Lecture 9

November 21, 2025

1. Submersions

Theorem 1.1. *The following are equivalent:*

(1) $T_p f$ *is surjective.*
(2) *There exist open neighborhoods* U *of* p, V *of* q *and* W *of* 0 *(in* $\mathbf{R}^{m-n}$*) and a diffeomorphism* $\psi : U \to V \times W$ *such that:*
   (a) $f(U) = V$,
   (b) *If* π *denotes the projection* $V \times W \to V$*, then the following diagram commutes:*

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\psi"'] & V \\ V \times W \arrow[ur, "\pi"'] & \end{tikzcd}$$

(3) (M, N, p, q, f) *looks locally like a linear surjection* $\bar{f} : \mathbf{R}^m \to \mathbf{R}^n$.
(4) *There exist local coordinates* $\{x^i\}$ *at* p *and* $\{y^i\}$ *at* q *such that* $x^i = y^i \circ f$ *for* $1 \leqslant i \leqslant n$.
(5) *There exist open neighborhoods* U *of* p *and* V *of* q *and a smooth map* $\sigma : V \to U$ *such that* $f(U) \subset V$ *and* $f \circ \sigma = \mathbf{1}_V$.

Proof. The proof is similar to the proof of the corresponding theorem on immersions.

We only need to show (1) $\Rightarrow$ (2). Since the question is local, we may assume that the following conditions are satisfied:

a. M is an open subset of $\mathbf{R}^m$ and N is an open subset of $\mathbf{R}^n$,
b. $p = 0$ and $\operatorname{Ker} T_p f = \{0\} \times \mathbf{R}^{m-n} \subset \mathbf{R}^n \times \mathbf{R}^{m-n} = \mathbf{R}^m$.

Let W be $\{0\} \times \mathbf{R}^{m-n} \subset \mathbf{R}^m$. Define $\psi : M \to N \times W$ by $\psi(x) = (f(x), (x^{n+1}, \ldots, x^m))$. Then by the inverse function theorem, ψ is a local diffeomorphism at $p = 0$. Hence, by shrinking M, N and W, we may assume that ψ is a diffeomorphism. $\square$

Definition 1.2. *A smooth map* f *satisfying the equivalent conditions of the preceding theorem at* p *is called a submersion at* p*. A smooth map* f *which is a submersion at all* $p \in M$ *is called a submersion.*

Example 1.3 (Projection from product manifold)**.** *Let* M *and* N *be smooth manifolds. Consider the projection map* $\pi : M \times N \to M$ *defined by* $\pi(p, q) = p$.

This map is a submersion. To see this, note that for any point $(p, q) \in M \times N$*, the derivative* $T_{(p,q)}\pi$ *is surjective. Indeed, we can use condition (5) of the theorem by taking the section* $\sigma : M \to M \times N$ *defined by* $\sigma(p) = (p, q_0)$ *for any fixed* $q_0 \in N$*. Then* $\pi \circ \sigma = \mathbf{1}_M$*, so* π *is a submersion.*

Example 1.4 (Projection to projective space)**.** *Consider the map $\pi : \mathbf{R}^n \setminus \{0\} \to \mathbf{RP}^{n-1}$ defined by $\pi(x) = [x]$, where $[x]$ denotes the line through the origin containing x.*

This map is a submersion. To see this, consider the coordinate chart $U_1 = \{[x_1 : \cdots : x_n] \mid x_1 \neq 0\}$ on $\mathbf{RP}^{n-1}$. On U_1, we have coordinates $(y_2, \ldots, y_n)$ with $y_j = x_j/x_1$ for $j = 2, \ldots, n$.

Define a smooth section $\sigma : U_1 \to \mathbf{R}^n \setminus \{0\}$ by

$$\sigma(y_2, \ldots, y_n) = (1, y_2, \ldots, y_n).$$

Then we have

$$\pi \circ \sigma(y_2, \ldots, y_n) = [1 : y_2 : \cdots : y_n] = (y_2, \ldots, y_n),$$

where the last equality holds because in the coordinate chart U_1, the point $[1 : y_2 : \cdots : y_n]$ corresponds exactly to $(y_2, \ldots, y_n)$.

Therefore, $\pi \circ \sigma = \mathbf{1}_{U_1}$, and by condition (5) of the theorem, π is a submersion on $\pi^{-1}(U_1)$. Since $\mathbf{RP}^{n-1}$ is covered by similar coordinate charts $U_i = \{[x_1 : \cdots : x_n] \mid x_i \neq 0\}$, and on each such chart we can define an analogous section, it follows that π is a submersion on all of $\mathbf{R}^n \setminus \{0\}$.

More generally, if we consider the space of full-rank $m \times n$ matrices ($m \geqslant n$) and the projection to the Grassmannian $\mathrm{Gr}(n, m)$ which sends a matrix to its column space, this map is also a submersion, and a similar local section argument can be used to prove it.

1.4. **Remarks.**

(1) Sometimes the phrase "f has maximal rank" (meaning $T_p f$ is injective if $m \leqslant n$ and $T_p f$ is surjective if $m \geqslant n$) is used to include both concepts.

(2) An *embedding* is a smooth map f such that:
   (a) f is an immersion,
   (b) $f : M \to f(M)$ is a homeomorphism.

2. Subimmersions

Definition 2.1. *f is a subimmersion at p if the following equivalent conditions are satisfied:*

(1) f looks locally like a composition $\bar{M} \xrightarrow{s} \bar{Z} \xrightarrow{\iota} \bar{N}$ where s is a submersion and ι is an immersion.

(2) f looks locally like a linear map $\bar{f} : \mathbf{R}^m \to \mathbf{R}^n$.

A smooth map f which is a subimmersion at all $p \in M$ is called a subimmersion.

Remark 2.2. *(1) The set of points $p \in M$ where a smooth map $f : M \to N$ is an immersion (resp. a submersion, a subimmersion) is open in M.*

(2) The composition of two immersions (resp. submersions) is an immersion (resp. a submersion). The analogous statement for subimmersions is false.

(3) In the definition of subimmersion, the order (submersion then immersion) ensures stable rank behavior. For a linear map $\bar{f} : \mathbf{R}^m \to \mathbf{R}^n$ factoring as $\mathbf{R}^m \xrightarrow{s} \mathbf{R}^r \xrightarrow{\iota} \mathbf{R}^n$ with s surjective and ι injective, the rank is r. Reversing the order (immersion then submersion) may yield different ranks under local coordinates, as seen in the example $x \mapsto (x, 0)$ composed with different projections.

Theorem 2.3. *The following are equivalent:*

(1) f is a subimmersion at p.

(2) $\operatorname{rank} T_{p'} f$ is constant for $p' \in U$ and U some neighborhood of p.

Proof. (1) $\Rightarrow$ (2): Clear.

(2) $\Rightarrow$ (1): Let $r = \dim \operatorname{Im} T_p f$. Then, since the question is local, we may assume the following conditions are satisfied:

a. $N = V_1 \times V_2$ is open in $\mathbf{R}^r \times \mathbf{R}^{n-r}$,
b. $f(p) = 0$ and $\operatorname{Im} T_p f = \mathbf{R}^r \times \{0\}$.

Let $\pi : \mathbf{R}^r \times \mathbf{R}^{n-r} \to \mathbf{R}^r$ be the projection on the first factor. Then $\pi \circ f$ is a submersion. Hence we may further assume that:

a. $M = V_1 \times U_2$ is open in $\mathbf{R}^r \times \mathbf{R}^{m-r}$,
b. $\pi \circ f : V_1 \times U_2 \to V_1$ is the projection on the first factor.

The map f then has the following form:

$$f(x_1, x_2) = (x_1, \psi(x_1, x_2))$$

Finally, since $T_{p'} f$ has locally constant rank, we may assume that the rank of $T_{p'} f$ is in fact constant on $V_1 \times U_2$ (rank $= r$).

We claim that ψ must be independent of x_2 in a neighborhood of zero. Indeed, $D_2\psi(x_1, x_2) = 0$ since otherwise f would have rank greater than r at (x_1, x_2). Our claim is therefore a consequence of the following lemma:

Lemma 2.4. *Let $f : U \times V \to \mathbf{R}$ be a smooth function such that $D_2 f$ is identically 0. Then f is locally independent of the V coordinate.*

We conclude the proof of the theorem by noting that f may now be written as $V_1 \times U_2 \to V_1 \to V_1 \times V_2$ where the first map is pr_1 and the second is $\mathbf{1}_{V_1} \times \psi$. The first map is a submersion and the second is an immersion. □

Remark 2.5. *In differential geometry, the concept of subimmersion naturally arises in several important contexts. In this course, we will encounter one particularly fundamental example:*

Vector bundle theory*: Consider a vector bundle homomorphism $\Phi : E \to F$ between two vector bundles over the same base manifold M. If the rank of the linear map $\Phi_p : E_p \to F_p$ induced on each fiber is constant, then the bundle homomorphism Φ (when viewed as a map between smooth manifolds) is a subimmersion.*

This example is of fundamental importance. The subimmersion property ensures that we can define the ***kernel bundle*** $\ker(\Phi)$, *the* ***image bundle*** $\mathrm{image}(\Phi)$, *and the* ***cokernel bundle*** $\mathrm{coker}(\Phi)$. *These constructions form the foundation of linear algebra over vector bundles and play crucial roles in many areas of differential geometry.*

Another context *(which we may mention if time permits) is* ***Lie group theory****. For Lie groups (manifolds with group structure), every continuous Lie group homomorphism $\phi : G \to H$ is automatically smooth and a subimmersion. While Lie groups form an important class of examples, their theory is rich enough to warrant a separate course.*

In this course, the primary (and likely only) concrete examples of subimmersions we will encounter in detail will be constant-rank vector bundle homomorphisms.

3. Submanifold

Suppose M is a subspace of N (with the induced topology) and let

$$\iota : M \to N$$

be the inclusion map. We say M is **locally a submanifold of** N **at** $p \in M$ if either of the following equivalent conditions holds:

(1) There exists an open neighborhood U of p in M, a chart (V, ψ) of N about p, and a linear subspace $E \subset \mathbf{R}^n$ such that $U \subset V$ and $\psi(U) = E \cap \psi(V)$.
(2) There exist local coordinates $x^1, \ldots, x^n$ defined near p in N and an integer $0 \leqslant k \leqslant n$ such that M is locally given by $x^1 = \cdots = x^k = 0$.

If M is locally a submanifold of N at every $p \in M$, we say M is a **submanifold of** N.

We now justify this terminology.

Theorem 3.1. *The following are equivalent:*

(1) There exists a smooth structure on M such that ι is an immersion.
(2) M is a submanifold of N.

Proof. ($\Rightarrow$) This follows from the local coordinate characterization (item 4) of immersions.

($\Leftarrow$) Choose an open cover $\{U_i\}_{i\in I}$ of M such that for each $i \in I$, there exists a chart (V_i, ψ_i) of N and a linear subspace $E_i \subset \mathbf{R}^n$ satisfying $U_i \subset V_i$ and $\psi_i(U_i) = E_i \cap \psi_i(V_i)$.

Each U_i inherits a smooth structure making $\iota|_{U_i}$ an immersion. On overlaps $U_i \cap U_j$, these structures agree: if we take coordinates from (V_i, ψ_i) and (V_j, ψ_j), the transition map $\psi_j \circ \psi_i^{-1}$ is smooth on N and preserves the subspace structure defining M, hence $\{U_i, \varphi_i = \psi_i|_{U_i}\}_{i\in I}$ define a global smooth structure on M for which ι is an immersion.

Alternatively, once we prove the uniqueness theorem later, we could argue that both smooth structures make $\iota|_{U_i \cap U_j}$ an immersion, hence they must coincide by uniqueness. □

Now we address the uniqueness of this smooth manifold structure:

Theorem 3.2. *Let M be a topological space, N a smooth manifold, and $f : M \to N$ a continuous map. If there exists a smooth structure on M making f an immersion, then this smooth structure is unique.*

Proof. Suppose $\mathscr{A}$ is a smooth structure on M such that $f : (M, \mathscr{A}) \to N$ is an immersion. We claim that for any manifold P, a map $g : P \to (M, \mathscr{A})$ is smooth if and only if $f \circ g : P \to N$ is smooth.

The "only if" direction is clear. For the "if" direction, we work locally. Let $r \in P$, $p = g(r)$, and $q = f(p)$. Since f is an immersion, there exist neighborhoods U of p in M and V of q in N, and a smooth map $h : V \to U$ such that $h \circ f|_U = \mathbf{1}_U$.

By continuity of f, we can find a neighborhood W of r such that $g(W) \subset U$. Then on W we have:

$$g|_W = \mathrm{id}_U \circ g|_W = h \circ f \circ g|_W.$$

Since $f \circ g$ is smooth by assumption, $g|_W = h \circ (f \circ g|_W)$ is smooth. As smoothness is local, g is smooth everywhere.

Now, if $\mathscr{A}'$ is another smooth structure on M making f an immersion, then for any manifold P and map $g : P \to M$:

$$g \in C^\infty(P, (M, \mathscr{A}')) \iff f \circ g \in C^\infty(P, N) \iff g \in C^\infty(P, (M, \mathscr{A})).$$

By Theorem 3.4 in Lecture 4, this implies $\mathscr{A} = \mathscr{A}'$. □

Example 3.3. *Recall from Example 1.7 in Lecture 3 the two smooth structures on S^1: one defined by four charts via coordinate projections, and another by two charts via stereographic projection.*

One can verify that both atlases give S^1 the structure of a submanifold of $\mathbf{R}^2$, hence by the theorem they must be compatible.

Example 3.4. *Let M and N be topological manifolds with $f : M \to N$ a homeomorphism. If N has a smooth structure, then there exists a unique smooth structure on M making f a diffeomorphism.*

Proof. For existence, pull back the smooth structure from N via f: if $\{(V_\alpha, \psi_\alpha)\}$ is a smooth atlas for N, then $\{(f^{-1}(V_\alpha), \psi_\alpha \circ f)\}$ is a smooth atlas for M.

For uniqueness, if $\mathscr{A}_1$ and $\mathscr{A}_2$ both make f a diffeomorphism, then the identity map $\mathbf{1}_M = f^{-1} \circ f$ is a diffeomorphism between $(M, \mathscr{A}_1)$ and $(M, \mathscr{A}_2)$, so $\mathscr{A}_1 = \mathscr{A}_2$. Alternatively, this follows from the uniqueness theorem for immersions. □

The uniqueness theorem also has a submersion counterpart.

Theorem 3.5. *Let M be a smooth manifold, N a topological space, and $f : M \to N$ a surjective map (not necessarily continuous a priori). If there exists a smooth structure on N such that f is a submersion, then this smooth structure is unique.*

Proof. Suppose $\mathscr{A}$ is a smooth structure on N such that $f : M \to N_{\mathscr{A}}$ is a submersion. We claim that for any manifold P, a map $g : N_{\mathscr{A}} \to P$ is smooth if and only if $g \circ f : M \to P$ is smooth.

The "only if" direction is clear. For the "if" direction, we work locally. Let $q \in N$ and choose $p \in M$ with $f(p) = q$ by the surjectivity of f. Since f is a submersion, , there exist neighborhoods U of p in M and V of q in N, and a smooth map $h : V \to U$ such that $f \circ h|_V = \mathbf{1}_V$.

The map $g \circ f$ being smooth implies that $g \circ \mathbf{1}_V = g \circ f \circ h|_V$ is smooth. hence g is smooth in a neighborhood of q. As smoothness is local, g is smooth everywhere.

Now, if $\mathscr{A}'$ is another smooth structure on N making f a submersion, then for any manifold P and map $g : N \to P$:

$$g \in C^\infty(N_{\mathscr{A}'}, P) \iff g \circ f \in C^\infty(M, P) \iff g \in C^\infty(N_{\mathscr{A}}, P).$$

By Theorem 3.4 in Lecture 4 again, this implies $\mathscr{A} = \mathscr{A}'$. □

Remark 3.6. *From the proof of this theorem, we see that a surjective submersion plays a role analogous to a quotient map in topology. To verify that a map from a quotient manifold to another manifold is smooth, it suffices to check that its composition with the submersion is smooth.*

A direct example is provided by homogeneous polynomials on Euclidean space. Consider the projection $\pi : \mathbf{R}^{n+1} \setminus \{0\} \to \mathbf{RP}^n$ onto projective space. If $F : \mathbf{R}^{n+1} \to R$ is a homogeneous polynomial, then it naturally induces a map $f : \mathbf{RP}^n \to \mathbf{R}$ defined by $f([x]) = F(x)$. To verify that f is smooth, we only need to check that $f \circ \pi = F|_{\mathbf{R}^{n+1} \setminus \{0\}}$ is smooth, which is true since polynomials are smooth.

References

[1] Jean-Pierre Serre, *Lie Algebras and Lie Groups*, Lecture Notes in Mathematics, vol. 1500, Springer, 1992.

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