

Lecture 7

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1. TANGENT VECTORS ON MANIFOLDS

We now extend the concept of derivative from Euclidean spaces to smooth manifolds. In previous sections, we have studied the local and global properties of manifolds—their topology, compactness, partitions of unity, and approximation of continuous functions. Now we turn to studying the *infinitesimal* properties of manifolds, which will provide us with deeper insight into the local behavior of manifolds and smooth maps.

The key insight comes from observing how tangent vectors act in Euclidean space: given a vector v at a point $p \in \mathbf{R}^n$, we can define the directional derivative of any smooth function f in the direction v . This directional derivative operator $f \mapsto D_v f(p)$ is linear and satisfies the product rule (Leibniz rule).

This suggests that on manifolds, we can *define* tangent vectors as operators on smooth functions that capture this "directional derivative" behavior.

Definition 1.1. Let $p \in M$. A derivation at p is a linear map $v : C^\infty(M) \rightarrow \mathbf{R}$ satisfying

$$v(fg) = f(p)v(g) + g(p)v(f) \quad \text{for all } f, g \in C^\infty(M).$$

The tangent space $T_p M$ is the vector space of all derivations at p , and its elements are called tangent vectors.

Lemma 1.2. Let M be a smooth manifold, $p \in M$, $v \in T_p M$, and $f, g \in C^\infty(M)$.

- (1) If f is constant, then $v(f) = 0$
- (2) If $f(p) = g(p) = 0$, then $v(fg) = 0$

Proof. (1) For $f \equiv 1$, we have $v(1) = v(1 \cdot 1) = 2v(1)$, so $v(1) = 0$. Then $v(c) = cv(1) = 1$ for $\forall c \in \mathbf{R}$.

- (2) Direct computation: $v(fg) = f(p)v(g) + g(p)v(f) = 0$.

□

Example 1.3 (Tangent Space of Euclidean Space). We verify that the abstract definition of tangent vectors as derivations recovers the familiar notion in Euclidean space.

Let $M = \mathbf{R}^n$ and fix $p \in \mathbf{R}^n$. For $v \in \mathbf{R}^n$, define $D_v : C^\infty(\mathbf{R}^n) \rightarrow \mathbf{R}$ by:

$$D_v(f) = \nabla f(p) \cdot v.$$

This is a derivation since it is linear and satisfies the Leibniz rule:

$$D_v(fg) = g(p)D_v(f) + f(p)D_v(g).$$

Define $\Phi : \mathbf{R}^n \rightarrow T_p \mathbf{R}^n$ by $\Phi(v) = D_v$.

Injectivity: If $\Phi(v) = \Phi(w)$, then for coordinate functions x^i :

$$v^i = D_v(x^i) = D_w(x^i) = w^i,$$

so $v = w$.

Surjectivity: For $X \in T_p\mathbf{R}^n$, let $v = (X(x^1), \dots, X(x^n))$. For any $f \in C^\infty(\mathbf{R}^n)$, use the integral representation:

$$f(x) = f(p) + \sum_{i=1}^n (x^i - p^i) \int_0^1 \frac{\partial f}{\partial x^i}(p + t(x - p)) dt.$$

Define $g_i(x) = \int_0^1 \frac{\partial f}{\partial x^i}(p + t(x - p)) dt$, with $g_i(p) = \frac{\partial f}{\partial x^i}(p)$. Then:

$$X(f) = \sum_{i=1}^n g_i(p) X(x^i) = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(p) v^i = D_v(f).$$

Thus $X = D_v$, proving surjectivity.

Therefore, $T_p\mathbf{R}^n \cong \mathbf{R}^n$, with $e_i \in \mathbf{R}^n$ corresponding to $\frac{\partial}{\partial x^i}|_p \in T_p\mathbf{R}^n$.

Definition 1.4. For a smooth map $F: M \rightarrow N$ and $p \in M$, the differential $dF_p: T_pM \rightarrow T_{F(p)}N$ is defined by

$$dF_p(v)(f) = v(f \circ F) \quad \text{for } f \in C^\infty(N), v \in T_pM.$$

This is well-defined since $f \circ F \in C^\infty(M)$, and $dF_p(v)$ is a derivation:

$$\begin{aligned} dF_p(v)(fg) &= v((fg) \circ F) = v((f \circ F)(g \circ F)) \\ &= f(F(p))v(g \circ F) + g(F(p))v(f \circ F) \\ &= f(F(p))dF_p(v)(g) + g(F(p))dF_p(v)(f). \end{aligned}$$

Proposition 1.5 (Properties of the differential). Let M, N, P be smooth manifolds, $F: M \rightarrow N$ and $G: N \rightarrow P$ smooth maps, and $p \in M$.

$$(1) \quad d(G \circ F)_p = dG_{F(p)} \circ dF_p$$

$$(2) \quad d(\mathbf{1}_M)_p = \mathbf{1}_{T_pM}$$

$$(3) \quad \text{If } F \text{ is a diffeomorphism, then } dF_p \text{ is an isomorphism with } (dF_p)^{-1} = d(F^{-1})_{F(p)}$$

Proof. Properties (2) follow directly from definitions. For (1), given $v \in T_pM$ and $f \in C^\infty(P)$:

$$d(G \circ F)_p(v)(f) = v(f \circ G \circ F) = dF_p(v)(f \circ G) = dG_{F(p)}(dF_p(v))(f).$$

For (3), apply (1) and (2) to $F \circ F^{-1} = \mathbf{1}_N$ and $F^{-1} \circ F = \mathbf{1}_M$. □

2. THE LOCAL NATURE OF TANGENT SPACES

Intuitively, tangent vectors should be infinitesimal objects, and thus inherently local in nature. The following proposition confirms this intuition:

Proposition 2.1 (tangent vectors act locally). If $f, g \in C^\infty(M)$ agree near $p \in M$, then $v(f) = v(g)$ for any $v \in T_pM$.

Proof. Let $h = f - g$, which vanishes in a neighborhood U of p . Choose a bump function η supported in U with $\eta(p) = 1$. Then $\eta h \equiv 0$, so

$$0 = v(\eta h) = \eta(p)v(h) + h(p)v(\eta) = v(h),$$

hence $v(f) = v(g)$. □

This local property allows us to naturally identify tangent spaces of open submanifolds with those of the ambient manifold:

Theorem 2.2 (Tangent Space of Open Submanifolds). *Let M be a smooth manifold and $U \subset M$ an open subset. For any $p \in U$, the inclusion map $\iota : U \hookrightarrow M$ induces a natural isomorphism:*

$$d\iota_p : T_p U \xrightarrow{\cong} T_p M.$$

Proof. Define $\Phi : T_p U \rightarrow T_p M$ by $\Phi(v)(f) = v(f|_U)$ for $v \in T_p U$, $f \in C^\infty(M)$.

For injectivity: If $\Phi(v) = 0$, then for any $g \in C^\infty(U)$, extend g to $\tilde{g} \in C^\infty(M)$ agreeing with g near p . Then $v(g) = \Phi(v)(\tilde{g}) = 0$, so $v = 0$.

For surjectivity: Given $w \in T_p M$, define $v \in T_p U$ by $v(g) = w(\tilde{g})$ where $\tilde{g}$ is any smooth extension of $g \in C^\infty(U)$ to M . This is well-defined by the local nature of tangent vectors, and satisfies $\Phi(v) = w$.

Linearity and the derivation property are immediate, so Φ is an isomorphism with $\Phi = d\iota_p$. $\square$

Remark 2.3. *From our earlier example, we know that the tangent space of $\mathbf{R}^n$ is isomorphic to $\mathbf{R}^n$ hence has dimension n . Combined with this theorem, we conclude that the tangent space at any point of an n -dimensional manifold also has dimension n .*

In this course, we frequently encounter maps that are essentially identities, but writing them out explicitly can be quite cumbersome and technical. Therefore, we often introduce certain conventions to simplify the notation and make the underlying ideas more readable. Here are two such conventions:

Convention 2.4 (Extension of the Differential). *Let M be a smooth manifold and $U \subset M$ an open subset containing p . Although strictly speaking the differential $d\varphi_p$ maps $T_p U$ to $T_{\varphi(p)}\varphi(U)$, we will abuse notation and also denote by $d\varphi_p$ the composition:*

$$T_p M \cong T_p U \xrightarrow{d\varphi_p} T_{\varphi(p)}\varphi(U) \hookrightarrow T_{\varphi(p)}\mathbf{R}^n,$$

where the first isomorphism is the canonical identification $T_p U \cong T_p M$, and the last map is the natural inclusion. With this convention, we regard $d\varphi_p$ as an isomorphism from $T_p M$ to $T_{\varphi(p)}\mathbf{R}^n$.

Convention 2.5 (Coordinate Vectors). *Let $(U, \varphi = (x^1, x^2, \dots, x^n))$ be a coordinate chart on a smooth manifold M . For each point $p \in U$, we define the coordinate vectors $\frac{\partial}{\partial x^i} \Big|_p \in T_p M$ as:*

$$\frac{\partial}{\partial x^i} \Big|_p = (d\varphi_p)^{-1} \left(\frac{\partial}{\partial x^i} \Big|_{\varphi(p)} \right),$$

where the right-hand side $\frac{\partial}{\partial x^i}$ denotes the standard basis vectors of $T_{\varphi(p)}\mathbf{R}^n$.

This definition implies that for any $f \in C^\infty(M)$:

$$\frac{\partial}{\partial x^i} \Big|_p (f) = \frac{\partial (f \circ \varphi^{-1})}{\partial x^i} (\varphi(p)).$$

The coordinate vector fields form a natural basis for $T_p M$ and satisfy $\frac{\partial}{\partial x^i} (x^j) = \delta_i^j$.

That they form a basis. Since $\varphi : U \rightarrow \varphi(U)$ is a diffeomorphism, its differential $d\varphi_p : T_p M \rightarrow T_{\varphi(p)}\mathbf{R}^n$ is an isomorphism (under the identification convention above). The vectors $\frac{\partial}{\partial x^i}$ form a basis for $T_{\varphi(p)}\mathbf{R}^n$, so their preimages $\frac{\partial}{\partial x^i} \Big|_p = (d\varphi_p)^{-1} \left(\frac{\partial}{\partial x^i} \right)$ form a basis for $T_p M$. $\square$

Proposition 2.6 (Transformation Rule for Coordinate Vectors). *Under a coordinate change $(x^i) \rightarrow (y^j)$, the coordinate vectors transform according to:*

$$\frac{\partial}{\partial y^j} = \sum_{i=1}^n \frac{\partial x^i}{\partial y^j} \frac{\partial}{\partial x^i}.$$

Proof. Consider two coordinate charts $(U, \varphi = (x^1, \dots, x^n))$ and $(V, \psi = (y^1, \dots, y^n))$ with $U \cap V \neq \emptyset$. For any smooth function $f \in C^\infty(M)$ and any point $p \in U \cap V$, we have:

On one hand, by definition:

$$\left. \frac{\partial}{\partial y^j} \right|_p (f) = \frac{\partial(f \circ \psi^{-1})}{\partial y^j} (\psi(p)).$$

On the other hand, we can express $f \circ \psi^{-1}$ in terms of the x -coordinates. Note that:

$$f \circ \psi^{-1} = (f \circ \varphi^{-1}) \circ (\varphi \circ \psi^{-1}),$$

where $\varphi \circ \psi^{-1} : \psi(U \cap V) \rightarrow \varphi(U \cap V)$ is the transition map between coordinate charts.

Now apply the chain rule:

$$\frac{\partial(f \circ \psi^{-1})}{\partial y^j} = \sum_{i=1}^n \frac{\partial(f \circ \varphi^{-1})}{\partial x^i} \cdot \frac{\partial(\varphi \circ \psi^{-1})^i}{\partial y^j}.$$

Here, $(\varphi \circ \psi^{-1})^i$ is the i -th component of the transition map, which is exactly the function that gives x^i in terms of the y -coordinates. Therefore:

$$\frac{\partial(\varphi \circ \psi^{-1})^i}{\partial y^j} = \frac{\partial x^i}{\partial y^j}.$$

The Jacobian matrix $\left(\frac{\partial x^i}{\partial y^j} \right)$ appears naturally as the derivative of the coordinate transition map. Substituting back, we get:

$$\frac{\partial(f \circ \psi^{-1})}{\partial y^j} = \sum_{i=1}^n \frac{\partial x^i}{\partial y^j} \frac{\partial(f \circ \varphi^{-1})}{\partial x^i}.$$

Evaluating at p and recognizing that $\frac{\partial(f \circ \varphi^{-1})}{\partial x^i} (\varphi(p)) = \left. \frac{\partial}{\partial x^i} \right|_p (f)$, we obtain:

$$\left. \frac{\partial}{\partial y^j} \right|_p (f) = \sum_{i=1}^n \frac{\partial x^i}{\partial y^j} (p) \left. \frac{\partial}{\partial x^i} \right|_p (f).$$

Since this holds for all $f \in C^\infty(M)$, we conclude:

$$\frac{\partial}{\partial y^j} = \sum_{i=1}^n \frac{\partial x^i}{\partial y^j} \frac{\partial}{\partial x^i}.$$

□

Remark 2.7 (Physicist's Definition of Tangent Vectors). *Physicists define tangent vectors operationally by their transformation properties under coordinate changes. A tangent vector is characterized by its components transforming contravariantly:*

$$V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu$$

This matches our geometric construction: for $v = v^i \frac{\partial}{\partial x^i} = w^j \frac{\partial}{\partial y^j}$, the components transform as:

$$w^j = \sum_{i=1}^n \frac{\partial y^j}{\partial x^i} v^i$$

while the basis vectors transform complementarily:

$$\frac{\partial}{\partial y^j} = \sum_{i=1}^n \frac{\partial x^i}{\partial y^j} \frac{\partial}{\partial x^i}$$

This inverse relationship between component and basis transformations defines contravariant vectors in physics.

3. THE TANGENT BUNDLE

3.0. From Coordinate Transformations to the Tangent Bundle. We have understood tangent spaces in local coordinate charts and the transformation rules between different charts. When describing an object through local coordinate charts, our previous experience with manifolds tells us that there should be a manifold structure underlying it. This leads naturally to the following object:

Definition 3.1 (Tangent Bundle). *Let M be a smooth n -manifold. The tangent bundle of M is defined as the disjoint union of all tangent spaces:*

$$TM = \bigsqcup_{p \in M} T_p M = \{(p, v) : p \in M, v \in T_p M\}.$$

We define the projection map $\pi : TM \rightarrow M$ by $\pi(p, v) = p$.

The contravariant transformation rule for coordinate vector fields provides the transition maps for constructing the manifold structure on TM .

To make TM into a smooth manifold, we construct an atlas from the coordinate charts on M :

Theorem 3.2 (Manifold Structure on TM). *The tangent bundle TM has a natural structure of a smooth $2n$ -dimensional manifold.*

Proof. For each coordinate chart $(U, \varphi = (x^1, \dots, x^n))$ on M , define a chart $(\pi^{-1}(U), \tilde{\varphi})$ on TM by:

$$\tilde{\varphi}(p, v) = (x^1(p), \dots, x^n(p), v^1, \dots, v^n) \in \mathbf{R}^{2n},$$

where $v = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i} \big|_p$.

Given two such charts $(\pi^{-1}(U), \tilde{\varphi})$ and $(\pi^{-1}(V), \tilde{\psi})$ with $U \cap V \neq \emptyset$, the transition function $\tilde{\psi} \circ \tilde{\varphi}^{-1}$ acts as:

$$(y^1, \dots, y^n, w^1, \dots, w^n) = \left(\psi \circ \varphi^{-1}(x), \sum_{j=1}^n \frac{\partial y^i}{\partial x^j}(x) v^j \right),$$

which is smooth because $\psi \circ \varphi^{-1}$ is smooth and the partial derivatives depend smoothly on x .

These charts give TM the structure of a smooth $2n$ -dimensional manifold. $\square$

Remark 3.3 (The Tangent Bundle as a Vector Bundle). *The tangent bundle TM possesses a special structure beyond being merely a smooth manifold. For each coordinate chart (U, φ) on M , the induced chart $(\pi^{-1}(U), \tilde{\varphi})$ on TM satisfies compatibility with projections ($\text{pr}_1 \circ \tilde{\varphi} = \varphi \circ \pi$), linear fiber structure ($\tilde{\varphi}|_{T_p M}$ is a linear isomorphism), and smooth linear transitions. This defines a vector bundle—a smooth family of vector spaces parameterized by M .*

This structure enables the geometric extension of linear algebra constructions such as dual bundles, tensor bundles, Whitney sums, and exterior algebras. Each construction yields new manifolds from existing ones.

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