

Problem 1 (*Union of Nested Intervals in 1-Manifolds*)

Let M be a 1-dimensional topological manifold. Suppose $\{U_i\}_{i=1}^{\infty}$ is a sequence of open subsets of M such that:

$U_1 \subset U_2 \subset U_3 \subset \dots$, each U_i is homeomorphic to the open interval $(0, 1)$

- (a) Prove that the union $U = \bigcup_{i=1}^{\infty} U_i$ is also homeomorphic to $(0, 1)$.
- (b) Use part (a) to prove that for any open subset $I \subset M$ that is homeomorphic to $(0, 1)$, there exists a maximal open set $U \subset M$ containing I that is homeomorphic to $(0, 1)$.

Hint for part (b):

- Consider the collection of all open sets containing I that are homeomorphic to $(0, 1)$.
- Use Zorn's Lemma or a constructive argument using countable basis.

Problem 2+3 (*Closure of Intervals and Classification of 1-Manifolds*)

Let M be a 1-dimensional topological manifold and let $I \subset M$ be an open subset homeomorphic to $(0, 1)$.

- (a) Prove that the closure $\bar{I}$ of I in M is homeomorphic to one of the following:
 - The open interval $(0, 1)$
 - The half-open interval $[0, 1)$
 - The closed interval $[0, 1]$
 - The circle S^1
- (b) Use part (a) to prove that every connected 1-dimensional topological manifold is homeomorphic to either $\mathbf{R}$ or S^1 .

Hint for part (a):

- Consider the endpoints of I and how they can be approached in M .
- Analyze the possible behaviors using the local structure of 1-manifolds.

Hint for part (b):

- Start with a maximal interval and consider its closure.
- Show that if the closure is not the whole manifold, you can extend further.

Problem 4 (*Smooth Parametrization of Square Boundary*)

Construct a smooth map $f : \mathbf{R} \rightarrow \mathbf{R}^2$ such that the image $f(\mathbf{R})$ is exactly the boundary of the square $[0, 1] \times [0, 1]$.

Hint:

- Consider constructing the map piecewise for each side of the square.
- One approach is to use smooth transition functions.

Submit solutions by Tuesday, November 4th, before 6:00 PM to Ernst-Zermelo-Str. 1, mailbox on the 3rd floor, or directly to me during Tuesday's class.