setting of Euclidean space $\mathbf{R}^n$, convolution alone is insufficient for global uniform approximation on non-compact domains. While convolution provides excellent local approximations, the partition of unity technique remains essential for gluing these into a global approximation with uniform control.*

Indeed, convolution is not strictly necessary for this density theorem. An elementary alternative exists: take a locally finite open cover $\{U_i\}$ of $\operatorname{supp}(f)$ with each U_i sufficiently small, and construct

$$g(x) = \sum_i f(x_i) \eta_i(x)$$

using a subordinate partition of unity $\{\eta_i\}$ and points $x_i \in U_i$. This yields uniform approximation $\sup |f(x) - g(x)| < \epsilon$.

However, convolution provides a more robust approach. Crucially, when f is C^k, the convolution approximation $f_\delta = f * \rho_\delta$ converges to f in the C^k topology.

3. Tangent Vectors on Manifolds

We now extend the concept of derivative from Euclidean spaces to smooth manifolds. In previous sections, we have studied the local and global properties of manifolds–their topology, compactness, partitions of unity, and approximation of continuous functions. Now we turn to studying the *infinitesimal* properties of manifolds, which will provide us with deeper insight into the local behavior of manifolds and smooth maps.

The key insight comes from observing how tangent vectors act in Euclidean space: given a vector v at a point $p \in \mathbf{R}^n$, we can define the directional derivative of any smooth function f in the direction v. This directional derivative operator $f \mapsto D_v f(p)$ is linear and satisfies the product rule (Leibniz rule).

This suggests that on manifolds, we can *define* tangent vectors as operators on smooth functions that capture this "directional derivative" behavior.

Definition 3.1. *Let $p \in M$. A* derivation at p *is a linear map* $v : C^\infty(M) \to \mathbf{R}$ *satisfying*

$$v(fg) = f(p)v(g) + g(p)v(f) \quad \text{for all } f, g \in C^\infty(M).$$

The tangent space T_pM *is the vector space of all derivations at p, and its elements are called* tangent vectors.

Lemma 3.2. *Let M be a smooth manifold, $p \in M$, $v \in T_pM$, and $f, g \in C^\infty(M)$.*

(1) If f is constant, then $v(f) = 0$
(2) If $f(p) = g(p) = 0$, then $v(fg) = 0$

Proof. (1) For $f \equiv 1$, we have $v(1) = v(1 \cdot 1) = 2v(1)$, so $v(1) = 0$
(2) Direct computation: $v(fg) = f(p)v(g) + g(p)v(f) = 0$

□

Proposition 3.3 (Tangent vectors act locally)**.** *If $f, g \in C^\infty(M)$ agree near $p \in M$, then $v(f) = v(g)$ for any $v \in T_pM$.*

Proof. Let $h = f - g$, which vanishes in a neighborhood U of p. Choose a bump function η supported in U with $\eta(p) = 1$. Then $\eta h \equiv 0$, so

$$0 = v(\eta h) = \eta(p)v(h) + h(p)v(\eta) = v(h),$$

hence $v(f) = v(g)$. □

Definition 3.4. *For a smooth map $F : M \to N$ and $p \in M$, the* differential $dF_p : T_pM \to T_{F(p)}N$ *is defined by*

$$dF_p(v)(f) = v(f \circ F) \quad \text{for } f \in C^\infty(N), v \in T_pM.$$

This is well-defined since $f \circ F \in C^\infty(M)$, and $dF_p(v)$ is a derivation:

$$\begin{aligned} dF_p(v)(fg) &= v((fg) \circ F) = v((f \circ F)(g \circ F)) \\ &= f(F(p))v(g \circ F) + g(F(p))v(f \circ F) \\ &= f(F(p))dF_p(v)(g) + g(F(p))dF_p(v)(f). \end{aligned}$$

Proposition 3.5 (Properties of the differential)**.** *Let M, N, P be smooth manifolds, $F : M \to N$ and $G : N \to P$ smooth maps, and $p \in M$.*

(1) $dF_p : T_pM \to T_{F(p)}N$ is linear
(2) $d(G \circ F)_p = dG_{F(p)} \circ dF_p$
(3) $d(\mathbf{1}_M)_p = \mathbf{1}_{T_pM}$
(4) If F is a diffeomorphism, then dF_p is an isomorphism with $(dF_p)^{-1} = d(F^{-1})_{F(p)}$

Proof. Properties (1) and (3) follow directly from definitions. For (2), given $v \in T_pM$ and $f \in C^\infty(P)$:

$$d(G \circ F)_p(v)(f) = v(f \circ G \circ F) = dF_p(v)(f \circ G) = dG_{F(p)}(dF_p(v))(f).$$

For (4), apply (2) and (3) to $F \circ F^{-1} = \mathbf{1}_N$ and $F^{-1} \circ F = \mathbf{1}_M$. □

References

[1] John M. Lee, *Introduction to Smooth Manifolds*, 2nd ed., Graduate Texts in Mathematics, vol. 218, Springer, 2012.

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