Problem 1 *(The "Looks Locally Like" Convention)*

In class we used the convention of "looks locally like":
Let $\tilde{M}$ and $\tilde{N}$ be smooth manifolds, $\tilde{p} \in \tilde{M}$ and $\tilde{q} \in \tilde{N}$, and let $\tilde{f} : \tilde{M} \to \tilde{N}$ be a smooth map such that $\tilde{f}(\tilde{p}) = \tilde{q}$. Then (M, N, p, q, f) *looks locally like* $(\tilde{M}, \tilde{N}, \tilde{p}, \tilde{q}, \tilde{f})$ if there exist open neighborhoods U of p, V of q, $\tilde{U}$ of $\tilde{p}$, $\tilde{V}$ of $\tilde{q}$ and diffeomorphisms $g : U \to \tilde{U}$ and $h : V \to \tilde{V}$ such that:

1. $f(U) \subset V$ and $\tilde{f}(\tilde{U}) \subset \tilde{V}$,
2. $g(p) = \tilde{p}$ and $h(q) = \tilde{q}$,
3. The following diagram commutes:

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "g"'] & V \arrow[d, "h"] \\ \tilde{U} \arrow[r, "\tilde{f}"] & \tilde{V}. \end{tikzcd}$$

Consider the following simplest cases to get familiar with this convention:

(a) A local diffeomorphism looks locally like the identity map $\mathbf{R}^n \to \mathbf{R}^n$ (where $n = \dim M = \dim N$).

(b) The embedding map $M \to M \times N$ (defined by $x \mapsto (x, y_0)$ for some fixed $y_0 \in N$) looks locally like the linear injection $\mathbf{R}^m \to \mathbf{R}^m \times \mathbf{R}^n$ (defined by $x \mapsto (x, 0)$, where $m = \dim M$, $n = \dim N$).

(c) The projection map $M \times N \to M$ looks locally like the linear surjection $\mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^m$ (defined by $(x, y) \mapsto x$).

Check these directly using the definition, without using equivalent characterizations of immersions and submersions.

Problem 2 *(Lower Semi-continuity of the Rank)*

Let $f : M \to N$ be a smooth map between smooth manifolds. Consider the function $r : M \to \mathbf{Z}^+$ defined by $r(p) = \text{rank } d_p f$.
Recall that a function $g : X \to \mathbf{Z}$ on a topological space X is called *lower semi-continuous* if for every integer k, the set $\{x \in X : g(x) > k\}$ is open in X. Equivalently, for every $x_0 \in X$, there exists a neighborhood U of x_0 such that $g(x) \geq g(x_0)$ for all $x \in U$.

Prove that the rank function r is lower semi-continuous.

Problem 3 *(Cube Boundary Cannot Be an Immersion Image)*

Prove that the boundary of the cube $[0,1]^2$ cannot be the image of a smooth immersion $\gamma : \mathbf{R} \to \mathbf{R}^2$. That is, there does not exist a smooth map $\gamma : \mathbf{R} \to \mathbf{R}^2$ with $\gamma'(t) \neq 0$ for all $t \in \mathbf{R}$ such that $\gamma(\mathbf{R}) = \partial[0,1]^2$.

Problem 4 *(Fixed Point Set of a Finite Group Action)*

Let G be a finite group acting smoothly on a smooth manifold M. That is, for each $g \in G$, there is a smooth map $F_g : M \to M$ such that $F_e = \mathbf{1}_M$ (e is the unit of G) and $F_g \circ F_h = F_{gh}$ for all $g, h \in G$. Define the fixed point set of the action as $\mathrm{Fix}(G) = \{x \in M \mid F_g(x) = x \text{ for all } g \in G\}$.

(a) Consider a vector space V, which naturally has the structure of a smooth manifold with the canonical differential structure given by the atlas consisting of all linear isomorphisms $V \to \mathbf{R}^n$. For any $p \in V$, we have a canonical isomorphism $V \cong T_pV$. Make sure you understand this fact.

Now, for an arbitrary smooth manifold M and a point $p \in M$, we can choose a local diffeomorphism $f : U \to T_pM$ from an open neighborhood U of p to the tangent space T_pM such that $f(p) = 0$ and the differential $d_pf : T_pM \to T_0(T_pM) \cong T_pM$ is the identity map $\mathbf{1}_{T_pM}$.

(b) Prove that $\mathrm{Fix}(G)$ is a smooth submanifold of M.

Hint: Notice that the action of G on M naturally induces a linear action of G on T_pM for each $p \in M$. Choose an appropriate local diffeomorphism from M to T_pM near p that is equivariant with respect to these two actions.

Submit solutions by Tuesday, November 18th, before 6:00 PM to Ernst-Zermelo-Str. 1, mailbox on the 3rd floor, or directly to me during Tuesday's class.