Analysis 2: Parameterintegrale

$f : \Omega \times I \longrightarrow \mathbb{R}$, $f = f(x,y)$

Ω offen im $\mathbb{R}^n$, $x = (x_1, \dots, x_n)$ Parameter

$I = [a,b]$ kompaktes Intervall

Betrachte: $\Phi : \Omega \to \mathbb{R}$, $\Phi(x) = \int_I f(x,y)\,dy$.

Frage:
- Φ stetig?
- Φ diff'bar

Fakt: $f : K \to \mathbb{R}$ stetig, $K \subset \mathbb{R}^n$ kompakt

$\Rightarrow f$ gleichmäßig stetig

$$\operatorname{osc}(f,\delta) := \sup_{x,x' \in K} \{ |f(x) - f(x')| : |x - x'| < \delta \}$$

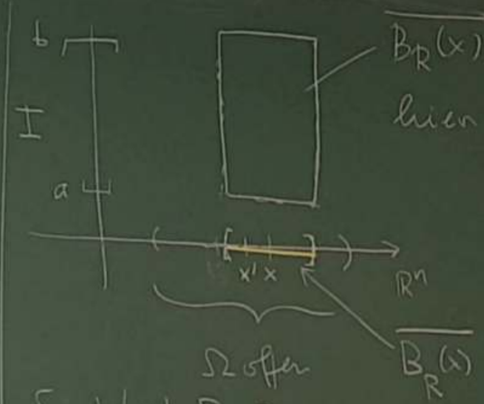
glm stetig $\Leftrightarrow \operatorname{osc}(f,\delta) \to 0$ mit $\delta \to 0$.

Satz $f \in C^0(\Omega \times I) \Rightarrow \Phi \in C^0(\Omega)$.

Beweis

- $\Phi(x)$ ist wohldefiniert
  $f(x,\cdot) : I \to \mathbb{R}$ ist stetig, also R-integrierbar
- seien $x, x' \in \Omega$

$$|\Phi(x') - \Phi(x)| = \left| \int_I \big(f(x',y) - f(x,y)\big)\,dy \right|$$



$\overline{B_R(x)} \times I$ = kompakt
hier f gleichmäßig stetig

Sei $|x' - x| < \delta \le R$

$$\le |I| \sup_{y \in I} |f(x',y) - f(x,y)| \le |I| \cdot \operatorname{osc}\left(f\big|_{\overline{B_R(x)} \times I}, \delta\right) \longrightarrow 0 \text{ mit } \delta \to 0.$$

Satz (Differentiation unter dem Integral)

Voraussetzungen:

(a) $f(x,\cdot)$ $\mathcal{R}$-integrierbar für alle $x \in \Omega$.

(b) $\frac{\partial f}{\partial x_j}$ existiert und ist stetig auf $\Omega \times I$.

Dann ist $\phi: \Omega \to \mathbb{R}$, $\phi(x) = \int_I f(x,y)\,dy$, nach x_j partiell diffbar, und zwar

$$\frac{\partial \phi}{\partial x_j}(x) = \int_I \frac{\partial f}{\partial x_j}(x,y)\,dy \qquad \forall x \in \Omega.$$

Sind $f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ in $C^0(\Omega \times I)$, so ist $\phi \in C^1(\Omega)$

$$\frac{f(x+he_j,y) - f(x,y)}{h} \underset{\text{HAUPTSATZ}}{=} \frac{1}{h}\int_0^1 \frac{d}{ds} f(x+she_j,y)\,ds$$

$$= \int_0^1 \frac{\partial f}{\partial x_j}(x+she_j,y)\,ds$$

Sei wieder $\overline{B_R(x)} \subset \Omega$, und $|h| < \delta \leq R$

$$\left| \frac{\phi(x+he_j)-\phi(x)}{h} - \int_I \frac{\partial f}{\partial x_j}(x,y)\,dy \right| = \left| \int_I \left(\frac{f(x+he_j,y)-f(x,y)}{h} - \frac{\partial f}{\partial x_j}(x,y) \right) dy \right|$$

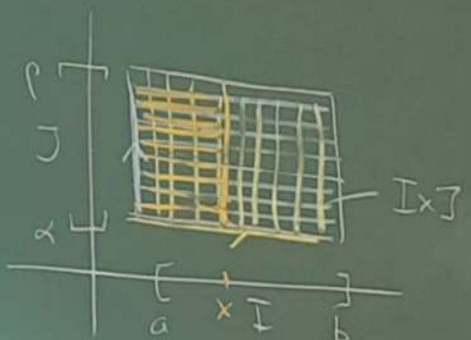
$$= \left| \int_I \int_0^1 \left(\frac{\partial f}{\partial x_j}(x+she_j,y) - \frac{\partial f}{\partial x_j}(x,y) \right) ds\,dy \right|$$

$$\leq |I| \operatorname{osc}\left(\frac{\partial f}{\partial x_j}\Big|_{\overline{B_R(x)} \times I}, \delta \right)$$

erinnere $|h| < \delta$.

$$\longrightarrow 0 \text{ mit } \delta \to 0$$

$\boxtimes$



Anwendung: Kleiner Fubini

<u>Satz</u> $f: I\times J \to \mathbb{R}$, $I=[a,b]$, $J=[\alpha,\beta]$, stetig.

$$\int_\alpha^\beta\left(\int_a^b f(x,y)\,dx\right)dy = \int_a^b\left(\int_\alpha^\beta f(x,y)\,dy\right)dx$$

<u>Beweis</u> Betrachte $\phi(x)=\int_\alpha^\beta \underbrace{\int_a^x f(\xi,y)\,d\xi}_{F(x,y)}\, dy$ (stetig in y nach Satz)

$$\frac{\partial F}{\partial x}(x,y) = f(x,y) = \text{stetig}.$$

$$\phi'(x)=\int_\alpha^\beta \frac{\partial F}{\partial x}(x,y)\,dy = \int_\alpha^\beta f(x,y)\,dy.$$

Betrachte jetzt

$$\psi(x)=\int_a^x \underbrace{\int_\alpha^\beta f(\xi,y)\,dy}_{\text{stetig in } \xi \text{ nach Satz}}\, dx$$

Hauptsatz

$$\psi'(x)=\int_\alpha^\beta f(x,y)\,dy.$$

$$\phi(b)=\int_\alpha^\beta\int_a^b f(\xi,y)\,d\xi\, dy$$

$$\psi(a)=0=\phi(a) \Rightarrow \phi(x)=\psi(x)\ \forall x\in I \overset{x=b}{\Rightarrow} \psi(b)=\int_a^b\int_\alpha^\beta f(\xi,y)\,dy\, dx.$$

$\boxtimes$

Variationsintegrale

$I = [a,b]$

$\mathcal{F} : C^1(I, \mathbb{R}^n) \longrightarrow \mathbb{R}$

$$\mathcal{F}(u) = \int_a^b f(\underbrace{t}_{\in I}, \underbrace{u(t)}_{\in \mathbb{R}^n}, \underbrace{u'(t)}_{\in \mathbb{R}^n})\, dt .$$

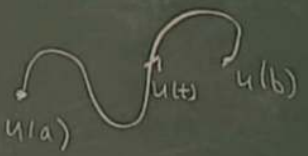
$f : I \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}.$

$f = f(t, x, v)$ Lagrangefunktion.

Beispiele

- $f(t,x,v) = |v|$

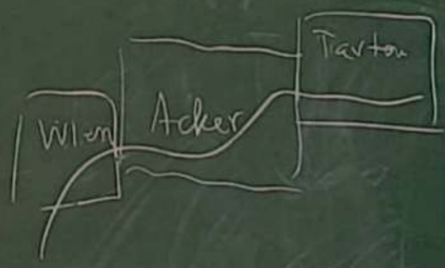
$$\Rightarrow \mathcal{F}(u) = \int_a^b |u'(t)|\, dt = \text{Bogenlänge von } u(t)$$



- $f(t,x,v) = \omega(x)|v|$

$$\Rightarrow \mathcal{F}(u) = \int_a^b \omega(u(t))\, |u'(t)|\, dt$$

$= \text{Bogenlänge mit Gewicht}$



- Klassische Mechanik

Teilchen der Masse $m > 0$.

Potential $V(x)$ (Gravitationspotential)

$$\mathcal{F}(u) = \int_a^b \Big(\underbrace{\tfrac{m}{2} |u'(t)|^2}_{\text{kinetische Energie}} - \underbrace{V(u(t))}_{\text{potentielle Energie}} \Big)\, dt \quad (= \text{Wirkung}).$$

Schar von (Vergleichs-)Wegen

$(\varepsilon,t) \mapsto u(\varepsilon,t)$ $\quad$ (ε = Scharparameter)

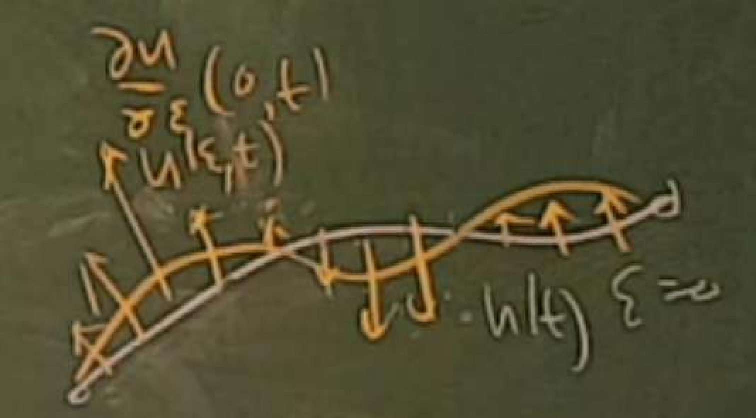
$$\mathcal{F}(u(\varepsilon,\cdot)) = \int_a^b f\left(t,u(\varepsilon,t),\frac{\partial u}{\partial t}(\varepsilon,t)\right)dt =: \phi(\varepsilon)$$

$$\phi'(0) = \int_a^b \frac{\partial}{\partial \varepsilon} f\left(t,\overset{x}{u(\varepsilon,t)},\overset{v}{\frac{\partial u}{\partial t}(\varepsilon,t)}\right)\Big|_{\varepsilon=0} dt$$

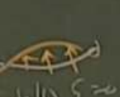
$u' = \frac{du}{dt}$

$$= \sum_{i=1}^{n} \int_a^b \left(\frac{\partial f}{\partial x_i}(t,u,u')\underbrace{\frac{\partial u_i}{\partial \varepsilon}(0,t)}_{\varphi_i(t)} + \frac{\partial f}{\partial v_i}(t,u,u')\frac{\partial}{\partial t}\underbrace{\frac{\partial u_i}{\partial \varepsilon}(0,t)}_{\varphi_i(t)}\right)dt$$

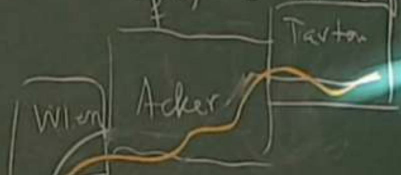
partielle Integration



$$= \sum_{i=1}^{n} \int_a^b \left(\frac{\partial f}{\partial x_i}(t,u,u') - \frac{d}{dt}\frac{\partial f}{\partial v_i}(t,u,u')\right)\varphi_i(t)\,dt + \sum_{i=1}^{n}\underbrace{\left[\frac{\partial f}{\partial v_i}(t,u,u')\varphi_i(t)\right]_{t=a}^{t=b}}$$



$$\int_a^b \Big\langle \underbrace{D_x f(t,u,u') - \frac{d}{dt}D_v f(t,u,u')}_{L_f(u)\,=\,\text{Euler-Lagrange-Operator}}, \varphi \Big\rangle dt + \underbrace{\Big\langle D_v f(t,u,u'), \varphi\Big\rangle\Big|_{t=a}^{t=b}}_{\text{Randterm, ist Null falls } \varphi(a)=\varphi(b)=0.}$$



Randterm, ist Null falls $\varphi(a)=\varphi(b)=0$.

Satz (Euler-Lagrange)

Sei u ein stationärer Punkt von $\mathcal{F}$:

$$\frac{d}{d\varepsilon}\mathcal{F}(u+\varepsilon\varphi)\Big|_{\varepsilon=0} = 0 \quad \forall \varphi \in C_c^\infty(I,\mathbb{R}^n).$$

$$\Rightarrow L_f(u) = 0: \quad \frac{\partial f}{\partial x_i}(t,u,u') - \frac{d}{dt}\frac{\partial f}{\partial v_i}(t,u,u') = 0 \quad \forall i = 1,\dots,n.$$