
Problem 1 *(Transversality of Linear Graphs)*

Let $A, B : \mathbf{R}^m \to \mathbf{R}^n$ be linear maps. Consider their graphs

$$\Gamma_A = \{(x, Ax) \in \mathbf{R}^{m+n} : x \in \mathbf{R}^m\}, \quad \Gamma_B = \{(x, Bx) \in \mathbf{R}^{m+n} : x \in \mathbf{R}^n\}.$$

Prove that Γ_A and Γ_B intersect transversely in $\mathbf{R}^{m+n}$ if and only if $A-B : \mathbf{R}^m \to \mathbf{R}^n$ is surjective.

Problem 2 *(Brieskorn Manifolds)*

Let $n \geq 2$ and let $k_1, \ldots, k_n$ be positive integers. Consider the set

$$\Sigma(k_1, \ldots, k_n) = \left\{(z_1, \ldots, z_n) \in \mathbf{C}^n : z_1^{k_1} + \cdots + z_n^{k_n} = 0,\ |z_1|^2 + \cdots + |z_n|^2 = 1\right\}.$$

Prove that $\Sigma(k_1, \ldots, k_n)$ is a smooth submanifold of $\mathbf{C}^n \cong \mathbf{R}^{2n}$.

Problem 3 *(Grassmannian Submanifolds)*

Consider the Grassmannian $G(n-1, n)$ of hyperplanes in $\mathbf{R}^n$ and define

$$Y = \left\{ W \in G(n-1, n) \;\middle|\; P \text{ and } W \text{ are not transverse in } \mathbf{R}^n \right\}.$$

Prove that Y is a smooth submanifold of $G(n-1, n)$.

Problem 4 *(Pullbacks and Immersions/Embeddings)*

Consider a transversal pullback square

$$\begin{array}{ccc} P & \xrightarrow{p_2} & N_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ N_1 & \xrightarrow[f_1]{} & M \end{array}$$

where $P = N_1 \times_M N_2$ is the fibre product, and f_1 and f_2 are transverse smooth maps into M.

(a) Show that if f_2 is an immersion, then $p_1 : P \to N_1$ is an immersion.

(b) Show that if f_2 is a submersion, then $p_1 : P \to N_1$ is a submersion.

(c) Show that if f_2 is an embedding, then $p_1 : P \to N_1$ is an embedding.

Hint: For (a) and (b), use the tangent space description of the fibre product: at a point $(x_1, x_2) \in P$ with $f_1(x_1) = f_2(x_2)$, the tangent space $T_{(x_1,x_2)}P$ can be identified with

$$\{(v_1, v_2) \in T_{x_1}N_1 \times T_{x_2}N_2 : Df_1(v_1) = Df_2(v_2)\} \subset T_{x_1}N_1 \times T_{x_2}N_2.$$

Use this description together with the assumption that Df_2 is injective (immersion) or surjective (submersion).
For (c), Work locally.

Submit solutions by Tuesday, November 25th, before 6:00 PM to Ernst-Zermelo-Str. 1, mailbox on the 3rd floor, or directly to me during Tuesday's class.