Problem 1 *(Hessian at a Critical Point)*

Let $f : M \to \mathbf{R}$ be a smooth function on a smooth manifold M, and let $p \in M$ be a critical point of f (so $df_p = 0$).
Recall that if $\{x^1, \dots, x^n\}$ is a coordinate chart around p, we may write f in these coordinates as $\tilde{f} = f \circ \varphi^{-1}$ and define the Hessian matrix by

$$\left(\frac{\partial^2 \tilde{f}}{\partial x^i \partial x^j}(0) \right).$$

(a) Show that the bilinear form

$$\mathrm{Hess}_p(f)(v, w) := \sum_{i,j} \frac{\partial^2 \tilde{f}}{\partial x^i \partial x^j}(0)\, v^i w^j, \quad v, w \in T_pM,$$

is independent of the choice of local coordinates. (Hint: Use $df_p = 0$ to show that all first-order terms in the coordinate change formula vanish at p.)

Thus the Hessian defines a well-defined symmetric bilinear form

$$\mathrm{Hess}_p(f) : T_pM \times T_pM \to \mathbf{R}.$$

(b) Show that if $\mathrm{Hess}_p(f)$ is nondegenerate, then there exists a coordinate chart (U, φ) around p in which f takes the standard form

$$f(x) = f(p) - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2.$$

(This is the local normal form of f at a nondegenerate critical point, i.e. the Morse lemma.)

Problem 2 *(Genericity of Morse Functions)*

A smooth function $f : M \to \mathbf{R}$ is called a *Morse function* if all of its critical points are nondegenerate.
Let $M \subset \mathbf{R}^n$ be a smooth submanifold, and let $f : M \to \mathbf{R}$ be a smooth function. For a fixed vector $h \in \mathbf{R}^n$, consider the function

$$f_h(x) := f(x) - \langle x, h \rangle.$$

Prove that for almost every $h \in \mathbf{R}^n$, the function f_h is a Morse function on M.

Problem 3 *(Transversality of Hyperplanes with Submanifolds)*

In Problem 3 from last week's assignment, we showed that the set of hyperplanes in $\mathbf{R}^n$ not transverse to a fixed subspace $P \subset \mathbf{R}^n$ forms a submanifold of $G(n-1, n)$. Let $M \subset \mathbf{R}^n$ be a smooth submanifold. Define the subset

$$Y = \{(x, W) \in M \times G(n-1, n) \mid W \text{ is not transverse to } T_x M\} .$$

Prove that Y is a smooth submanifold of $M \times G(n-1, n)$.

Problem 4 *(Bertini's Theorem)*

Let $\mathrm{Gr}_k(\mathbf{P}^n)$ denote the Grassmannian of k-dimensional projective subspaces in projective space $\mathbf{P}^n$. Note that $\mathrm{Gr}_k(\mathbf{P}^n)$ is diffeomorphic to $G(k+1, n+1)$, the Grassmannian of $(k+1)$-planes in $\mathbf{R}^{n+1}$.

(a) State (without proof) the analogue of Problem 3 for $\mathrm{Gr}_{n-1}(\mathbf{P}^n)$.

(b) Deduce that for almost every hyperplane $H \in \mathrm{Gr}_{n-1}(\mathbf{P}^n)$, the intersection $M \cap H$ is transverse.

Submit solutions by Tuesday, December 2nd, before 6:00 PM to Ernst-Zermelo-Str. 1, mailbox on the 3rd floor, or directly to me during Tuesday's class.