Lecture 8

November 21, 2025

1. Alternative Definition: Tangent Vectors as Curve Equivalence Classes

There is another intuitive and geometric approach to defining tangent vectors, which views them as equivalence classes of curves through a point. This definition essentially captures the classical notion of velocity vectors–the instantaneous direction and speed of motion along a curve.

Definition 1.1 (Tangent Vectors as Curve Equivalence Classes)**.** *Let M be a smooth manifold and $p \in M$. Consider the set of all smooth curves $\gamma : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = p$. Define an equivalence relation on these curves by:*

$$\gamma_1 \sim \gamma_2 \iff \textit{for some coordinate chart } (U, \varphi) \textit{ around } p\textit{, } (\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0).$$

A tangent vector *at p is defined as an equivalence class $[\gamma]$ under this relation. The* tangent space *$T_p^{curve} M$ is the set of all such equivalence classes.*

Remark 1.2. *The equivalence relation is well-defined because if two curves have the same derivative in one coordinate chart, they have the same derivative in any other chart by the chain rule. This makes the definition chart-independent.*

We can define a vector space structure on $T_p^{\text{curve}} M$ using coordinates: given $[\gamma_1], [\gamma_2] \in T_p^{\text{curve}} M$ and $a \in \mathbf{R}$, define

$$[\gamma_1] + [\gamma_2] = [\gamma] \quad \text{where } \varphi \circ \gamma(t) = \varphi \circ \gamma_1(t) + \varphi \circ \gamma_2(t) - \varphi(p),$$
$$a[\gamma_1] = [\gamma_a] \quad \text{where } \varphi \circ \gamma_a(t) = \varphi(p) + a(\varphi \circ \gamma_1(t) - \varphi(p)).$$

[Think this through why this make sense, i.e. why it is independent of the choice of representative of curve and of the chart.]

Now we establish the equivalence with our previous definition:

Theorem 1.3 (Equivalence of Definitions)**.** *There is a natural isomorphism between the tangent space defined via derivations and the tangent space defined via curve equivalence classes:*

$$T_p M \cong T_p^{curve} M.$$

Proof. We construct mutually inverse maps between the two spaces.

From curves to derivations: Define $\Phi : T_p^{\text{curve}} M \to T_p M$ by

$$\Phi([\gamma])(f) = (f \circ \gamma)'(0) \quad \text{for } f \in C^\infty(M).$$

This is well-defined and preserves the derivation structure.

From derivations to curves: Define $\Psi : T_p M \to T_p^{\text{curve}} M$ as follows. Given $v \in T_p M$, choose a coordinate chart (U, φ) with $\varphi(p) = 0$ and let $v^i = v(x^i)$. Define the curve $\gamma_v(t) = \varphi^{-1}(tv^1, \dots, tv^n)$ and set $\Psi(v) = [\gamma_v]$. This is independent of the coordinate choice.

A straightforward verification shows that $\Phi \circ \Psi = \mathrm{id}$ and $\Psi \circ \Phi = \mathrm{id}$, establishing the isomorphism. $\square$

2. Parameterized Contraction Principle and the Inverse Function Theorem

Theorem 2.1 (Parameterized Contraction Mapping Principle)**.** *Let X be a complete metric space and Y a metric space. Let $\Phi : X \times Y \to X$ be a continuous map such that for some $0 \leqslant \rho < 1$ and all $y \in Y$:*

$$d(\Phi(x_1, y), \Phi(x_2, y)) \leqslant \rho d(x_1, x_2) \quad \text{for all } x_1, x_2 \in X.$$

Then for each $y \in Y$ there exists a unique $x \in X$ with $\Phi(x, y) = x$. If we denote this fixed point by $x = S(y)$, then the map $S : Y \to X$ is continuous.

Proof. Fix $x_0 \in X$ and define the sequence $\{x_n(y)\}$ by $x_0(y) = x_0$, $x_{n+1}(y) = \Phi(x_n(y), y)$.

By the contraction property, $d(x_{n+1}(y), x_n(y)) \leqslant \rho^n d(x_1(y), x_0(y))$. For $m > n$, the triangle inequality gives:

$$d(x_m(y), x_n(y)) \leqslant \sum_{k=n}^{m-1} \rho^k d(x_1(y), x_0(y)) \leqslant \frac{\rho^n}{1-\rho} d(x_1(y), x_0(y))$$

Since $\rho < 1$, $\{x_n(y)\}$ is Cauchy and converges to some $x(y) \in X$, which satisfies $\Phi(x(y), y) = x(y)$ by continuity. Uniqueness follows from the contraction property.

For continuity of $S(y) = x(y)$, given $\epsilon > 0$, choose N so that $d(x_N(y), S(y)) < \epsilon/3$ for all y. By continuity of Φ, $x_N(y)$ is continuous in y, so there exists $\delta > 0$ such that $d(y, y_0) < \delta$ implies $d(x_N(y), x_N(y_0)) < \epsilon/3$. Then:

$$d(S(y), S(y_0)) \leqslant d(S(y), x_N(y)) + d(x_N(y), x_N(y_0)) + d(x_N(y_0), S(y_0)) < \epsilon$$

$\square$

Let $p \in M$ and let $x^1, \ldots, x^m$ be differentiable functions on a neighborhood U of p. Let $\varphi(q) = (x^1(q), \ldots, x^m(q))$ for $q \in U$. We say that $\{x^i\}_{1 \leqslant i \leqslant m}$ defines a coordinate system at p if there exists an open neighborhood U' of p, contained in U, such that $(U', \varphi|_{U'}, m)$ is a chart on M.

Theorem 2.2. *The following are equivalent:*

(1) $\{x^i\}$ defines a coordinate system at p.
(2) $\frac{\partial}{\partial x^i}$ form a basis of T_pM.

Theorem 2.2 is a consequence of the following more general theorem:

Theorem 2.3. *Let M and N be manifolds, $p \in M$ and $q \in N$, and let $F : M \to N$ be a smooth map such that $F(p) = q$. Then the following are equivalent:*

(1) F is a local diffeomrophism at p.
(2) $d_pF : T_pM \to T_qN$ is an isomorphism.

Theorem 2.4 (Inverse Function Theorem on Euclidean Space)**.** *Let $P : U \subset \mathbf{R}^m \to \mathbf{R}^m$ be a C^∞ map. Suppose that at some point $a \in U$, the derivative $DP(a) : \mathbf{R}^m \to \mathbf{R}^m$ is an invertible linear map. Then there exist neighborhoods $\tilde{U}$ of a and $\tilde{V}$ of $b = P(a)$ such that P restricts to a C^∞-diffeomorphism from $\tilde{U}$ onto $\tilde{V}$.*

Proof. Assume $a = 0$, $P(a) = 0$, and by composing with $[DP(0)]^{-1}$, assume $DP(0) = I$.

Define $\Phi(x, y) = x - P(x) + y$. Then $P(x) = y$ if and only if $\Phi(x, y) = x$.

Since $DP(0) = I$ and P is C^1, choose $\varepsilon > 0$ such that for $\|x_1\|, \|x_2\| \leqslant \varepsilon$:

$$\|P(x_1) - P(x_2) - (x_1 - x_2)\| \leqslant \tfrac{1}{2}\|x_1 - x_2\|$$

Then $\|\Phi(x_1, y) - \Phi(x_2, y)\| \leqslant \frac{1}{2}\|x_1 - x_2\|$, so $\Phi(\cdot, y)$ is a contraction.

Let $X = \{x : \|x\| \leqslant \varepsilon\}$, $Y = \{y : \|y\| \leqslant \varepsilon/2\}$. For $x \in X$, $y \in Y$:

$$\|\Phi(x, y)\| \leqslant \|x - P(x)\| + \|y\| \leqslant \tfrac{1}{2}\|x\| + \|y\| \leqslant \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon$$

so $\Phi(X \times Y) \subset X$.

By the contraction mapping theorem, for each $y \in Y$ there exists a unique $x \in X$ with $\Phi(x, y) = x$, i.e., $P(x) = y$. Denote this by $x = S(y)$. Then $S : Y \to X$ is continuous and $P \circ S = \mathrm{id}_Y$.

To show S is C^1:

Let $y_0 \in Y$ and $x_0 = S(y_0)$. Since $DP(x_0)$ is invertible (by continuity of DP and $DP(0) = I$), consider the difference quotient:

$$\frac{S(y) - S(y_0) - [DP(x_0)]^{-1}(y - y_0)}{\|y - y_0\|}$$

Using $P(S(y)) = y$ and $P(S(y_0)) = y_0$, we have:

$$\begin{aligned}
&S(y) - S(y_0) - [DP(x_0)]^{-1}(y - y_0)\\
&= S(y) - S(y_0) - [DP(x_0)]^{-1}(P(S(y)) - P(S(y_0)))\\
&= [DP(x_0)]^{-1}\big(DP(x_0)(S(y) - S(y_0)) - (P(S(y)) - P(S(y_0)))\big)
\end{aligned}$$

By the differentiability of P at x_0:

$$P(S(y)) - P(S(y_0)) = DP(x_0)(S(y) - S(y_0)) + o(\|S(y) - S(y_0)\|)$$

Thus:

$$\frac{\|S(y) - S(y_0) - [DP(x_0)]^{-1}(y - y_0)\|}{\|y - y_0\|} \leqslant \|[DP(x_0)]^{-1}\| \cdot \frac{o(\|S(y) - S(y_0)\|)}{\|y - y_0\|}$$

Since S is continuous and $DP(x_0)$ is invertible, $\|S(y) - S(y_0)\|/\|y - y_0\|$ is bounded. Therefore the right-hand side tends to 0 as $y \to y_0$, proving that S is differentiable at y_0 with $DS(y_0) = [DP(x_0)]^{-1}$.

The continuity of DS follows from the continuity of DP and S.

Higher regularity: The C^k case for $k \geqslant 2$ follows by induction. We already have the derivative formula $DS(y) = [DP(S(y))]^{-1}$. If P is C^k, then DP is C^{k-1}, and since matrix inversion is smooth, the composition $[DP(S(y))]^{-1}$ is C^{k-1} by the chain rule and the induction hypothesis that S is C^{k-1}. Thus DS is C^{k-1}, meaning S is C^k. □

Remark 2.5. *The contraction mapping principle is not strictly necessary here, as the inverse mapping is clearly Lipschitz continuous. Nevertheless, it is a convenient tool in other contexts, such as proving continuous dependence on initial conditions for ordinary differential equations.*

3. Immersions, Submersions, and Subimmersions

Let M and N be smooth manifolds, $p \in M$ and $q \in N$, and let $f : M \to N$ be a smooth map such that $f(p) = q$. Let $m = \dim M$ and $n = \dim N$.

Definition 3.1. *Let $\tilde{M}$ and $\tilde{N}$ be smooth manifolds, $\tilde{p} \in \tilde{M}$ and $\tilde{q} \in \tilde{N}$, and let $\tilde{f} : \tilde{M} \to \tilde{N}$ be a smooth map such that $\tilde{f}(\tilde{p}) = \tilde{q}$. Then (M, N, p, q, f) looks locally like $(\tilde{M}, \tilde{N}, \tilde{p}, \tilde{q}, \tilde{f})$ if there exist* *open neighborhoods U of p, V of q, $\tilde{U}$ of $\tilde{p}$, $\tilde{V}$ of $\tilde{q}$ and diffeomorphisms $g : U \to \tilde{U}$ and $h : V \to \tilde{V}$ such that:*

(1) $f(U) \subset V$ *and* $\tilde{f}(\tilde{U}) \subset \tilde{V}$,
(2) $g(p) = \tilde{p}$ *and* $h(q) = \tilde{q}$,
(3) *The following diagram commutes:*

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "g"'] & V \arrow[d, "h"] \\ \tilde{U} \arrow[r, "\tilde{f}"] & \tilde{V}. \end{tikzcd}$$

Remark 3.2. *We shall apply this definition mainly when $\bar{M}$ is a vector space $\mathbf{R}^m$, $\bar{N}$ is a vector space $\mathbf{R}^n$ and $\bar{f}$ is a linear map. In this case, we will take $\bar{p} = 0$, $\bar{q} = 0$ without explicit mention.*

3.2. **Immersions.**

Notation 3.3. *We use both $d_p f$ and $T_p f$ to denote the differential (or the tangent map) of a smooth map f at point p. Both notations are standard in the literature and will be used interchangeably in these notes.*

Theorem 3.4. *The following are equivalent:*

(1) $T_p f$ *is injective.*
(2) *There exist open neighborhoods U of p, V of q, and W of 0 (in $\mathbf{R}^{n-m}$) and a diffeomorphism $\psi : V \to U \times W$ such that:*
 (a) $f(U) \subset V$,
 (b) *If ι denotes the inclusion $U \to U \times \{0\} \subset U \times W$, then the following diagram commutes:*

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[dr, "\iota"'] & V \arrow[d, "\psi"] \\ & U \times W \end{tikzcd}$$

(3) (M, N, p, q, f) *looks locally like a linear injection* $\bar{f} : \mathbf{R}^m \to \mathbf{R}^n$.
(4) *There exist local coordinates $\{x^i\}$ at p and $\{y^j\}$ at q such that $x^i = y^i \circ f$ for $1 \leqslant i \leqslant m$ and $0 = y^j \circ f$ for $m+1 \leqslant j \leqslant n$.*
(5) *There exist open neighborhoods U of p and V of q, and a smooth map $\sigma : V \to U$ such that $f(U) \subset V$ and $\sigma \circ f = \mathbf{1}_U$.*

Proof. The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1) are elementary.

We show (1) $\Rightarrow$ (2). Since the question is local, we may assume that the following conditions are satisfied:

a. N is an open subset of $\mathbf{R}^n$,
b. $f(p) = 0$ and $\operatorname{Im} T_p f = \mathbf{R}^m \times \{0\} \subset \mathbf{R}^m \times \mathbf{R}^{n-m} = \mathbf{R}^n$.

Let W be $\{0\} \times \mathbf{R}^{n-m} \subset \mathbf{R}^n$. Define $f' : M \times W \to N$ by $f'(p, w) = f(p) + w$. Then by the inverse function theorem, f' is a local diffeomorphism at $(p, 0)$. Hence, by shrinking M, N and W, we may assume that f' is a diffeomorphism. Then the inverse ψ of f' satisfies the condition of (2).

Let us verify the commutativity $\psi \circ f = \iota$ in detail. By the definition of f' and the fact that f' is a diffeomorphism between $U \times W$ and V, we know that $f : U \to V$ is an injection. If $y \in f(U)$, then $y \in f'(U \times \{0\})$ and $f'^{-1}(y) = (f^{-1}(y), 0)$ by the definition of f'. Therefore, $\psi(f(x)) = (x, 0)$ for all $x \in U$, which means $\psi \circ f = \iota$. □

Definition 3.5. *A smooth map f satisfying the equivalent conditions of the preceding theorem at p is called an immersion at p. A smooth map f which is an immersion at all $p \in M$ is called an immersion.*

Example 3.6 (Inclusion into product manifold)**.** *Let M and N be smooth manifolds, and fix a point $q_0 \in N$. Consider the inclusion map $\iota : M \to M \times N$ defined by*

$$\iota(p) = (p, q_0).$$

This map is an immersion. To see this, consider the projection map $\pi : M \times N \to M$ defined by $\pi(p, q) = p$. Then we have

$$\pi \circ \iota = \mathbf{1}_M.$$

This means that π is a smooth left inverse for ι on the entire manifold M. By condition (5) of the theorem, with $U = M$ and $V = M \times N$, it follows immediately that ι is an immersion at every point $p \in M$.

Example 3.7 (Dense immersion of $\mathbf{R}^1$ into T^2)**.** *Consider the 2-torus $T^2 = S^1 \times S^1$ and let α be a real number. Define the map $f : \mathbf{R}^1 \to T^2$ by*

$$f(t) = (e^{2\pi i t}, e^{2\pi i \alpha t}).$$

This map is an immersion. To see this, consider the natural angular coordinates on T^2. Let (θ, ϕ) be coordinates on the universal cover $\mathbf{R}^2$ of T^2, with the identification $(\theta, \phi) \sim (\theta + m, \phi + n)$ for $m, n \in \mathbf{Z}$.

In these coordinates, the map f becomes

$$f(t) = (t, \alpha t) \mod \mathbf{Z}^2.$$

Now, at any point $t_0 \in \mathbf{R}$, we can choose a neighborhood U of t_0 small enough so that the projection $\mathbf{R}^2 \to T^2$ restricts to a diffeomorphism on $(t_0 - \epsilon, t_0 + \epsilon) \times (\alpha t_0 - \epsilon, \alpha t_0 + \epsilon)$ for some $\epsilon > 0$. In this local coordinate chart, the map is simply

$$f(t) = (t, \alpha t),$$

and its derivative is

$$T_{t_0} f = \begin{pmatrix} 1 \\ \alpha \end{pmatrix},$$

which has full rank 1. Therefore, f is an immersion.

Note that when α is irrational, the image $f(\mathbf{R}^1)$ is dense in T^2. This follows from the fact that the irrational rotation on the circle is minimal (every orbit is dense). More precisely, for any open set $U \subset T^2$, there exists some $t \in \mathbf{R}$ such that $f(t) \in U$.

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