

Problem 1 (*Straightening Handles in One Dimension*)

Let $a < b < c$ be real numbers and let $f : U \rightarrow \mathbf{R}$ be a continuous injection, where U is an open neighborhood of $[a, c]$ in $\mathbf{R}$.

(a) (*0-handle*) Prove that there exists a continuous injection $g : U \rightarrow \mathbf{R}$ and an open interval J containing b with $\overline{J} \subset (a, c)$ such that:

- $g(x) = f(x)$ for all $x \in U \setminus J$,
- and g is a diffeomorphism on J .

(b) (*1-handle*) Assume additionally that there exist open neighborhoods U_a of a and U_c of c such that f is a diffeomorphism on U_a and U_c . Prove that there exists a smooth injection $g : U \rightarrow \mathbf{R}$ and open neighborhoods $V_a \subset U_a$ of a , $V_c \subset U_c$ of c such that:

- $g(x) = f(x)$ for all $x \in V_a \cup V_c$,
- and g is a diffeomorphism on U .

Problem 2 (*Uniqueness of Smooth Structure on $\mathbf{R}$*)

Let M be a 1-dimensional smooth manifold that is homeomorphic to $\mathbf{R}$, i.e., there exists a homeomorphism $f : M \rightarrow \mathbf{R}$. Prove that M is actually diffeomorphic to $\mathbf{R}$.

Hint:

- Choose a sequence of points $\dots < t_{i-1} < t_i < t_{i+1} < \dots$ in $\mathbf{R}$ such that each interval $[t_i, t_{i+2}]$ is contained in the image of a smooth coordinate chart of M .
- Apply the results from Problem 1 to construct a diffeomorphism by "straightening" the homeomorphism f .

Problem 3 (*Tangent Space and Germs of Functions*)

Let M be a smooth manifold and $p \in M$.

- The *germ* of a smooth function at p is an equivalence class of pairs (U, f) , where U is an open neighborhood of p and $f : U \rightarrow \mathbf{R}$ is a smooth function. Two such pairs (U, f) and (V, g) are equivalent if there exists an open neighborhood $W \subset U \cap V$ of p such that $f|_W = g|_W$.

- The set of all germs at p forms a ring, denoted by C_p , with operations defined by:

$$\begin{aligned}[f]_p + [g]_p &= [f + g]_p, \\ [f]_p \cdot [g]_p &= [f \cdot g]_p,\end{aligned}$$

where $[f]_p$ denotes the germ represented by f .

- The set of germs that vanish at p forms an ideal of C_p , denoted by $\mathfrak{m}_p$. That is,

$$\mathfrak{m}_p = \{[f]_p \in C_p : f(p) = 0\}.$$

This is in fact a maximal ideal of C_p .

Prove that the tangent space $T_p M$ is isomorphic to the dual space of $\mathfrak{m}_p/\mathfrak{m}_p^2$, i.e.,

$$T_p M \cong (\mathfrak{m}_p/\mathfrak{m}_p^2)^*.$$

Hint:

- Recall that a tangent vector $v \in T_p M$ can be defined as a derivation on smooth functions at p , i.e., a linear map $v : C_p \rightarrow \mathbf{R}$ satisfying the Leibniz rule: $v(fg) = f(p)v(g) + g(p)v(f)$.
- Show that if $f \in \mathfrak{m}_p^2$, then $v(f) = 0$ for any derivation v . Thus, v induces a linear map on the quotient $\mathfrak{m}_p/\mathfrak{m}_p^2$.
- Conversely, given a linear map $\ell : \mathfrak{m}_p/\mathfrak{m}_p^2 \rightarrow \mathbf{R}$, define a derivation v by setting $v(f) = \ell([f - f(p)]_p)$ for any $f \in C_p$, and verify that this satisfies the derivation properties.
- Finally, establish that these constructions are mutually inverse and linear.

Problem 4 (*Projectivization of the Tangent Bundle and Descent of Diffeomorphism*)

- (a) Let M and N be smooth manifolds, and let $\mathbf{R}^n$ be the Euclidean space with $n \geq 1$. Suppose there exists a diffeomorphism $\Phi : M \times \mathbf{R}^n \rightarrow N \times \mathbf{R}^n$ such that:
- For each $p \in M$, $\Phi(\{p\} \times \mathbf{R}^n) = \{q\} \times \mathbf{R}^n$ for some $q \in N$,
 - and the restriction $\Phi|_{\{p\} \times \mathbf{R}^n} : \{p\} \times \mathbf{R}^n \rightarrow \{q\} \times \mathbf{R}^n$ is a linear isomorphism for each $p \in M$.

Prove that Φ descends to a diffeomorphism $\tilde{\Phi} : M \times \mathbf{RP}^{n-1} \rightarrow N \times \mathbf{RP}^{n-1}$.

- (b) Let M be an n -dimensional smooth manifold. The projectivization of the tangent bundle TM , denoted by $\mathbf{P}(TM)$, is defined as the set of all 1-dimensional linear subspaces of the tangent spaces T_pM for all $p \in M$. That is,

$$\mathbf{P}(TM) = \bigcup_{p \in M} \mathbf{P}(T_pM),$$

where $\mathbf{P}(T_pM)$ is the projective space of the tangent space T_pM (which is isomorphic to $\mathbf{RP}^{n-1}$). Prove that $\mathbf{P}(TM)$ is a smooth manifold of dimension $2n - 1$.

- (c) (*Optional Discussion*) One may consider generalizing these constructions. For instance, one could define the Grassmannization of the tangent bundle as the bundle of k -dimensional linear subspaces of TM , and formulate an analogous descent statement for Grassmann manifolds.

Hint for (b): Use the smooth structure on M and local trivializations of TM to construct charts for $\mathbf{P}(TM)$. Then apply the method of patching smooth structures in Lecture 4.

Submit solutions by Tuesday, November 11th, before 6:00 PM to Ernst-Zermelo-Str. 1, mailbox on the 3rd floor, or directly to me during Tuesday's class.