Problem $1+2$ *(Product Manifolds)*
Let M, N be topological spaces. A set $\Omega \subset M \times N$ is called open if for every $p = (x, y) \in \Omega$ there exist open neighborhoods U, V of x, y respectively such that $U \times V \subset \Omega$.

(a) Show that this defines a topology on $M \times N$.

(b) If M, N are Hausdorff, then so is $M \times N$.

(c) If M, N have countable bases, then so does $M \times N$.

(d) The projections from $M \times N$ onto the factors M, N are continuous.

(e) If M,N are C^∞-manifolds, then so is $M \times N$.

Problem 3 *(Path Connectedness Implies Connectedness)*

Show that the closed interval $[0, 1]$ is connected, and use this to prove that path connectedness implies connectedness for topological spaces.
Hint for showing $[0, 1]$ *is connected:*

- Assume for contradiction that $[0, 1]$ is not connected, so it can be written as a disjoint union of two non-empty closed sets A and B.
- Without loss of generality, assume $0 \in A$. Consider the supremum of the set $\{x \in [0, 1] : [0, x] \subseteq A\}$.

Problem 4 *(Compact Exhaustion)*
Assume I, J are bounded open intervals and $K \subset I, L \subset J$ are bounded closed (and thus compact) intervals. Prove that there exists a homeomorphism $f : I \to J$ such that $f(K) = L$. Furthermore, if $f_0 : K_0 \to L_0$ is a homeomorphism between closed subintervals $K_0 \subset K$ and $L_0 \subset L$, show that f can be chosen such that $f|_{K_0} = f_0$.
Using this result, prove that if M is a manifold admitting a compact exhaustion

$$K_1 \subset K_2 \subset K_3 \subset \cdots$$

where each K_i is homeomorphic to $[0, 1]$, then M is homeomorphic to $\mathbf{R}$.
Hints for second part:

- Construct the homeomorphism inductively. Start by mapping K_1 to a closed interval in $\mathbf{R}$.

- Use the first part to extend the homeomorphism from K_i to K_{i+1} in a compatible way.
- Consider the union of these compatible homeomorphisms to define a global homeomorphism $F : M \to \mathbf{R}$.
- Pay attention to the "overlap" regions to ensure consistency throughout the construction.

Submit solutions by Tuesday, October 28st, before 6:00 PM to Ernst-Zermelo-Str. 1, mailbox on the 3rd floor, or directly to me during Tuesday's class.