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Geometric Measure Theory

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Contents

Introduction

0.1. **Motivation.** Geometric Measure Theory is a branch of Geometric Analysis which was born around the middle of the 20th century, out of the desire to find a mathematical framework in which one can prove the existence of solutions to the Plateau's problem.

0.2. Plateau's problem (Lagrange, 1760). *Among all* surfaces *with a prescribed* boundary, *find one which minimizes the* area.

The problem was raised by Joseph Louis Lagrange in 1760. Nevertheless it takes its name from the Belgian physicist Joseph Plateau, due to his extensive experiments on soap films in the middle of 19th century. Plateau also formulated some laws describing the structure of soap films:

(1) soap films are pieces of smooth surfaces;
(2) the pieces have constant mean curvature;
(3) when some pieces meet they do it in groups of three (Plateau's borders) and they meet with angles of 120°;
(4) when Plateau's borders meet they do it in groups of four and they form a tetrahedral angle.

Nowadays by "solving the Plateaus's problem we mean finding a mathematical framework in which the existence of area-minimizing surfaces with prescribed boundary can be proved in any dimension and codimension. The terms *surface, boundary*, and *area* are in principle not clearly defined: providing suitable notions for these terms is then part of the solution to the problem.

0.3. **Direct method in the Calculus of Variations.** As in several Variational Problems, the most robust way to prove the existence of minimizers when the competitors vary in an infinite dimensional space is to use the "Direct Method". This works generally as follows.
Let X be a topological space. Let $F : X \to \mathbb{R}$ be a *sequentially lower-semicontinuous* functional, i.e.

$$F(x) \leq \liminf_{n\to\infty} F(x_n), \quad \text{whenever } x_n \to x.$$

Assume that the sublevel-sets

$$\{x \in X : F(x) \leq M\}$$

are sequentially compact. Then F admits a minimizer in X.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a minimizing sequence, i.e. $F(x_n)$ converges to

$$m := \inf\{F(x) : x \in X\}.$$

In particular there exists $M \in \mathbb{R}$ such that $\{x_n\}_{n\in\mathbb{N}} \subset F^{-1}((-\infty, M])$. By assumption this set is sequentially compact, hence there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ which converges to a point $x \in X$. By the sequential lower-semicontinuity it holds

$$-\infty < F(x) \leq \liminf_{k\to\infty} F(x_{n_k}) = m,$$

hence x is a minimum for F. $\square$

0.4. **Functional analytic approach.** The main idea is to replace oriented surfaces with the elements of a larger Banach space. On an m-dimensional surface $\Sigma \subset \mathbb{R}^d$, oriented by a tangent field τ_Σ, one can integrate a differential m-form ω. Moreover the map

$$\omega \mapsto \int_\Sigma \omega := \int_{\mathbb{R}^d} \langle \omega(x); \tau_\Sigma(x) \rangle d\Sigma(x)$$

is a linear and continuous functional (we remain vague on the notion of norm in the space of differential forms, for the moment). This suggests that one can enlarge the class of oriented m-surfaces considering elements of the dual of the space of differential m-forms (*generalized surfaces*, or *currents*). A natural replacement of the notion of area is the notion of *mass* $\mathbb{M}(T)$ of a current T, i.e. simply its dual norm, i.e.

$$\mathbb{M}(T) := \sup_{\|\omega\| \le 1} \{\langle T; \omega \rangle\}$$

The sequential compactness (with respect to the weak* topology) of the sets $\{T : \mathbb{M}(T) \le M\}$ is an immediate consequence of Banach-Alaoglu theorem, and the sequential lower-semicontinuity of the mass is a trivial fact (being the mass a supremum of linear functionals). Lastly, one should define a notion of boundary. This can be done "imposing" the validity of the Stoke's theorem for currents. If T is an m-dimensional current, then ∂T is an $(m-1)$-dimensional current, which is defined by

$$\langle \partial T, \phi \rangle := \langle T; d\phi \rangle, \qquad \text{for every differential } (m-1)\text{-form } \phi,$$

where d is the *differential* operator. It is easy to see that the notion of boundary is continuous with respect to the weak* convergence, which allows to complete the proof of the existence of generalized surfaces minimizing the mass among those with a prescribed boundary.

0.5. Remark. In principle the solution to this problem might be "weird" objects. Observe that for example any discrete or "continuous" convex combination of surfaces with the same boundary is a generalized surface with that boundary. In particular the solutions might have non-integer multiplicities and moreover the *support* of a solution might fail to be a set of the correct dimension.

0.6. **Geometric measure theory.** In order to be able to guarantee better properties of the solutions, one could require additional *a priori* regularity assumptions to the class of generalized surfaces. This is achieved by considering the class of *integral currents*. Their support is a *rectifiable set* (i.e. a set which is essentially contained in a countable union of C^1 submanifolds) and multiplicities are allowed, but only with integer values.
The major difficulty is to prove that this class is closed with respect to weak* convergence, in order to recover compactness properties. The closure theorem for integral currents is the main topic of this series of lectures.

0.7. Remark (Regularity of solutions). In general, area-minimizing integral currents are not regular. In some cases, singularities are unavoidable. The regularity theory, then aims at estimating the dimension of the set of *singular points*, i.e. those points for which there is no neighbourhood in which the current is represented by a smooth submanifold.

- In $\mathbb{R}^d$, for $d \le 7$, area-minimizing hypersurfaces are regular in the interior (i.e. far from the boundary)
- for $d > 7$, area minimizing hypersurfaces might have a singular set of dimension at most $d - 8$ (and countable in dimension $d = 8$)
- Higher codimension area-minimizing surfaces might have a "larger" singular set. An m-dimensional area-minimizing surfaces in $\mathbb{R}^d$ $(d \ge m + 2)$ might have a singular set of dimension up to $m - 2$.

CHAPTER 1

Preliminaries

In this chapter we collect some basic definitions and results in measure theory and real analysis.

1. Measure Theory

1.1. Definition. *Let X be any set. An* outer measure *on X is a set function*

$$\mu^* : 2^X \to [0, +\infty]$$

such that

(1) $\mu^*(\emptyset) = 0$*;*
(2) μ^* *is σ-subadditive, i.e.*

$$\mu^*(E) \le \sum_{i\in\mathbb{N}} \mu^*(E_i), \quad \text{whenever } E \subset \textstyle\bigcup_{i\in\mathbb{N}} E_i.$$

1.2. Definition. *Let (X, τ) be a topological space. A* (Borel) measure *on X is a set function*

$$\mu : \mathcal{B}(X) \to [0, +\infty],$$

where $\mathcal{B}(X)$ denotes the σ-algebra of Borel subsets of X, such that

(1) $\mu(\emptyset) = 0$*;*
(2) μ *is σ-additive, i.e.*

$$\mu(E) = \sum_{i\in\mathbb{N}} \mu(E_i), \quad \text{whenever } E \subset \textstyle\bigcup_{i\in\mathbb{N}} E_i \text{ and } E_i \cap E_j = \emptyset \text{ for } i \ne j.$$

1.3. Definition. *Let μ^* be an outer measure on a set X. A set $E \subset X$ is* measurable (in the sense of Caratheodory) *if*

$$\mu^*(F) = \mu^*(E \cap F) + \mu^*(E \setminus F), \quad \text{for every } F \subset X.$$

1.4. Proposition. *Let μ^* be an outer measure on a set X. The class $\mathcal{S}$ of measurable sets is a σ-algebra and μ^* is σ-additive on $\mathcal{S}$.*

1.5. EXERCISE. $(*)$ *Prove Proposition 1.4.*

1.6. Theorem (Caratheodory's Criterion, see [7], Theorem 1.2). *Let (X, d) be a metric space and let μ^* be an outer measure on X, which is additive on distant sets, i.e.*

$$\mu^*(E_1 \cup E_2) = \mu^*(E_1) + \mu^*(E_2) \quad \text{if } dist(E_1, E_2) > 0.$$

Then the σ-algebra $\mathcal{S}$ of measurable sets contains the Borel σ-algebra $\mathcal{B}(X)$.

1.7. Definition (Hausdorff measures). *Let (X, d) be a metric space, $s \in [0, +\infty]$, $\delta \in (0, +\infty]$. For every $E \subset X$, denote*

$$\mathscr{H}^s_\delta(E) := \omega_s \inf_{E_i} \left\{ \sum_{i\in\mathbb{N}} \left(\frac{diam(E_i)}{2} \right)^s \right\},$$

where $\{E_i\}_{i\in\mathbb{N}}$ is a covering of E made by sets satisfying $diam(E_i) \le \delta$ and ω_s is a geometric constant, which coincides with the s-dimensional volume of the unit ball in $\mathbb{R}^s$, when s is an integer. Denote also

$$\mathscr{H}^s(E) := \sup_{\delta>0} \mathscr{H}^s_\delta(E).$$

*The supremum is called the s-*dimensional Hausdorff measure *of the set E.*

Notice that the function $\mathscr{H}^s_\delta(E)$ is trivially monotone decreasing in δ, hence the supremum above coincides with the limit $\lim_{\delta\to 0} \mathscr{H}^s_\delta(E)$.

1.8. EXERCISE.

(1) *Prove that $\mathscr{H}^s$ is a (Borel) measure for every s.*
(2) *Let $E \subset \mathbb{R}^d$ be bounded; prove that for every $\delta > 0$ it holds $\mathscr{H}^s_\delta(E) < +\infty$.*
(3) *Find a compact set $E \subset \mathbb{R}^2$ such that $\mathscr{H}^1(E) = +\infty$ but E is a countable union of sets with finite $\mathscr{H}^1$-measure.*
(4) *Prove that the value of $\mathscr{H}^s(E)$ is unchanged if the sets E_i in the definition are required to be open/closed/convex.*

1.9. Remark. The value of $\mathscr{H}^s(E)$ might change if the sets E_i are required to be balls. The resulting measure is called *spherical Hausdorff measure*.

1.10. EXERCISE. *Is the Hausdorff measure sequentially lower-semicontinuous with respect to the Hausdorff distance $d_{\mathscr{H}}$? Here we denoted*

$$d_{\mathscr{H}}(E,F) := \max\{\sup_{x\in F}\{dist(x,E)\}; \sup_{y\in E}\{dist(y,F)\}\}.$$

1.11. EXERCISE. *Prove that if $s < t$, then for every $\delta > 0$ it holds*

$$\mathscr{H}^s_\delta(E) \ge C(s,t)\delta^{s-t}\mathscr{H}^t_\delta(E),$$

where $C(s,t) > 0$ depends only on s and t. Deduce that if $\mathscr{H}^s(E) < +\infty$ then $\mathscr{H}^t(E) = 0$.

The last property stated in the previous exercise motivates the following definition.

1.12. Definition (Hausdorff dimension). *Let (X,d) be a metric space and let $E \subset X$ be any set. Then we define*

$$\dim_{\mathscr{H}}(E) := \sup\{s \ge 0 : \mathscr{H}^s(E) = +\infty\}.$$

1.13. EXERCISE.

(1) *Prove that if E is a finite set then $\mathscr{H}^0(E)$ is the number of its elements and if E is an infinite set, then $\mathscr{H}^0(E) = +\infty$.*
(2) *Prove that if E is countable, then $\dim_{\mathscr{H}}(E) = 0$.*

1.14. EXERCISE. $(*)$ *Find an uncountable set $E \subset \mathbb{R}$ such that $\dim_{\mathscr{H}}(E) = 0$.*

1.15. EXERCISE. $(*)$ *Prove that the Cantor middle-third set C has $\dim_{\mathscr{H}}(C) = \log_3(2) =: s$ and $\mathscr{H}^s(C) = \omega_s$.*

The (non-trivial) next result states that for smooth m-dimensional surfaces, the m-dimensional Hausdorff measure agrees with the natural notion of m-dimensional area.

1.16. Theorem (see [1], Theorem 2.53). *If $E \subset \mathbb{R}^n$, then $\mathscr{H}^n(E) = \mathscr{L}^n(E)$, where $\mathscr{L}^n$ is the Lebesgue measure.*

2. Lipschitz Maps

2.1. Definition. *Let $(X,d),(X',d')$ be metric spaces. A function $f : X \to X'$ is L-Lipschitz $(L \geq 0)$ if*

$$d'(f(x), f(y)) \leq Ld(x, y), \quad \text{for every } x, y \in X.$$

The least constant L such that the inequality holds is called Lipschitz constant of f and denoted $\mathrm{Lip}(f)$.

2.2. EXERCISE. *For $e \in \mathbb{R}^{n+1}$ with $|e| = 1$, and $\alpha \in (0, \frac{\pi}{2})$, denote*

$$C(e, \alpha) := \{x \in \mathbb{R}^{n+1} : |\langle x, e\rangle| > \cos(\alpha)\}$$

the open cone with axis e and angle α. Prove that a function $f : \mathbb{R}^n \to \mathbb{R}$ is $\cot(\alpha)$-Lipschitz if and only if, for every $x \in \mathbb{R}^n$ it holds

$$graph(f) \cap [C(e_{n+1}, \alpha) + (x, f(x))] = \{(x, f(x))\}.$$

2.3. Proposition. *Let $f : X \to X'$ be L-Lipschitz and let $E \subset X$. Then for every $s \geq 0$ it holds $\mathscr{H}_s(f(E)) \leq L^s \mathscr{H}_s(E)$.*

2.4. EXERCISE. *Prove Proposition 2.3.*

2.5. EXERCISE. *The "Cantor dust" is a subset C of $\mathbb{R}^2$ constructed as follows.*

- *Let $C_0 := [0, 1] \times [0, 1]$.*
- *The set C_1 is the subset of C_0 obtained as the union of four disjoint squares $Q^1_1, \dots, Q^1_4$ of edge length $\frac{1}{4}$. The set is identified by the condition that each square Q^1_j has one vertex coinciding with one of the vertices of C_0.*
- *For $i = 1, 2, \dots,$ the set C_{i+1} is obtained from the set C_i replacing each of the 4^i squares Q^i_j such that $C_i = \cup_{j=1}^{4^i} Q^i_j$ with an homothetic copy of C_1 contained in Q^i_j and with homothety ratio 4^{-i}.*
- *We define $C := \cap_{i\in\mathbb{N}} C_i$.*

Prove that $\dim_{\mathscr{H}}(C) = 1$.

2.6. Theorem (Arzelà-Ascoli). *Let X be a separable metric space, Y be a compact metric space and $L \geq 0$. Let $(f_k : X \to Y)_{k\in\mathbb{N}}$ be L-Lipschitz functions. Then there exists a subsequence $(f_{k_h})_{h\in\mathbb{N}}$ and an L-Lipschitz function f such that f_{k_h} converges uniformly to f as $h \to \infty$.*

2.7. Theorem (Mc Shane). *Let X be a metric space, $E \subset X$ and $f : E \to \mathbb{R}$ be a Lipschitz function. Then there exists an Lipschitz extension $F : X \to \mathbb{R}$ such that $F_{|E} = f$ and* $\mathrm{Lip}(F) = \mathrm{Lip}(f)$.

2.8. EXERCISE. *Prove Theorem 2.7. More precisely, prove that the functions*

$$F_1(x) := \sup_{y\in E}\{f(y) - \mathrm{Lip}(f)d(x, y)\}$$

$$F_2(x) := \inf_{y\in E}\{f(y) + \mathrm{Lip}(f)d(x, y)\}$$

are extensions of f with the same Lipschitz constant and that any other extension F of f with the same Lipschitz constant satisfies

$$F_1 \leq F \leq F_2.$$

2.9. EXERCISE. *Let $\varepsilon > 0$ and let $f_1, f_2 : E \subset X \to \mathbb{R}$ be L-Lipschitz with $|f_1 - f_2| \leq \varepsilon$. Prove that if $F_1 : X \to \mathbb{R}$ is an L-Lipschitz extension of f_1, then there exists an L-Lipschitz extension F_2 of f_2 such that $|F_1 - F_2| \leq \varepsilon$.*

If $f : E \subset X \to \mathbb{R}^m$ is Lipschitz, then application of Theorem 2.7 fo every component of f gives a Lipschitz extension (with Lipschitz constant possibly larger than $\mathrm{Lip}(f)$). The next Theorem shows that, if X is Hilbert, then also an extension with the same Lipschitz constant can be found. This is not true in general if X is not Hilbert.

2.10. Theorem (Kirszbraun, see [3] 2.10.43). *Let X, Y be Hilbert spaces, $E \subset X$ and $f : E \to Y$ be a Lipschitz function. Then there exists an Lipschitz extension $F : X \to Y$ such that $F_{|E} = f$ and* $\mathrm{Lip}(F) = \mathrm{Lip}(f)$.

2.11. EXERCISE. $(*)$ *Let $K \subset \mathbb{R}^n$ be compact. Let $f_1, f_2 : K \subset \mathbb{R}^n \to \mathbb{R}^m$ be both L-Lipschitz, for some $L > 0$. Prove that for every $\varepsilon > 0$ there exists $\delta > 0$ and L-Lipschitz extensions F_1, F_2 of f_1 and f_2 respectively, such that if $|f_1 - f_2| \le \delta$ then $|F_1 - F_2| \le \varepsilon$.*

(Hint: use Theorem 2.10 where $E \subset \mathbb{R}^{n+1}$ is the union of two "parallel" copies of K.*)*

Next we prove the main result (for our purposes) of this section.

2.12. Theorem (Rademacher). *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz function. Then f is differentiable $\mathscr{L}^n$-almost everywhere.*

Proof. We can assume $m = 1$, the proof for general m is obtained applying the result for $m = 1$ to every component of f.

Step 1: we want to prove that the set of points x where

$$\mathrm{grad} f(x) := (D_{e_1} f(x), \dots, D_{e_n} f(x))$$

does not exist has measure zero. Fix $v \in \mathbb{R}^n$ and denote

$$E^v := \left\{ x : \limsup_{t \to 0} \frac{f(x + tv) - f(x)}{t} \neq \liminf_{t \to 0} \frac{f(x + tv) - f(x)}{t} \right\}.$$

Since E^v is Borel, we can apply Fubini's theorem to get

$$|E^v| = \int_{v^\perp} |E^v_x| d\mathscr{L}^{n-1}(x),$$

where $E^v_x := E^v \cap (x + \langle v \rangle)$. Since f is Lipschitz on each line $x + \langle v \rangle$, the version of the theorem for $n = 1$ (due to Lebesgue) implies that $|E^v_x| = 0$ for every $x \in v^\perp$, hence $|E^v| = 0$. In particular $|E^{e_1} \cup \dots \cup E^{e_n}| = 0$, which completes the proof of this step.

Step 2: we want to prove that for every $v \in \mathbb{R}^n$ the set of points x for which

$$D_v(x) = \mathrm{grad} f(x) \cdot v$$

has full measure. Fix $v \in \mathbb{R}^n$ with $|v| = 1$ and a test function $g \in C^\infty_c(\mathbb{R}^n)$. Take a sequence $(t_k)_{k \in \mathbb{N}}$ of non-zero numbers, which converges to 0. By a simple change of coordinates, for every $k \in \mathbb{N}$, we can write

$$\int_{\mathbb{R}^n} \frac{f(x + t_k v) - f(x)}{t_k} g(x) dx = -\int_{\mathbb{R}^n} \frac{g(x - t_k v) - g(x)}{t_k} f(x) dx.$$

Since f is Lipschitz, the integrands on the LHS are dominated, for every k, by $\mathrm{Lip}(f)|g|$, hence by the dominated convergence theorem we can conclude, denoting $v = (v_1, \dots, v_n)$,

$$\begin{aligned}
\int_{\mathbb{R}^n} D_v f(x) g(x) dx &= -\int_{\mathbb{R}^n} D_v g(x) f(x) dx \\
&= -\sum_{i=1}^n v_i \int_{\mathbb{R}^n} D_{e_i} g(x) f(x) dx \\
&= \sum_{i=1}^n v_i \int_{\mathbb{R}^n} D_{e_i} f(x) g(x) dx \\
&= \int_{\mathbb{R}^n} (\mathrm{grad} f(x) \cdot v) g(x) dx.
\end{aligned}$$

Since g is arbitrary, we can conclude that the equality

$$\mathrm{grad} f(x) \cdot v = D_v f(x)$$

holds almost everywhere.

Step 3: conclusion. Let $(v_k)_{k\in\mathbb{N}}$ be a sequence which is dense in $\mathbb{S}^{n-1}$. By the previous steps there exist a set $A \subset \mathbb{R}^n$ such that $|\mathbb{R}^n \setminus A| = 0$ and

- $\mathrm{grad} f(x)$ exists for every $x \in A$;
- $\mathrm{grad} f(x) \cdot v_k = D_{v_k} f(x)$, for every $x \in A$.

Fix now $v \in \mathbb{S}^{n-1}$ and $x \in A$. For every $t \neq 0$ and for every $k \in \mathbb{N}$ we estimate:

$$\begin{aligned} &\left|\frac{f(x+tv)-f(x)}{t} - \mathrm{grad} f(x)\cdot v\right| \le \left|\frac{f(x+tv)-f(x)}{t} - \frac{f(x+tv_k)-f(x)}{t}\right| + \\ &+ \left|\frac{f(x+tv_k)-f(x)}{t} - \mathrm{grad} f(x)\cdot v_k\right| + |\mathrm{grad} f(x)\cdot(v_k - v)|. \end{aligned} \tag{2.1}$$

Now fix $\varepsilon > 0$. Let $K \in \mathbb{N}$ be such that for every $w \in \mathbb{S}^{n-1}$ it holds

$$dist(w, \{v_1, \dots, v_K\}) \le \varepsilon.$$

Let $\delta > 0$ be such that, for every $j = 1, \dots, K$ it holds

$$\left|\frac{f(x+tv_j)-f(x)}{t} - \mathrm{grad} f(x)\cdot v_j\right| \le \varepsilon$$

whenever $0 < |t| < \delta$. Finally choose $v_k \in \{v_1, \dots, v_K\}$ such that $|v - v_k| \le \varepsilon$. Plugging these in (2.1), we obtain the estimate, for $0 < |t| < \delta$,

$$\left|\frac{f(x+tv)-f(x)}{t} - \mathrm{grad} f(x)\cdot v\right| \le \mathrm{Lip}(f)\varepsilon + \varepsilon + n\mathrm{Lip}(f)\varepsilon = C(\mathrm{Lip}(f), n)\varepsilon,$$

which proves that $D_v f(x) = \mathrm{grad} f(x)\cdot v$. Moreover the dependence of δ in terms of ε is uniform in $v \in \mathbb{S}^{n-1}$, hence f is differentiable at x with $Df(x) = \mathrm{grad} f(x)$. $\square$

The next Theorem shows that if f is a continuous function on a closed subset of $\mathbb{R}^n$ which admits a fist order Taylor expansion with a continuous first order linear term, then it can be extended to a function of class C^1. Let us introduce some preliminary notation.
Let $C \subset \mathbb{R}^n$ be a closed set, and let $f : C \to \mathbb{R}$, $d : C \to \mathbb{R}^n$. Define

$$R(x,y) := \frac{f(y) - f(x) - d(x)(y-x)}{|y-x|} \quad \text{for } x, y \in C, x \neq y.$$

$$\rho_K(\delta) := \sup\{|R(x,y)| : 0 < |x-y| < \delta, x, y \in K\},$$

where $K \subset C$ is compact, $\delta > 0$

2.13. Theorem (Whitney's extension theorem, see [2] Theorem 6.10). *Let f, d as above and continuous. Assume that for every compact set $K \subset C$ it holds*

$$\rho_K(\delta) \to 0 \quad \text{as } \delta \to 0.$$

Then there exists a function $\bar{f} : \mathbb{R}^n \to \mathbb{R}$ of class C^1 such that $\bar{f} = f$ and $D\bar{f} = d$ on the set C. Moreover if f is L-Lipschitz, one can choose $\bar{f}$ to be cL-Lipschitz, for some constant $c > 0$.

2.14. Theorem (Lusin type approximation of Lipschitz functions with C^1 functions). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz. Then for every $\varepsilon > 0$ there exist an open set A with $\mathscr{L}^n(A) < \varepsilon$ and a function $g : \mathbb{R}^n \to \mathbb{R}$ of class C^1 such that $g = f$ on $\mathbb{R}^n \setminus A$.*

Proof. Let $E \subset \mathbb{R}^n$ the set

$$E := \{x \in \mathbb{R}^n : f \text{ is not differentiable at } x\}.$$

By Theorem 2.12 it holds $\mathscr{L}^n(E) = 0$, moreover the function Df is measurable on $\mathbb{R}^n \setminus E$. Now fix $\varepsilon > 0$. By Lusin's theorem there exists an open set $A \supset E$ such that $\mathscr{L}(A) < \varepsilon$ and Df is continuous on $C := \mathbb{R}^n \setminus A$. Hence Df is uniformly continuous on any compact set $K \subset C$, which implies that $\rho_k(\delta) \to 0$ as $\delta \to 0$. The conclusion follows from Theorem 2.13. $\square$

2.15. EXERCISE.

(1) *Prove that Lipschitz functions are* weakly differentiable *i.e. for every* $f : \mathbb{R}^n \to \mathbb{R}$ *Lipschitz, there exist functions* $g_1, \ldots, g_n \in L^1_{loc}(\mathbb{R}^n)$ *such that*

$$\int_{\mathbb{R}^n} f \frac{\partial \phi}{\partial x_i} d\mathscr{L}^n = -\int_{\mathbb{R}^n} g_i \phi d\mathscr{L}^n, \quad \textit{for every } i = 1, \ldots, n.$$

(2) *Exhibit a function* $f : \mathbb{R} \to \mathbb{R}$ *which is differentiable almost everywhere but not weakly differentiable.*

3. Area Formula

The aim of this section is to recall the area formula for Lipschitz maps, which allows to compute the n dimensional Hausdorff measure of the image of a measurable set $E \subset \mathbb{R}^n$ through a Lipschitz map $f : \mathbb{R}^n \to \mathbb{R}^m$ $(m \geq n)$. We need to recall some further notions and results from measure theory and some facts from linear algebra.

A *Radon measure* μ is a *locally finite* measure which is *inner regular*, i.e. for every measurable set E with finite measure and for every $\varepsilon > 0$, there exists a compact set $K \subset E$ such that $\mu(E \setminus K) < \varepsilon$. Remember that on $\mathbb{R}^n$ every finite measure is a Radon measure. If λ and μ are Radon measures, we say that λ is *absolutely continuous* with respect to μ, and we write $\lambda \ll \mu$ if $\lambda(E) = 0$ whenever $\mu(E) = 0$. We say that λ is *supported* on E if $\lambda(E^c) = 0$ and we say that λ is *singular* with respect to μ, and we write $\lambda \perp \mu$, if λ is supported on a set E such that $\mu(E) = 0$. The *support* of λ is the set

$$\operatorname{supp}(\lambda) := \bigcap C \text{ closed: } \lambda \text{ is supported on } C.$$

Finally if $f \in L^1(\lambda)$ we write $f\lambda$ for the measure

$$(f\lambda)(E) := \int_E f d\lambda.$$

3.1. EXERCISE. *Find a Radon measure* μ *on* $\mathbb{R}^n$ *which is singular with respect to* $\mathscr{L}^n$ *but* $supp(\mu) = \mathbb{R}^n$.

Given two Radon measures λ, μ on $\mathbb{R}^n$ the function

$$f(x) := \lim_{\rho \to 0} \frac{\lambda(B(x,\rho))}{\mu(B(x,\rho))}$$

is defined and it is finite μ-a.e. and it is called the Radon-Nikodym density of λ with respect to μ.

3.2. Theorem (Besicovitch differentiation theorem, see [1] Theorem 2.22). *Let* λ *and* μ *be Radon measures. Then* λ *can be split in a unique way as* $\lambda = \lambda_a + \lambda_s$, *where* $\lambda_a \ll \mu$ *and* $\lambda_s \perp \mu$. *Moreover* $\lambda_a = f\mu$ *and* $\lambda_s = \mathbf{1}_E \lambda$, *where*

$$E := \operatorname{supp}(\mu)^c \cup \{x \in \operatorname{supp}(\mu) : f(x) = +\infty\}.$$

3.3. EXERCISE. *A Radon measure* μ *on* $\mathbb{R}^n$ *is called* α-uniform $(\alpha > 0)$ *if* $\mu(B(x,r)) = \omega_\alpha r^\alpha$ *for every* $x \in \operatorname{supp}(\mu)$ *and for every* $r > 0$. *Prove that there exists no* α*-uniform measure on* $\mathbb{R}^n$ *and that the only* α*-uniform measure on* $\mathbb{R}^n$ *for* $\alpha = n$ *is the Lebesgue measure* $\mathscr{L}^n$.

3.4. EXERCISE. *Prove the Lebesgue differentiation theorem. Let* μ *be a Radon measure and* $f \in L^1(\mu)$. *Then*

$$\lim_{rho \to 0} \frac{1}{\mu(B(x,\rho))} \int_{(B(x,\rho))} |f(y) - f(x)| d\mu(y) = 0, \quad \textit{for } \mu\textit{- a.e. } x.$$

(<u>Hint</u>: for every $q \in \mathbb{Q}$, apply Theorem 3.2 with $\lambda := |f - q|\mu$.*)*

A linear map:

- $O : \mathbb{R}^n \to \mathbb{R}^m$ is *orthogonal* if $\langle Ox; Oy \rangle = \langle x; y \rangle$ for every $x, y \in \mathbb{R}^n$;
- $S : \mathbb{R}^n \to \mathbb{R}^n$ is *symmetric* if $\langle x; Sy \rangle = \langle Sx; y \rangle$ for every $x, y \in \mathbb{R}^n$.

3.5. Theorem (Polar decomposition, see [2], Theorem 3.5). *Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be linear*

(i) *if $n \leq m$, then there exists $S : \mathbb{R}^n \to \mathbb{R}^n$ symmetric and $O : \mathbb{R}^n \to \mathbb{R}^m$ orthogonal such that $L = O \circ S$;*

(ii) *if $n \geq m$, then there exists $S : \mathbb{R}^m \to \mathbb{R}^m$ symmetric and $O : \mathbb{R}^m \to \mathbb{R}^n$ orthogonal such that $L = S \circ O^*$, where $O^* : \mathbb{R}^n \to \mathbb{R}^m$ is the* adjoint *of O, namely*

$$\langle x; O^* y \rangle = \langle Ox; y \rangle \qquad \textit{for every } x \in \mathbb{R}^m, y \in \mathbb{R}^n.$$

The previous theorem allow to give the following definition.

3.6. Definition. *Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be linear. Let S and O be as in Theorem 3.5. We define the* Jacobian *of L to be*

$$J_L := |\det(S)|.$$

From now on let us assume that $n \leq m$. We begin with the proof of the area formula for linear maps.

3.7. Lemma. *Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be linear. Then*

$$\mathscr{H}^n(L(A)) = J_L \mathscr{L}^n(A), \qquad \textit{for every Borel set } A.$$

Proof. Let $L = O \circ S$ ans in Theorem 3.5. If $J_L = 0$, then $\dim(S(\mathbb{R}^n)) \leq n - 1$, hence $\mathscr{H}^n(L(A)) = 0$. Assume now $J_L > 0$. We can compute, for every $x \in \mathbb{R}^n, r > 0$,

$$\frac{\mathscr{H}^n(L(B(x,r)))}{\mathscr{L}^n(B(x,r))} = \frac{\mathscr{L}^n(O^* \circ L(B(x,r)))}{\mathscr{L}^n(B(x,r))} = \frac{\mathscr{L}^n(S(B(x,r)))}{\mathscr{L}^n(B(x,r))} = \frac{\mathscr{L}^n(S(B(0,1)))}{\omega_n} = |\det S| = J_L,$$

where the first equality follows from the equality between $\mathscr{L}^n$ and $\mathscr{H}^n$ and from the fact that O^* is an isometry and the second inequality follows from the fact that $O^* = O^{-1}$.
Let us now define the measure ν, where

$$\nu(A) := \mathscr{H}^n(L(A)), \qquad \text{for every } A \text{ Borel.}$$

Clearly ν is a Radon measure, $\nu \ll \mathscr{L}^n$, and by the previous computation

$$\lim_{\rho \to 0} \frac{\nu(B(x,\rho))}{\mathscr{L}^n(B(x,\rho))} = J_L.$$

It follows from Theorem 3.2 that $\mathscr{H}^n(L(A)) = J_L \mathscr{L}^n(A)$, for every Borel set A. □

3.8. Lemma. *Let $D \subset \mathbb{R}^n$ be an open set and let $f : D \to f(D) \subset \mathbb{R}^m$ be a homeomorphism of class C^1 with $\mathrm{rk}(df) = n$ everywhere. Then for every $E \subset D$ Borel it holds*

$$\mathscr{H}^n(f(E)) = \int_E J_{df(x)} d\mathscr{L}^n(x).$$

Idea of the proof. Locally around every point f is the composition of a linear map and an "almost-isometry", hence the statement follows from Lemma 3.7 and the fact that isometries preserve the measure $\mathscr{H}^n$. □

3.9. EXERCISE. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a map of class C^1. Denote*

$$S := \{x \in \mathbb{R}^n : \mathrm{rk}(df) < n\}.$$

Prove that $\mathscr{H}^n(f(S)) = 0$.

Combining Theorem 2.14, Lemma 3.8 and Exercise 3.9, we obtain the main theorem of this section.

3.10. Theorem (Area Formula for Lipschitz functions, see [2] Theorem 3.8). *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz.*

(i) *For every Borel set $A \subset \mathbb{R}^n$, it holds*

$$\int_A J_{df(x)} d\mathscr{L}^n(x) = \int_{\mathbb{R}^m} N(f_{|A}, y) d\mathscr{H}^n(y),$$

where $N(f_{|A}, y) := \sharp\{x \in A : f(x) = y\}$.

(ii) *For every function $u \in L^1(\mathbb{R}^n)$ it holds*

$$\int_{\mathbb{R}^n} u(x) J_{df(x)} d\mathscr{L}^n(x) = \int_{\mathbb{R}^m} \sum_{x \in f^{-1}(y)} u(x) d\mathscr{H}^n(y).$$

CHAPTER 2

Rectifiable Sets

The aim of this chapter is to introduce and study the class of rectifiable sets, which is the largest class of sets for which one can define a (weak) notion of tangent bundle. Rectifiable sets are building blocks for the construction of rectifiable currents, which is the main object of study of this course.

1. Definitions and basic properties

1.1. Definition. *Let $0 \le k \le n$. A Borel set $E \subset \mathbb{R}^n$ is said to be* countably k-rectifiable *(or simply* k-rectifiable*) if $E \subset E_0 \cup (\bigcup_{j=1}^{\infty} E_j)$, where*

(i) *$\mathscr{H}^k(E_0) = 0$;*
(ii) *For $j \ge 1$, $E_j \subset F_j(\mathbb{R}^k)$, where $F_j : \mathbb{R}^k \to \mathbb{R}^n$ are Lipschitz maps.*

1.2. EXERCISE. *Prove that the definition of k-rectifiable set does not change if* (ii) *is replaced by one of the following properties.*

(ii)$'$ *For $j \ge 1$, $E_j \subset F_j(A_j)$ with $A_j \subset \mathbb{R}^k$ and F_j are Lipschitz;*
(ii)$''$ *For $j \ge 1$ $E_j \subset F_j(\mathbb{R}^k)$, where $F_j : \mathbb{R}^k \to \mathbb{R}^n$ are maps of class C^1;*
(ii)$'''$ *$E_j \subset M_j$, where M_j are k-dimensional submanifolds of $\mathbb{R}^n$ of class C^1.*

1.3. EXERCISE (Why E_0?). *Find a set $E \subset \mathbb{R}^2$ such that $\mathscr{H}^1(E) = 0$, but E_0 cannot be covered by countably many Lipschitz curves.*
(Hint: it is sufficient to prove that E cannot be covered by countably many graphs of type $(x, f(x))$ or $(y, f(y))$ (Why?). To find a set E with such property it is sufficient to take the product of two uncountable sets.*)*

1.4. Remark.

- The class of k-rectifiable sets is closed under countable union and set inclusion (in the class of Borel sets);
- k-rectifiable sets have Hausdorff dimension less than or equal to k.

1.5. Definition. *A Borel set $E \subset \mathbb{R}^n$ is said to be* k-purely unrectifiable *if $\mathscr{H}^k(E \cap F) = 0$ whenever F is k-rectifiable.*

1.6. Proposition. *Let $E \subset \mathbb{R}^n$ Borel, with $\mathscr{H}^k(E) < +\infty$. Then E can be written as a disjoint union $E = E_r \cup E_{pu}$, where E_r is k-rectifiable and E_{pu} is purely k-unrectifiable. Moreover the writing is unique, up to sets of $\mathscr{H}^k$ measure zero.*

Proof. Let $\{E_i\}_{i\in\mathbb{N}}$ be a sequence of k-rectifiable subsets of E, maximizing the quantity $\mathscr{H}^k(E_i)$. Set $E_r := \bigcup_{i\in\mathbb{N}} E_i$. Clearly E_r is rectifiable and maximizes $\mathscr{H}^k$ among all k-rectifiable subsets of E. Denote $E_{pu} := E \setminus E_r$. Then E_{pu} is k-purely unrectifiable. Indeed, assume the contrary; then there exists a k-rectifiable set E_∞ such that $\mathscr{H}^k(E_\infty \cap E_{pu}) > 0$. Then the set $E_r \cup (E_\infty \cap E_{pu}$ would violate the maximality property of E_r.
Regarding the uniqueness, assume that $E = E_r^1 \cup E_{pu}^1 = E_r^2 \cup E_{pu}^2$ are two distinct decompositions of E. Since E_{pu}^2 is k-purely unrectifiable, then $\mathscr{H}^k(E_r^1 \cap E_{pu}^2) = 0$. Then

$$\mathscr{H}^k(E_r^1 \cap E_r^2) = \mathscr{H}^k(E_r^1 \cap (E \setminus E_{pu}^2)) = \mathscr{H}^k(E_r^1 \cap E) - \mathscr{H}^k(E_r^1 \cap E_{pu}^2) = \mathscr{H}^k(E_r^1).$$

Similarly we can prove that $\mathscr{H}^k(E_r^1 \cap E_r^2) = \mathscr{H}^k(E_r^2)$. This implies that $\mathscr{H}^k(E_r^1 \Delta E_r^2) = 0$, which completes the proof. □

1.7. EXERCISE. *Prove the proposition when E is σ-finite wrt $\mathscr{H}^k$.*

1.8. EXERCISE (A bad rectifiable set with a tangent field). *Let $E := \bigcup_{(p,q)\in\mathbb{Q}^2\times\mathbb{Q}^2} S_{p,q}$, where $S_{p,q}$ is the line segment in the plane joining p and q. Notice that E is 1-rectifiable. Define*

$$\tau(x) := \operatorname{span}(q-p) \qquad \text{if } x \in S_{p,q}.$$

Prove that τ is well defined $\mathscr{H}^1$-almost everywhere.

1.9. Proposition. *Let $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$ be of class C^1. Then $Df_1 = Df_2$, $\mathscr{L}^n$-a.e. on the set $\{f_1 = f_2\}$. More precisely the set*

$$I := \{x \in \mathbb{R}^n : f_1(x) = f_2(x), \quad \text{and} \quad Df_1(x) \neq Df_2(x)\}$$

has Hausdorff dimension at most $n-1$.

Proof. Let $f := f_1 - f_2$. Observe that f is of class C^1, $f \equiv 0$ on I, and $Df \neq 0$ on I. Hence by the implicit function theorem I is an hypersurface of class C^1. □

In the next proposition we denote by $\operatorname{Gr}(k,n)$ the Grassmannian of k-dimensional vector subspaces of $\mathbb{R}^n$. This is a metric space, endowed with the distance

$$d(V,W) := \operatorname{dist}_{\mathscr{H}}(V \cap B(0,1), W \cap B(0,1)).$$

If S is a k-dimensional surface of class C^1, we denote by $\operatorname{Tan}(S,x)$ the classical tangent space of S at the point x.

1.10. Proposition (Existence of the weak tangent bundle). *Let $E \subset \mathbb{R}^n$ be k-rectifiable. Then there exists a Borel map $\tau_E : E \to \operatorname{Gr}(k,n)$ with the following property. For every k-dimensional surface S of class C^1 it holds*

$$\operatorname{Tan}(S,x) = \tau_E(x), \qquad \text{for } \mathscr{H}^k \text{ a.e. } x \in S \cap E.$$

Moreover the map τ_E is uniquely defined, up to sets of $\mathscr{H}^k$ measure zero.

Idea of the proof. Write $E = E_0 \cup (\bigcup_{i=1}^\infty E_i)$ with $\mathscr{H}^k(E_0) = 0$ and $E_i \subset S_i$ for $i \geq 1$, where S_i are k-dimensional surfaces of class C^1. Fix any k-pane V and set

$$\tau_e(x) := \begin{cases} \operatorname{Tan}(S_i,x), & \text{if } i \geq 1 \text{ is the first index such that } x \in E_i; \\ V, & \text{if } x \in E_0 \setminus \bigcup_{i=1}^\infty E_i. \end{cases}$$

□

1.11. EXERCISE. *Complete the proof.*

- *Use Proposition 1.9 to prove that τ_E has the claimed property;*
- *Prove the uniqueness.*

2. Characterizations of rectifiability

2.1. **Characterization through densities.** Let $E \subset \mathbb{R}^n$ be a set with locally finite $\mathscr{H}^s$ measure, for some $0 \leq s \leq n$. For every point $x \in E$ we define the *s-dimensional upper density* of x wrt E by

$$\theta^{*s}(x,E) := \limsup_{\rho\searrow 0} \frac{\mathscr{H}^s(E \cap B(x,\rho))}{\omega_s \rho^s}.$$

Similarly we define the *s-dimensional lower density* of x wrt E by

$$\theta^s_*(x,E) := \liminf_{\rho\searrow 0} \frac{\mathscr{H}^s(E \cap B(x,\rho))}{\omega_s \rho^s}.$$

When the two quantities coincide, we call the limit the *s-dimensional density* of x wrt E and we denote it by $\theta^s(x,E)$.

2.2. Theorem (Preiss, see [5] Theorem 16.7). *Let $E \subset \mathbb{R}^n$ and let $0 \le s \le n$ be such that $\mathscr{H}^s(E) < +\infty$ and $\theta^s(x, E)$ exists and belongs to $(0, +\infty)$ for $\mathscr{H}^s$-a.e. $x \in E$. Then $s \in \mathbb{N}$ and moreover E is s-rectifiable.*

2.3. EXERCISE. *Let $C \subset \mathbb{R}^2$ be the "Cantor dust", i.e. the set defined in Exercise 2.5. Prove that there are two constants $c_1, c_2 > 0$ such that for every $x \in C$ it holds*

$$c_1 \le \theta^1_*(x, E) \le \theta^{*1}(x, E) \le c_2.$$

A measure μ on a group $(G, \cdot)$ is said to be *invariant* if $\mu(A) = \mu(g \cdot A) = \mu(A \cdot g)$ for every measurable set A, where

$$g \cdot A := \{g \cdot a : a \in A\} \quad \text{and} \quad A \cdot g := \{a \cdot g : a \in A\}.$$

2.4. Characterization through projections.

2.5. Theorem (Haar measure, see [4] Theorem 3.1.1). *Let G be a compact topological group. Then there exists a unique invariant Radon measure μ on G such that $\mu(G) = 1$.*

In particular there exists an invariant probability measure on the orthogonal group O_n of linear isometries on $\mathbb{R}^n$. This induces a probability measure on the space $\mathrm{Gr}(k, n)$ which is invariant under the action of $O(n)$. We call such measure $\gamma_{k,n}$. Given $V \in \mathrm{Gr}(k, n)$, we denote by π_V the orthogonal projection from $\mathbb{R}^n$ onto V.

2.6. Theorem (Besicovitch-Federer projection theorem, see [5] Theorem 18.1). *Let $A \subset \mathbb{R}^n$ with $\mathscr{H}^k(A) < +\infty$*

(i) *A is k-rectifiable if and only if $\mathscr{H}^k(\pi_V(B)) > 0$ for $\gamma_{k,n}$-a.e. $V \in \mathrm{Gr}(k, n)$, whenever $B \subset A$ and $\mathscr{H}^k(B) > 0$.*
(i) *A is k-purely rectifiable if and only if $\mathscr{H}^k(\pi_V(A)) = 0$ for $\gamma_{k,n}$-a.e. $V \in \mathrm{Gr}(k, n)$.*

2.7. Remark. If $k = 1$ and $n = 2$ the expression "for $\gamma_{1,2}$-almost every $V \in \mathrm{Gr}(1, 2)$", can be replaced with "for every $V \in \mathrm{Gr}(1, 2)$ except at most one".

2.8. EXERCISE.

- *Use the previous remark to prove that if $A \subset \mathbb{R}^2$ is the product of two Lebesgue null sets, then A is 1-purely unrectifiable.*
- *Use the previous remark to prove that the "Cantor dust" C is 1-purely unrectifiable.*

3. Local behaviour of rectifiable sets

3.1. Measure inside/outside local cones.

3.2. Definition. *Let $V \in \mathrm{Gr}(k, n)$, $x \in \mathbb{R}^n$, and $\alpha > 0$. We denote by $C(x, V, \alpha)$ the closed cone*

$$C(x, V, \alpha) := x + \{y \in \mathbb{R}^n : \mathrm{dist}(y, V) \le \alpha|y|\}.$$

3.3. Theorem (Local behaviour of rectifiable sets). *Let $E \subset \mathbb{R}^n$ be k-rectifiable, with $\mathscr{H}^k(E) < +\infty$ (locally finiteness would suffice). Let τ_E be the weak tangent bundle defined in Proposition 1.10. Then for any $\alpha > 0$ and for $\mathscr{H}^k$-a.e. $x \in E$ the following holds:*

(i) $\mathscr{H}^k[E \cap B(x, r) \cap C(x, \tau_E(x), \alpha)] = \omega_k r^k + o(r^k)$.
(ii) $\mathscr{H}^k[E \cap (B(x, r) \setminus C(x, \tau_E(x), \alpha))] = o(r^k)$.

3.4. EXERCISE.

(1) *Prove the theorem for a set $E \subset \mathbb{R}^2$ which is a finite union of segments.*
(2) *$(*)$ Prove the theorem for a set $E \subset \mathbb{R}^2$ which is a countable union of segments S_i with $\sum \mathscr{H}^1(S_i) < +\infty$.*

Proof of Theorem 3.3. For every $i \in \mathbb{N}$ let S_i be k-dimensional surfaces of class C^1 such that $\mathscr{H}^k(E \setminus \bigcup_{i\in\mathbb{N}} S_i) = 0$. Denote $E_i := S_i \cap E$. By Proposition 1.10 it is sufficient to prove that for every $i \in \mathbb{N}$ (i) and (ii) hold, for $\mathscr{H}^k$-a.e. $x \in E_i$, where $\tau_E(x)$ is replaced by $\mathrm{Tan}(S_i, x)$. Fix $i \in \mathbb{N}$ and define the following three measures

$$\mu := \mathscr{H}^k \llcorner S_i, \quad \lambda_1 := \mathscr{H}^k \llcorner E_i, \quad \lambda_2 := \mathscr{H}^k \llcorner (E \setminus E_i),$$

and observe that $\mu \perp \lambda_2$, because $E \setminus E_i = E \setminus S_i$. Since S_i is a k-surface of class C^1, it holds

$$\mu(B(x,r)) = \omega_k r^k + o(r^k), \quad \text{for every } x \in S_i. \tag{3.1}$$

Moreover, since $\lambda_1 \ll \mu$ (because $E_i \subset S_i$), by Theorem 3.2 it holds

$$\lim_{\rho\to 0} \frac{\lambda_1(B(x,\rho))}{\mu(B(x,\rho))} = \mathbf{1}_{E_i}(x), \quad \text{for } \mu\text{-a.e. } x, \text{ i.e. for } \mathscr{H}^k\text{-a.e. } x \in S_i.$$

In particular, by (3.1) it holds

$$\lambda_1(B(x,r)) = \omega_k r^k + o(r^k), \quad \text{for } \mathscr{H}^k\text{-a.e. } x \in E_i. \tag{3.2}$$

Since $\mu \ll \lambda_2$, again by Theorem 3.2 it holds

$$\lim_{\rho\to 0} \frac{\lambda_2(B(x,\rho))}{\mu(B(x,\rho))} = 0, \quad \text{for } \mu\text{-a.e. } x, \text{ i.e. for } \mathscr{H}^k\text{-a.e. } x \in S_i.$$

In particular, by (3.1), we have

$$\lambda_2(B(x,r)) = o(r^k), \quad \text{for } \mathscr{H}^k\text{-a.e. } x \in E_i. \tag{3.3}$$

Lastly, observe that, since S_i is a k-dimensional surface of class C^1, for every $\alpha > 0$ and for every $x \in S_i$ there exists $r_0 = r_0(\alpha, x)$ such that for every $r \le r_0$ it holds

$$S_i \cap B(x,r) \subset C(x, \mathrm{Tan}(S_i,x), \alpha),$$

and in particular, since $E_i \subset S_i$,

$$E_i \cap B(x,r) \subset C(x, \mathrm{Tan}(S_i,x), \alpha). \tag{3.4}$$

Now, to obtain (i) we can compute

$$\begin{aligned}
&\mathscr{H}^k[E \cap B(x,r) \cap C(x, \mathrm{Tan}(S_i,x),\alpha)] \\
&= \mathscr{H}^k[E_i \cap B(x,r) \cap C(x, \mathrm{Tan}(S_i,x),\alpha)] + \mathscr{H}^k[(E\setminus E_i) \cap B(x,r) \cap C(x, \mathrm{Tan}(S_i,x),\alpha)] \\
&= \lambda_1[B(x,r) \cap C(x, \mathrm{Tan}(S_i,x),\alpha)] + \lambda_2[B(x,r) \cap C(x, \mathrm{Tan}(S_i,x),\alpha)] \\
&\overset{(3.4)}{=} \lambda_1(B(x,r)) + \lambda_2[B(x,r) \cap C(x, \mathrm{Tan}(S_i,x),\alpha)] \\
&\overset{(3.2)}{=} \omega_k r^k + o(r^k) + \lambda_2[B(x,r) \cap C(x, \mathrm{Tan}(S_i,x),\alpha)] \le \omega_k r^k + o(r^k) + \lambda_2(B(x,r)) \\
&\overset{(3.3)}{=} \omega_k r^k + o(r^k).
\end{aligned}$$

□

Regarding (ii), we can compute

$$\begin{aligned}
&\mathscr{H}^k[E \cap (B(x,r) \setminus C(x, \mathrm{Tan}(S_i,x),\alpha))] \\
&= \mathscr{H}^k[E_i \cap (B(x,r) \setminus C(x, \mathrm{Tan}(S_i,x),\alpha))] + \mathscr{H}^k[(E\setminus E_i) \cap (B(x,r) \setminus C(x, \mathrm{Tan}(S_i,x),\alpha))] \\
&= \lambda_1[B(x,r) \setminus C(x, \mathrm{Tan}(S_i,x),\alpha)] + \lambda_2[B(x,r) \setminus C(x, \mathrm{Tan}(S_i,x),\alpha)] \\
&\overset{(3.3)}{=} \lambda_2[B(x,r) \setminus C(x, \mathrm{Tan}(S_i,x),\alpha)] \le \lambda_2(B(x,r)) \overset{(3.3)}{=} o(r^k).
\end{aligned}$$

3.5. **Blow-up properties of the measure.** We recall some classical results about Radon measures. Given a locally compact, separable metric space (X, d), we denote $\mathcal{K}(X)$ the vector space of continuous functions with compact support, endowed with the strongest locally convex topology such that the inclusions $\mathcal{K}(K) \to \mathcal{K}(X)$ are continuous for every compact set $K \subset X$ and on $\mathcal{K}(K)$ we consider the supremum distance. The space $\mathcal{K}(X)$ can be also endowed with the supremum norm, and in this case we denote by $\overline{\mathcal{K}}(X)$ its completion.

3.6. EXERCISE. *Prove that $u \in \overline{\mathcal{K}}(X)$ if and only if u is continuous on X and for every $\varepsilon > 0$ there exists a compact set K such that $|u(x)| < \varepsilon$ for every $x \in X \setminus K$.*

3.7. Theorem (Riesz representation theorem, see [1] Theorem 1.54). *Let X be a locally compact, separable metric space and let $L : \mathcal{K}(X) \to \mathbb{R}$ be a linear functional which is positive (i.e. $L(f) \geq 0$ whenever $f \geq 0$) and continuous, with respect to the topology described above. Then there exists a unique Radon measure μ on X such that*

$$L(f) = \int_X f d\mu, \qquad \text{for every } f \in \mathcal{K}(X).$$

3.8. EXERCISE. *Prove the vice-versa: the linear functional L_μ induced by a Radon measure μ via integration is continuous and positive. Does continuity hold also with respect to the topology induced by the supremum distance on $\mathcal{K}(X)$, if X is only locally compact?*

3.9. Remark. The Riesz theorem can be restated by saying that the dual of the space $\mathcal{K}(X)$ is the space of (locally finite) Radon measures. A "global" version of the theorem states the dual of the space $\overline{\mathcal{K}}(X)$ is the space of finite Radon measures.

Application of Banach-Alaoglu theorem gives the following compactness result. A sequence of measures $\{\mu_j\}_{j\in\mathbb{N}}$ on a locally compact metric space (X, d) is said to be *uniformly locally finite* if for every $K \subset X$ compact there exists a constant $C = C(K)$ such that $\mu_j(K) \leq C$ for every $j \in \mathbb{N}$.

3.10. Theorem (Compactness of Radon measures, see [1] Theorem 1.59). *Let $\{\mu_j\}_{j\in\mathbb{N}}$ be a uniformly locally finite sequence of measures on a locally compact, separable metric space (X, d). Then there exists a subsequence $\{\mu_{j_k}\}_{k\in\mathbb{N}}$ and a Radon measure μ such that $\mu_{j_k} \overset{*}{\rightharpoonup} \mu$, where the latter means that*

$$\lim_{k\to\infty} \int_X f d\mu_{j_k} = \int_X f d\mu, \qquad \text{for every } f \in \mathcal{K}(X).$$

3.11. EXERCISE. *Find a sequence of probability measures on $\mathbb{R}^n$ which does not converge to a probability measure.*

3.12. Proposition (Useful properties of weak* convergence). *Let $\{\mu_j\}_{j\in\mathbb{N}}$ and μ be measures on a locally compact metric space (X, d), such that $\mu_j \overset{*}{\rightharpoonup} \mu$. Then:*

(1) *for every lower semicontinuous function $f : X \to \mathbb{R}^+$ it holds*
$$\liminf_{j\to\infty} \int_X f d\mu_j \geq \int_X f d\mu;$$
(2) *for every upper semicontinuous function $f : X \to \mathbb{R}^+$ with compact support it holds*
$$\limsup_{j\to\infty} \int_X f d\mu_j \leq \int_X f d\mu;$$
(3) *for every open set U it holds*
$$\liminf_{j\to\infty} \mu_j(U) \geq \mu(U);$$
(4) *for every compact set K it holds*
$$\limsup_{j\to\infty} \mu_j(K) \geq \mu(K);$$
(5) $\mu(A) = \lim_{j\to\infty} \mu_j(A)$ *for every A s.t. $\mu(\partial A) = 0$;*

(6) *if $X = \mathbb{R}^n$, then for every $x \in \mathbb{R}^n$ there exists I_x at most countable such that $\mu(B(x,r)) = \lim_{j\to\infty} \mu_j(B(x,r))$ for every $r \notin I_x$.*

3.13. EXERCISE. $(*)$ *Prove Proposition 3.12.*

3.14. EXERCISE.

(1) *Prove that if K is compact then every converging sequence μ_j of probability measures on K converges to a probability measure on K.*

(2) *Prove that if μ_j is a sequence of probability measures on $\mathbb{R}^n$ and for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ and $j_0 \in \mathbb{N}$ such that $\mu_j(\mathbb{R}^n \setminus B(0,N)) \leq \varepsilon$, for every $j \geq j_0$, then μ_j converges to a probability measure, up to subsequences.*

Let $x \in \mathbb{R}^n$ and $r > 0$. Let $\psi_{x,r}$ be the homothety which maps $B(x,r)$ onto $B(0,1)$, i.e. $\psi_{x,r}(y) = \frac{y-x}{r}$. Given a set $E \subset \mathbb{R}^n$ let $E_{x,r} := B(0,1) \cap \psi_{x,r}(E)$. Denote $\mu_{x,r} := \mathscr{H}^k \llcorner E_{x,r}$.

3.15. Definition. *We say that $V \in \mathrm{Gr}(k,n)$ is the* approximate tangent k-plane to E at x *if*

$$\mu_{x,r} \stackrel{*}{\rightharpoonup} \mathscr{H}^k \llcorner (V \cap B(0,1)).$$

3.16. Theorem. *Let $E \subset \mathbb{R}^n$ be such that $\mathscr{H}^k(E) < \infty$. Then E is k-rectifiable if and only if the approximate tangent plane to E at x exists for $\mathscr{H}^k$-a.e. $x \in E$.*

Proof. The proof of the "only if" part is very similar to that of Theorem 3.3. Let $\{S_i\}_{i\in\mathbb{N}}$ be a sequence of k-dimensional submanifolds of class C^1 such that $\mathscr{H}^k(E \setminus \bigcup_{i\in\mathbb{N}} S_i) = 0$. Fix $i \in \mathbb{N}$ and denote

$$\lambda_{x,r} := \mathscr{H}^k \llcorner (S_i)_{x,r}, \quad \lambda'_{x,r} := \mathscr{H}^k \llcorner (S_i \setminus E)_{x,r}, \quad \text{and} \quad \lambda''_{x,r} := \mathscr{H}^k \llcorner (E \setminus S_i)_{x,r}.$$

Observe that $\mathbf{1}_{E_{x,r}} = \mathbf{1}_{(S_i)_{x,r}} - \mathbf{1}_{(S_i\setminus E)_{x,r}} + \mathbf{1}_{(E\setminus S_i)_{x,r}}$, hence

$$\mu_{x,r} = \mathscr{H}^k \llcorner (E_{x,r}) = \lambda_{x,r} - \lambda'_{x,r} + \lambda''_{x,r}.$$

We split the proof in the following three claims.

Claim 1: $\lambda_{x,r} \stackrel{*}{\rightharpoonup} \mathscr{H}^k \llcorner (\mathrm{Tan}(S_i,x) \cap B(0,1))$, as $r \to 0$. The proof is left as an exercise.
Claim 2: $\lambda'_{x,r} \stackrel{*}{\rightharpoonup} 0$, as $r \to 0$. Actually one can prove more, namely that $\lambda'_{x,r}(B(0,1)) \to 0$. Indeed

$$\begin{aligned}\lambda'_{x,r}(B(0,1)) &= \mathscr{H}^k((S_i \setminus E)_{x,r}) = r^{-k}\mathscr{H}^k((S_i \setminus E) \cap B(x,r)) \\ &= \frac{\mathscr{H}^k(S_i \cap B(x,r))}{r^k} - \frac{\mathscr{H}^k(S_i \cap E \cap B(x,r))}{r^k} \to 0 \quad \text{as } r \to 0,\end{aligned}$$

because the two terms both converge to ω_k (see (3.1)).
Claim 3: $\lambda''_{x,r} \stackrel{*}{\rightharpoonup} 0$, as $r \to 0$. Again one can prove that $\lambda''_{x,r}(B(0,1)) \to 0$. Indeed

$$\lambda''_{x,r}(B(0,1)) = \mathscr{H}^k((E \setminus S_i)_{x,r}) = r^{-k}\mathscr{H}^k((E \setminus S_i) \cap B(x,r)) \to 0 \quad \text{as } r \to 0,$$

(see (3.3)).

Now we prove the "if" part. Denote

$$E^1 := \{x \in E : \text{ there exists an apporximate tangent } k\text{-plane at } x, \text{ called } V_x\}.$$

By assumption, $\mathscr{H}^k(E\setminus E^1) = 0$, hence it suffices to prove that E^1 is k-rectifiable. Let $m := n-k$ and let $V_1^\perp, \dots, V_N^\perp$ (where N depends on n and k) be m-dimensional vector subspaces such that for every $W \in \mathrm{Gr}(m,n)$ there holds $d(W, V_j^\perp) < \frac{1}{16}$ for at least one index j. Denote

$$E_j^1 := \{x \in E^1 : d(V_x^\perp, V_j^\perp) < \frac{1}{16}\}.$$

Since $E^1 = \bigcup_j E_j^1$ it suffices to prove that each E_j^1 is k-rectifiable. Observe that, by definition of weak tangent k-plane, for every $x \in E^1$ we have that for every $\varepsilon > 0$ there exists $\delta = \delta(x,\varepsilon)$ such that

$$\frac{\mathscr{H}^k(E^1 \cap B(x,\rho))}{\omega_k \rho^k} \geq 1 - \varepsilon, \qquad \text{for every } \rho < \delta \tag{3.5}$$

and
$$\frac{\mathscr{H}^k(E^1 \cap B(x,\rho) \cap C(x, V_x^\perp, \frac{1}{2}))}{\omega_k \rho^k} \leq \varepsilon, \qquad \text{for every } \rho < \delta. \tag{3.6}$$
Moreover, by Egorov's theorem, there exists $E^2 \subset E^1$ such that $\mathscr{H}^k(E^1 \setminus E^2) \leq \frac{1}{2}\mathscr{H}^k(E^1)$ and (3.5) and (3.6) hold uniformly on E^2, i.e. δ can be chosen depending only on ε (and independent of $x \in E^2$). Let $E_j^2 := E^2 \cap E_j^1$. The proof boils down to the following claim.

Claim 4: There exists $\varepsilon_0 > 0$ such that if $\delta_0 := \delta(\varepsilon_0)$ is as above, i.e. (3.5) and (3.6) hold for every $x \in E^2$, then
$$B(x, \frac{\delta_0}{2}) \cap E_j^2 \cap C(x, V_j^\perp, \frac{1}{4}) = \{x\}, \qquad \text{for every } x \in E_j^2, \text{ for every } j = 1, \dots, N.$$
We leave the proof of the claim as an exercises (guided). Now, by a straightforward modification of Exercise 2.2, we conclude that for every $j = 1, \dots, N$ and for every $x \in E_j^2$, the set $B(x, \frac{\delta_0}{4}) \cap E_j^2$ is contained in the graph of a Lipschitz function on V_j. The prove can be completed via a standard iteration argument. □

3.17. EXERCISE. *Prove Claim 1 in the proof of Theorem 3.16.*
(Hint: To prove it it is sufficient to write, for r small enough, $S_i \cap B(x,r)$ as the intersection between $B(x,r)$ and the graph of a C^1-function $f_i : \mathrm{Tan}(S - i, x) \to \mathbb{R}^{n-k}$ with small Lipschitz constant. The action of $\lambda_{x,r}$ on a test function can be computed via the Area formula.*)*

3.18. EXERCISE. *Prove Claim 4 in the proof of Theorem 3.16.*
(Hint: If there was a point $y \neq x$ in the set $B(x, \frac{\delta_0}{2}) \cap E_j^2 \cap C(x, V_j^\perp, \frac{1}{4})$, then at scale $\rho := \frac{|y-x|}{4}$ we would have, by (3.5),
$$\mathscr{H}^k(E^1 \cap B(y,\rho)) \geq C\omega_k \rho^k, \tag{3.7}$$
for some $C > 0$, but since
$$y \in B(x, \frac{\delta_0}{2}) \cap C(x, V_j^\perp, \frac{1}{4}),$$
then
$$B(y,\rho) \subset B(x, 8\rho) \cap C(x, V_x^\perp, \frac{1}{2}),$$
(check it!) hence (3.7) would contradict (3.6).*)*

3.19. EXERCISE. *Complete the proof of Theorem 3.16 (why is it sufficient to prove that each E_j^2 is k-rectifiable?).*

3.20. **Area formula for rectifiable sets.** We begin with the following proposition

3.21. Proposition (see [3] 3.2.19). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz and $E \subset \mathbb{R}^n$ be k-rectifiable. Then for $\mathscr{H}^k$-a.e. $x \in E$ there exists a linear map $d_\tau f(x) : \tau_E(x) \to \mathbb{R}^n$, called* tangential differential*, such that*
$$f(x+h) = f(x) + \langle d_\tau f(x); h\rangle + o(|h|), \quad , \text{ for every } h \in \tau_E(x).$$

Note that it is sufficient to prove the proposition when E is a k-dimensional submanifold of class C^1, in which case the proposition can be proved parametrizing locally the submanifold by a function $\phi : \mathbb{R}^k \to \mathbb{R}^n$ of class C^1. A similar statement holds for Lipschitz maps $f : \mathbb{R}^n \to \mathbb{R}^m$, just applying the previous proposition to each component of f.

3.22. EXERCISE. *Prove that the tangential differential $d_\tau f$ depends only on the restriction of f to E. More precisely if $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$ satisfy $f_1(x) = f_2(x)$ for $\mathscr{H}^k$-a.e. $x \in E$, then $d_\tau f_1(x) = d_\tau f_2(x)$, for $\mathscr{H}^k$-a.e. $x \in E$.*

3.23. Proposition. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz and $E \subset \mathbb{R}^n$ be k-rectifiable. Then $f(E)$ is k-rectifiable and for $\mathscr{H}^k$-a.e. $x \in E$ there holds*
$$d_\tau f(x)(\tau_E(x)) \subset \tau_{f(E)}(f(x)).$$

3.24. Remark. Observe that if $E \subset \mathbb{R}^n$ is a Borel set and $f : \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz function, $f(E)$ might fail in general to be Borel. Hence when we say that $f(E)$ is k-rectifiable we just mean that it can be covered, up to an $\mathscr{H}^k$-negligible subset, with countably many k-submanifolds of class C^1. Nevertheless $f(E)$ can be written as the union of a Borel set and an $\mathscr{H}^k$-null set.

3.25. EXERCISE. *Prove Proposition 3.23, using the following lemma.*

3.26. Lemma. *Let $S_1 \subset \mathbb{R}^n, S_2 \subset \mathbb{R}^m$ be two k-dimensional surfaces of class C^1 and let $E \subset S_1$. Let $f : S^1 \to \mathbb{R}^m$ be of class C^1 and satisfying $f(E) \subset S_2$. Then $d_\tau f(x)$ maps $\mathrm{Tan}(S_1, x)$ into $\mathrm{Tan}(S_2, f(x))$, for $\mathscr{H}^k$-a.e. $x \in E$.*

3.27. EXERCISE. *Prove Lemma 3.26.*

We can now state the version of the area formula for rectifiable sets. The *tangential Jacobian* $J_{d_\tau f}(x)$ is defined as for the smooth case.

3.28. Theorem (Area formula for rectifiable sets, see [3] 3.2.22). *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz and $E \subset \mathbb{R}^n$ be k-rectifiable.*

(i) *For every Borel set $A \subset E$, it holds*

$$\int_A J_{df(x)} d\mathscr{H}^k(x) = \int_{\mathbb{R}^m} N(f_{|A}, y) d\mathscr{H}^k(y),$$

where $N(f_{|A}, y) := \sharp\{x \in A : f(x) = y\}$.

(ii) *For every Borel function $u : E \to [0, \infty)$ it holds*

$$\int_E u(x) J_{d_\tau f(x)} d\mathscr{H}^k(x) = \int_{\mathbb{R}^m} \sum_{x \in f^{-1}(y) \cap E} u(x) d\mathscr{H}^k(y).$$

CHAPTER 3

Currents

1. Prerequisites from multilinear algebra

1.1. **Vectors, covectors, and differential forms.** Let us denote by $\{\mathbf{e_1}, \dots, \mathbf{e_n}\}$ an orthonormal basis of $\mathbb{R}^n$ and by $\{dx_1, \dots, dx_n\}$ its dual basis.

1.2. Definition (k-covectors). *A* k-covector *on* $\mathbb{R}^n$ *is a function* $t : (\mathbb{R}^n)^k \to \mathbb{R}$ *with the following properties:*

(i) t *is multilinear, i.e.*

$$t(v_1, \dots, c_1 v_i + c_2 w_i, \dots, v_k) = c_1 t(v_1, \dots, v_i, \dots, v_k) + c_2 t(v_1, \dots, w_i, \dots, v_k);$$

(ii) t *is alternating. i.e.*

$$t(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = -t(v_1, \dots, v_j, \dots, v_i, \dots, v_k).$$

The space of k*-covectors in* $\mathbb{R}^n$ *is denoted* $\Lambda^k(\mathbb{R}^n)$*, with the convention* $\Lambda^0(\mathbb{R}^n) := \mathbb{R}$.

1.3. EXERCISE. *Prove that the definition of* k*-covector is unchanged if (ii) is replaced by*

(ii)$'$ $t(\mathbf{e}_{\sigma(\mathbf{1})}, \dots, \mathbf{e}_{\sigma(\mathbf{k})}) = \mathrm{sign}(\sigma) t(\mathbf{e_1}, \dots, \mathbf{e_k})$, *for every permutation* $\sigma \in S^k$;
(ii)$''$ $t(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \mathrm{sign}(\sigma) t(v_1, \dots, v_k)$, *for every permutation* $\sigma \in S^k$;
(ii)$'''$ $t(v_1, \dots, v_k) = 0$, *if* $v_i = v_j$ *for some* $i \neq j$;
(ii)$''''$ $t(v_1, \dots, v_k) = 0$, *if* $v_1, \dots, v_k$ *are linearly dependent.*

Deduce that $\Lambda^k(\mathbb{R}^n) = \{0\}$*, for every* $k > n$.

1.4. Definition (exterior product). *Given* $\alpha \in \Lambda^h(\mathbb{R}^n), \beta \in \Lambda^k(\mathbb{R}^n)$*, we define their* exterior product $\alpha \wedge \beta \in \Lambda^{h+k}(\mathbb{R}^n)$ *as*

$$(\alpha \wedge \beta)(v_1, \dots, v_{h+k}) := \frac{1}{h!k!} \sum_{\sigma \in S^{h+k}} \alpha(v_{\sigma(1)}, \dots, v_{\sigma(h)}) \beta(v_{\sigma(h+1)}, \dots, v_{\sigma(h+k)}).$$

1.5. EXERCISE. *Prove that* $\alpha \wedge \beta$ *is actually a covector. Prove also that*

$$(dx_{i_1} \wedge dx_{i_k})(\mathbf{e_{i_1}}, \dots, \mathbf{e_{i_k}}) = 1, \qquad \textit{whenever } i_l \neq i_m, \textit{ for } l \neq m.$$

1.6. EXERCISE. *Let* α *be the 2-covector in* $\mathbb{R}^{2n}$ *defined by* $\alpha := \sum_{i=1}^n dx_{2i-1} \wedge dx_{2i}$*. Compute*

$$\underbrace{\alpha \wedge \dots \wedge \alpha}_{n \textit{ times}}.$$

We denote by $I_{k,n}$ the set of *multiindices* of length k in $\mathbb{R}^n$, i.e. sequences of the form

$$I = (i_1, \dots, i_k), \qquad \text{with } 1 \leq i_1 < \dots < i_k \leq n.$$

For every $I \in I_{k,n}$, we also denote $dx_I := dx_{i_1} \wedge, \dots, \wedge dx_{i_k}$.

1.7. Theorem. *The collection* $\{dx_I\}_{I \in I_{k,n}}$ *is a basis for* $\Lambda^k(\mathbb{R}^n)$*. Moreover, for every* $\alpha \in \Lambda^k(\mathbb{R}^n)$ *one can write* $\alpha = \sum_{I \in I_{k,n}} \alpha_I dx_I$*, where*

$$\alpha_I := \alpha(\mathbf{e_{i_1}}, \dots, \mathbf{e_{i_k}}), \qquad \textit{for every } I = (i_1, \dots, i_k).$$

1.8. Definition (simple k-vectors). *Consider the following equivalent relation on $(\mathbb{R}^n)^k$. We say that*

$$(v_1, \ldots, v_k) \sim (w_1, \ldots, w_k) \quad \text{if} \quad \alpha(v_1, \ldots, v_k) = \alpha(w_1, \ldots, w_k), \quad \text{for every } \alpha \in \Lambda^k(\mathbb{R}^n).$$

The equivalence classes are called simple k-vectors.

1.9. Proposition. *Assume $(v_1, \ldots, v_k) \sim (w_1, \ldots, w_k) \not\sim (0, \ldots, 0)$. Then it holds*

$$W := \operatorname{span}(v_1, \ldots, v_k) = \operatorname{span}(w_1, \ldots, w_k).$$

Moreover if M denotes the matrix of change of base between $(v_1, \ldots, v_k)$ and $(w_1, \ldots, w_k)$ on W (i.e. $w_i = \sum_j M_{i,j} v_j$), then $\det(M) = 1$.

Notice that the set of simple k-vectors is not a linear space. Nonetheless we define the following "norm" on it. We define

$$\|(v_1, \ldots, v_k)\| := J_L, \quad \text{where } L : \mathbb{R}^k \to \mathbb{R}^n \text{ is the matrix with columns } v_1, \ldots, v_k.$$

With this definition, Proposition 1.9 can be read as follows.

1.10. Proposition. *Simple k-vectors of unit norms are in bijection with oriented k-planes.*

1.11. Definition (differential forms). *Let $\Omega \subset \mathbb{R}^n$ be an open set. A* differential k-form on Ω *is a map $\omega : \mathbb{R}^n \to \Lambda^k(\mathbb{R}^n)$. We can write the form omega "in coordinates" as*

$$\omega(x) = \sum_{I \in I_{k,n}} \omega_I(x) dx_I.$$

We say that the form ω is of class C^j (or smooth) if all ω_I's are so.

1.12. Definition (exterior derivative). *Let $\omega(x) = \sum_{I \in I_{k,n}} \omega_I(x) dx_I$. be a differential k-form of class C^1. We denote $d\omega$ the differential $(k+1)$-form*

$$d\omega(x) := \sum_{I \in I_{k,n}} d\omega_I(x) \wedge dx_I, \qquad \text{where } d\omega_I(x) := \textstyle\sum_{i=1}^n \frac{\partial \omega_I}{\partial x_i}(x) dx_i.$$

1.13. **Stokes' theorem.** Let S be a k-dimensional surface in $\mathbb{R}^n$ of class C^1. An *orientation* of S is a continuous map which Associates with every point $x \in S$ a simple k-vector which "spans" the tangent k-plane $\operatorname{Tan}(S, x)$.

If S is has boundary ∂S which is of class C^1, the *orientation of the boundary* is defined as follows. Let $(v_1, \ldots, v_k)$ be an orientation of S and let η be the exterior normal to S at ∂S. The induced orientation on the boundary is a simple $(k-1)$-vector $(w_1, \ldots, w_{k-1})$, satisfying

$$(\eta, w_1, \ldots, w_{k-1}) \sim (v_1, \ldots, v_k).$$

Let S be a k-dimensional surface in $\mathbb{R}^n$ of class C^1, oriented by a simple vectorfield τ, and let ω be a differential k-form, defined in a neighbourhood of S. Then we define

$$\int_S \omega := \int_S \omega(x)(\tau(x)) d\mathscr{H}^k(x).$$

1.14. Theorem (Stokes, see [4] Theorem 6.2.11). *S be a compact, oriented, k-dimensional surface in $\mathbb{R}^n$ of class C^1, and let ϕ be a differential $(k-1)$-form of class C^1, defined in a neighbourhood of S. Assume that ∂S is of class C^1. Then we have*

$$\int_{\partial S} \phi = \int_S d\phi,$$

where the orientation on ∂S is that induced by the orientation of S.

1.15. **Mass and comass.** Observe that up to now we never defined general k-vectors, but only simple k-vectors. This can be done analogously to what we did for k-covectors, via the identification between $(\mathbb{R}^n)^*$ and $\mathbb{R}^n$. In particular a basis for the space $\Lambda_k(\mathbb{R}^n)$ of k-vectors is the collection

$$\{\mathbf{e_I} := \mathbf{e_{i_1}} \wedge \cdots \wedge \mathbf{e_{i_k}}\}_{I \in I_{k,n}}$$

and its dual basis is $\{dx_I\}_{I \in I_{k,n}}$.

1.16. Proposition. *The space $\Lambda^k(\mathbb{R}^n)$ can be identified with the dual of $\Lambda_k(\mathbb{R}^n)$, via the pairing*

$$\langle \alpha, \mathbf{e_{i_1}} \wedge \cdots \wedge \mathbf{e_{i_k}} \rangle := \alpha(\mathbf{e_{i_1}}, \ldots, \mathbf{e_{i_k}}),$$

which extends to every k-vector by linearity.

1.17. EXERCISE.

(1) *Prove that a k-vector v is simple if and only if it can be written in the form $v = v_1 \wedge \cdots \wedge v_k$ for some $v_1, \ldots, v_k \in \mathbb{R}^n$;*
(2) *Prove that $v = \mathbf{e_1} \wedge \mathbf{e_2} + \mathbf{e_3} \wedge \mathbf{e_4} \in \Lambda_2(\mathbb{R}^4)$ is not simple.*

1.18. Definition. *Given $\alpha \in \Lambda^k(\mathbb{R}^n)$, we define its* comass norm *as*

$$\|\alpha\| := \sup\{\langle \alpha; v\rangle : v \textit{ is a simple vector with } \|v\| \leq 1\}.$$

Denote by $|w|$ the Euclidean norm of a k-vector $w \in \Lambda_k(\mathbb{R}^n)$. We define the mass norm *of a k-vector v as*

$$\|v\| := \inf\left\{\sum_{i=1}^N t_i|v_i| : v = \sum_{i=1}^N t_i v_i \textit{ is a convex combination}\right\}.$$

1.19. EXERCISE.

(1) *Prove that $\|v\| \geq |v|$ for every $v \in \Lambda_k(\mathbb{R}^n)$, and equality holds if v is simple;*
(2) *Prove that $\|\alpha\| \leq |\alpha|$ for every $\alpha \in \Lambda^k(\mathbb{R}^n)$ and equality holds if and only if α is simple (i.e. $\alpha = \phi_1 \wedge \cdots \wedge \phi_k$, for some $\phi_1, \ldots, \phi_k \in (\mathbb{R}^n)^*$).*

2. Currents

2.1. **Convergence on the space of k-forms.** Let $\mathscr{D}(\mathbb{R}^n)$ be the space of smooth, compactly supported (test) functions on $\mathbb{R}^n$. We consider the following notion of convergence on $\mathscr{D}(\mathbb{R}^n)$. We say that a sequence $\{\phi_i\}_{i\in\mathbb{N}}$ converges to ϕ, and we write $\phi_i \to \phi$ if

(i) there exists a compact set $K \subset \mathbb{R}^n$ such that every ϕ_i has support contained in K;
(ii) for every multiindex α it holds

$$\partial^\alpha \phi_i \to_{i\to\infty} \partial^\alpha \phi, \quad \text{uniformly.}$$

By writing a form in components, we can identify the space $\mathscr{D}^k(\mathbb{R}^n)$ of smooth and compactly supported differential k-forms on $\mathbb{R}^n$ with the product $[\mathscr{D}(\mathbb{R}^n)]^{I_{k,n}}$. Hence we can extend the previous notion of convergence to the space $\mathscr{D}^k(\mathbb{R}^n)$. A functional T on that space is continuous if and only if $\langle T; \omega_i\rangle \to \langle T; \omega\rangle$, whenever $\omega_i \to \omega$ in $\mathscr{D}^k$.

2.2. Definition (k-current). *The elements of the dual space $\mathscr{D}_k(\mathbb{R}^n)$ of $\mathscr{D}^k(\mathbb{R}^n)$ are called* k-dimensional currents *(or simply* k-currents*). The action of a current T on a form ω is denoted $\langle T; \omega\rangle$. We say that a sequence of currents T_i weakly* converges to a current T, and we write $T_i \to T$, if*

$$\langle T_i; \omega\rangle \to_{i\to\infty} \langle T; \omega\rangle, \quad \textit{for every } \omega \in \mathscr{D}^k(\mathbb{R}^n).$$

The basic example of k-current (and the reason why currents are also called *generalized surfaces*) is the following. Let S be a closed, oriented, k-dimensional surface in $\mathbb{R}^n$ of class C^1, with (possibly empty) boundary. We associate with S the current $[S]$, defined by

$$\langle [S]; \omega \rangle := \int_S \omega d\mathscr{H}^k.$$

2.3. Definition (support, boundary, mass). *Let $T \in \mathscr{D}_k(\mathbb{R}^n)$. The* support *of T is the set*

$$\operatorname{supp}(T) := \bigcap \{C \text{ closed: } \langle T; \omega \rangle = 0, \text{ whenever } \operatorname{supp}(\omega) \subset \mathbb{R}^n \setminus C\}.$$

The boundary *of T (defined only for $k \geq 1$) is the current $\partial T \in \mathscr{D}_{k-1(\mathbb{R}^n)}$ satisfying*

$$\langle \partial T; \phi \rangle := \langle T; d\phi \rangle, \qquad \text{for every } \phi \in \mathscr{D}^{k-1}(\mathbb{R}^n).$$

The mass *of T is the quantity*

$$\mathbb{M}(T) := \sup\{\langle T; \omega \rangle : \omega \in \mathscr{D}^k(\mathbb{R}^m), \|\omega(x)\| \leq 1 \; \forall x \in \mathbb{R}^n\}.$$

It is easy to observe that if S is a k-surface, oriented by a simple, unit vectorfield τ_s, then $S = \operatorname{supp}([S])$ and, by Stokes theorem, $\partial[S] = [\partial S]$ (with respect to the orientation on ∂S described in §1.13).

2.4. EXERCISE.

(1) *Find a non-trivial k-current with infinite mass and whose support is $\mathscr{H}^k$-null.*
(2) *Prove that $d(d\phi) = 0$, for every $\phi \in \mathscr{D}^k(\mathbb{R}^n)$. Deduce that $\partial(\partial T) = 0$, for every $T \in \mathscr{D}_k(\mathbb{R}^n)$.*
(3) *Prove that $\mathbb{M}([S]) = \mathscr{H}^k(S)$.*

(*Hint for (3):* to prove the inequality "$\geq$", define a form ω on S such that $\|\omega(x)\| \leq 1$ and $\langle \omega(x); \tau_S(x)) \rangle \equiv 1$. Extend it to a continuous form on $\mathbb{R}^n$ and make it smooth and compactly supported, by convolution and cutoff).

Let μ be a finite, positive measure on $\mathbb{R}^n$ and τ be a k-vectorfield with $\|\tau\| = 1$ μ-a.e. Then we denote $T = \tau\mu$ the current

$$\langle T; \omega \rangle := \int_{R^n} \langle \omega(x); \tau(x) \rangle d\mu. \tag{2.1}$$

One can show that $\mathbb{M}(T) = \mu(\mathbb{R}^n)$ (see [7, 26.7]).

2.5. EXERCISE. *Prove that supp(∂T) $\subset$supp(T).*

2.6. EXERCISE. *Prove that every current T with finite mass can be written as $T = \tau\mu$ in the sense of (2.1). Prove that μ is unique and τ is unique up to μ-negligible sets.*

2.7. **Normal currents.** A current $T \in \mathscr{D}^k$ is said to be *normal* if $\mathbb{M}(T) + \mathbb{M}(\partial T) < \infty$. The space of normal k-currents is denoted $\mathbf{N}_k(\mathbb{R}^n)$

2.8. EXERCISE. *Prove that the following currents are not normal. Let $T = \tau\mu$, where*

(i) $\tau = e_1, \quad \mu = \mathscr{H}^1 \llcorner (\{0\} \times [0, 1])$;
(ii) $\tau = e_1 \wedge e_2, \quad \mu = \delta_0$
(iii) $\tau = e_1, \quad \mu = \mathscr{H}^1 \llcorner (C_\alpha \times \{0\})$, *where C_α is a Cantor set on $\mathbb{R}$ of positive Lebesgue measure.*

2.9. EXERCISE. *Compute the boundary of the 2-current $T = \tau\mu$ on $\mathbb{R}^2$, where $\mu = \mathscr{L}^2 \llcorner B(0, 2)$ and $\tau = e_1 \wedge e_2$ on $B(0, 1)$ $\tau = -e_1 \wedge e_2$ on $B(0, 2) \setminus B(0, 1)$.*

2.10. Proposition (Compactness for currents of finite mass). *Let $\{T_i\}_{i \in \mathbb{N}} \subset \mathscr{D}_k(\mathbb{R}^n)$ be such that*

$$\sup_i \{\mathbb{M}(T_i)\} < \infty.$$

Then there exists a current $T \in \mathscr{D}_k(\mathbb{R}^n)$ and a subsequence $\{t_{i_j}\}_{j\in\mathbb{N}}$ such that $T_{i_j} \stackrel{*}{\rightharpoonup} T$ as $j \to \infty$. Moreover

$$\mathbb{M}(T) \leq \liminf_j \mathbb{M}(T_{i_j}).$$

In particular if $T_i \in \mathbf{N}_k(\mathbb{R}^n)$ and

$$\sup_i\{\mathbb{M}(T_i) + \mathbb{M}(\partial T_i)\} < \infty,$$

then, up to subsequences, T_i converges weakly* to a normal current.

Proof. The existence of a subsequential limit is a consequence of Theorem 3.10. The lower semicontinuity of the mass follows from the fact that the mass is a supremum of linear functionals. The only thing which is left to show to prove the second part of the theorem is that the boundary operator is continuous with respect to the weak* convergence of currents, which is an easy exercise. □

2.11. EXERCISE. *Prove that if $T_i \stackrel{*}{\rightharpoonup} T$ then $\partial T_i \stackrel{*}{\rightharpoonup} \partial T$.*

2.12. EXERCISE. *For $t \in [-1, 1]$ let $\gamma_t : [0, 1] \to \mathbb{R}^2$ be the curve*

$$\gamma_t(s) = (\cos s, t \sin s).$$

Define a functional T on $\mathscr{D}^1(\mathbb{R}^2)$ as follows:

$$\langle T; \omega\rangle := \int_{-1}^{1} \langle[\gamma_t]; \omega\rangle dt.$$

Prove that $T \in \mathbf{N}_1(\mathbb{R}^2)$ and $supp(T) = B(0, 1)$.

2.13. **Rectifiable currents.** A *rectifiable k-current* on $\mathbb{R}^n$ is a current T whose action on forms is given by

$$\langle T; \omega\rangle := \int_E \langle\omega(x); \tau(x)\rangle\theta(x)d\mathscr{H}^k,$$

where:

(i) $E \subset \mathbb{R}^n$ is a rectifiable set;

(ii) $\tau : E \to \Lambda_k(\mathbb{R}^n)$ is called the *orientation* of T and it is a simple, k-vector field which at $\mathscr{H}^k$-a.e. $x \in E$ spans the weak tangent field τ_E (see Proposition 1.10) and satisfies $\|\tau(x)\| = 1$;

(iii) $\theta \in L^1_{loc}(\mathscr{H}^k \llcorner E, \mathbb{R})$ is called the *multiplicity* of T.

Such current is also denoted by $[E, \tau, \theta]$. The class of rectifiable k-currents on $\mathbb{R}^n$ is denoted $\mathbf{R}_k(\mathbb{R}^n)$. When the multiplicity takes only integer values T is called *integer rectifiable current*. An *integral current* is an integer rectifiable current with finite mass whose boundary is also an integer rectifiable current with finite mass. The class of integral k-currents on $\mathbb{R}^n$ is denoted $\mathbf{I}_k(\mathbb{R}^n)$.

2.14. Remark. A rectifiable current $T = [E, \tau, \theta]$ with finite mass can be written, in the sense of (2.1), as $T = (\mathrm{sign}(\theta)\tau)(\theta\mathscr{H}^k \llcorner E)$, hence its mass is the quantity $\mathscr{M}(T) = \int_E |\theta| d\mathscr{H}^k$.

The aim of the rest of this series of lectures is to prove the following theorem.

2.15. Theorem (Closure theorem for integral currents). *Let $\{T_i\}_{i\in\mathbb{N}} \subset \mathbf{I_k}(\mathbb{R}^n)$ be such that*

$$\sup_i\{\mathbb{M}(T_i) + \mathbb{M}(\partial T_i)\} < \infty.$$

Then, up to subsequences, T_i converges weakly* to an integral current.

2.16. Remark (Non-counterexamples to the closure theorem).

(i) For every $i \in \mathbb{N}$, let T_i be the integer rectifiable 1-current associated to the sum of i^2 segments of length i^{-2} "equi-distributed" on the unit square $Q = [0, 1] \times [0, 1]$ in $\mathbb{R}^2$ and with direction (and orientation) e_1. One can check that the sequence of currents converge to the normal current $T = \tau\mu$, where $\tau = e_1$ and $\mu = \mathscr{L}^2 \llcorner Q$ (which is not rectifiable). Note that $\mathbb{M}(T_i) = 1$ for every i, but $\mathbb{M}(\partial T_i) = 2i^2$.

(ii) For every $i \in \mathbb{N}$, let T_i be the rectifiable 1-current associated to the sum of i segments of length 1 which are "vertically equi-distributed" on the unit square Q and with direction (and orientation) e_1. Moreover take multiplicity i^{-1} on each segment. One can check that the sequence of currents converge to the same normal current T of point (i). Note that $\mathbb{M}(T_i) = 1$, and $\mathbb{M}(\partial T_i) = 1$, for every i but T_i are not integer rectifiable.

(iii) For every $i \in \mathbb{N}$, let S_i be the integer rectifiable 2-current associated to the sum of i^2 squares Q^i_j of side-length i^{-2} "equi-distributed" on the unit square Q and with orientation $e_1 \wedge e_2$. Let T_i be the integral 1-current ∂S_i. Observe that $\mathbb{M}(T_i) = 4$ and $\partial T_i = 0$, for every i, hence the closure theorem applies. Note also that the measure $\mathscr{H}^1 \llcorner (\bigcup_j \partial Q^i_j)$ converges (as $i \to \infty$) to $4\mathscr{L}^2 \llcorner Q$ (which is not 1-rectifiable). What is the limit of T_i?

2.17. EXERCISE. *Answer the question in point (iii) of the previous remark.*

An important ingredient for the proof of Theorem 2.15 is the following result, which we will prove later.

2.18. Theorem (Boundary rectifiability theorem). *Let $T \in \mathbf{R}_k(\mathbb{R}^n)$ be an integer rectifiable current with $\mathbb{M}(\partial T) < \infty$. Then ∂T is also integer rectifiable.*

2.19. Remark (integer is needed). Let $f \in C^1([0,1])$. Let $T := [[0,1], e_1, f]$ be a rectifiable 1-current on $\mathbb{R}$. Then we have, for every 0-form $\phi \in \mathscr{D}(\mathbb{R})$

$$\langle \partial T, \phi \rangle = \langle T, d\phi \rangle = \langle T, \phi' dx \rangle = \int_0^1 \phi' f \, dx = \phi(1) f_1 - \phi(0) f(0) - \int_0^1 \phi f' \, dx.$$

Hence we can represent

$$\partial T = f(1)\delta_1 - f(0)\delta_0 - f' \mathscr{L}^1 \llcorner [0,1].$$

Observe that $\partial T \notin \mathbf{R}_0(\mathbb{R})$, unless f is constant.

3. Polyhedral approximation

3.1. Definition (Polyhedral current). *A current $T \in \mathscr{D}_k(\mathbb{R}^n)$ is called* polyhedral *if it can be written as*

$$T = \sum_{i \in I} [S_i, \tau_i, \theta_i],$$

where

(i) *I is finite and S_i are k-dimensional* simplexes *in $\mathbb{R}^n$, i.e. the convex envelopes of $(k+1)$ affinely independent points;*

(ii) *τ_i is a constant orientation of S_i;*

(iii) *θ_i is constant on S^i.*

The vector space of polyhedral k-currents in $\mathbb{R}^n$ is denoted $\mathbf{P}_k(\mathbb{R}^n)$. If the multiplicity θ_i is integer-valued for all i, then T is called integer polyhedral.

3.2. EXERCISE. *Verify that if T is polyhedral, then ∂T is also polyhedral and that an integer polyhedral current is an integral current.*

This section is devoted to prove the following result

3.3. Theorem (Polyhedral approximation theorem). *Let $T \in \mathbf{N}_k(\mathbb{R}^n)$. Then there exists a constant $C = C(n,k)$ and a sequence $\{P_i\}_{i \in \mathbb{N}} \subset \mathbf{P}_k(\mathbb{R}^n)$ such that*

$$P_i \overset{*}{\rightharpoonup} T; \quad \mathbb{M}(P_i) \le C\mathbb{M}(T); \quad \mathbb{M}(\partial P_i) \le C\mathbb{M}(\partial P).$$

Moreover, if $T \in \mathbf{I}_k(\mathbb{R}^n)$, then P_i can be chosen integral polyhedral.

3.4. Remark.

(i) The constant C can be chosen equal to 1, but the proof of this result is not part of this lectures. It can be found in [**3**, 4.2.20].
(ii) The convolution technique which in many function spaces is used to approximate general functions with smooth functions, is available also in our context. Nevertheless the convolution of a k-current does not produce a k-dimensional smooth surface, but an object whose support is n-dimensional.

3.5. **Product of currents.**

3.6. Definition (Product of currents). *Let $T \in \mathscr{D}_k(\mathbb{R}^n)$ and $S \in \mathscr{D}^h(\mathbb{R}^d)$. Then there is a unique current $T \times S \in \mathscr{D}_{k+h}(\mathbb{R}^{n+d})$, satisfying*

(i) $\langle T \times S; \phi(x)\psi(y)dx_I \wedge dy_J\rangle = \langle T; \phi(x)dx_I\rangle\langle S; \psi(y)dy_J\rangle$, *whenever* $\phi \in \mathscr{D}(\mathbb{R}^n), \psi \in \mathscr{D}(\mathbb{R}^d)$, $I \in I_{k,n}$, $J \in I_{h,d}$;
(ii) $\langle T \times S; \rho(x,y)dx_I \wedge dy_J\rangle = 0$, *whenever* $I \notin I_{k,n}$ *or* $J \notin I_{h,d}$.

3.7. EXERCISE. *Prove that (i) and (ii) define the action of $T\times S$ on all forms in $\mathscr{D}_{k+h}(\mathbb{R}^{n+d})$. (Hint: You can assume that forms which are finite sum of forms of the type $\phi(x)\psi(y)dx_I \wedge dy_J$ are dense in $D^{k+h}(\mathbb{R}^{n+d})$.)*

3.8. EXERCISE. $(*)$ *Prove that supp$(T \times S) \subset$supp$(T)\times$supp(S).*

3.9. EXERCISE. *Let $\omega \in \mathscr{D}^k(\mathbb{R}^n), \sigma \in \mathscr{D}^h(\mathbb{R}^n)$ be of class C^1. Prove that*

$$d(\omega \wedge \sigma) = d\omega \wedge \sigma + (-1)^k \omega \wedge d\sigma. \tag{3.1}$$

3.10. Proposition. *Let T, S be as in Definition 3.6, then*

$$\partial(T \times S) = \partial T \times S + (-1)^k T \times \partial S. \tag{3.2}$$

Part 1 of the proof. Let $\omega \in \mathscr{D}^{k+h-1}(\mathbb{R}^{n+d})$ be of the form $\omega := \phi(x)\psi(y)dx_I \wedge dy_j$, where $I \in I_{k-1,n}$, $J \in I_{h,d}$. Then we compute

$$\begin{aligned}\langle \partial(T\times S); \omega\rangle = \langle T\times S; d\omega\rangle &\overset{(3.1)}{=} \langle T\times S; d(\phi(x)dx_I)\wedge\psi(y)dy_J\rangle + (-1)^k\langle T\times S; \phi(x)dx_I\wedge d(\psi(y)dy_J)\rangle \\ &\overset{(ii)}{=} \langle T \times S; d(\phi(x)dx_I) \wedge \psi(y)dy_J\rangle \overset{(i)}{=} \langle T; d(\phi(x)dx_I)\rangle\langle S; \psi(y)dy_J\rangle \\ &= \langle \partial T; \phi(x)dx_I\rangle\langle S; \psi(y)dy_J\rangle = \langle \partial T \times S; \omega\rangle.\end{aligned}$$

□

3.11. EXERCISE. $(*)$ *Complete the proof: prove the formula firstly for the remaining interesting case in which $\omega := \phi(x)\psi(y)dx_I \wedge dy_j$, where $I \in I_{k,n}$, $J \in I_{h-1,d}$ and then extend it to the general case.*

3.12. Remark (Product of currents with finite mass). Let T, S be as in Definition 3.6, with $T = \tau\mu$, and $S = \tau'\mu'$ currents with finite mass. Then one can prove that

$$T \times S = (\tau \wedge \tau')(\mu \times \mu').$$

Notice that there is a small abuse of notation, since in the expression $\tau \wedge \tau'$, the two vectors are in $\mathbb{R}^{n+d}$. Since the norm of $\tau \wedge \tau'$ splits in the product of the corresponding norms, we have that $\mathbb{M}(T \times S) = \mathbb{M}(T)\mathbb{M}(S)$. Moreover, if T and S are integer rectifiable, so is $T \times S$.

3.13. Remark (Product of rectifiable currents). Let $T := [E\tau\theta], S := [E'\tau'\theta']$. Then

$$T \times S = [E \times E', \tau \wedge \tau', \theta\theta'].$$

3.14. EXERCISE. *Prove that if E and E' are rectifiable, then $E \times E'$ is rectifiable and that $\tau \wedge \tau'$ is an orientation. Prove that $\mathscr{H}^{k+h} \llcorner (E \times E') = (\mathscr{H}^k \llcorner E) \times (\mathscr{H}^h \llcorner E')$.*

3.15. **Push-forward of currents.** We want to define the "image" of a current $T \in \mathscr{D}_k(\mathbb{R}^n)$ according to a map $f : \mathbb{R}^n \to \mathbb{R}^d$ with nice properties. We begin with the following definitions.

3.15.1. *Pull-back of covectors.* Let $L : \mathbb{R}^n \to \mathbb{R}^d$ be a linear map. Let $\alpha \in \Lambda^k(\mathbb{R}^d)$ be a k-covector. We define the *pull-back* $L^\sharp\alpha$ of α according to L to be the k-covector in $\mathbb{R}^n$ satisfying

$$L^\sharp\alpha(v_1, \ldots, v_k) := \alpha(Lv_1, \ldots, Lv_k),$$

for every k-tuple of vectors $(v_1, \ldots, v_n)$ in $\mathbb{R}^n$.

3.16. EXERCISE. *Let $|L|$ be the operator norm (i.e. the Lipschitz constant) of L. Prove that*

$$\|L^\sharp\alpha\| \leq |L|^k\|\alpha\|,$$

where we denoted by $\|\cdot\|$ the comass norm (see Definition 1.18).

3.16.1. *Push-forward of vectors.* Let L be as above and $v \in \Lambda_k(\mathbb{R}^n)$ be a k-vector. We define the *push-forward* $L_\sharp v$ of v according to L to be the k-vector in $\mathbb{R}^d$ satisfying

$$\langle \alpha, L_\sharp v\rangle := \langle L^\sharp\alpha, v\rangle,$$

for every $\alpha \in \Lambda^k(\mathbb{R}^d)$.

3.17. EXERCISE. *Prove that*

$$\|L_\sharp v\| \leq |L|^k\|v\|,$$

where we denoted by $\|\cdot\|$ the mass norm (see Definition 1.18).

3.17.1. *Pull-back of a form.* Let $f : \mathbb{R}^n \to \mathbb{R}^d$ be of class C^1. Let $\omega \in \mathscr{D}^k(\mathbb{R}^d)$ be a differential k-form. We define the *pull-back* $f^\sharp\omega$ of ω according to f, to be the differential k-form on $\mathbb{R}^n$ satisfying

$$(f^\sharp\omega)(x) := (df(x))^\sharp\omega(f(x)),$$

for every $x \in \mathbb{R}^n$

3.17.2. *Push-forward of a current.* Let $f : \mathbb{R}^n \to \mathbb{R}^d$ be smooth and proper (i.e. the preimage of a compact set is compact). Let $T \in \mathscr{D}_k(\mathbb{R}^n)$ be a k-current. We define the *push-forward* $f_\sharp T$ of T according to f, to be the k-current on $\mathbb{R}^d$ satisfying

$$\langle f_\sharp T, \omega\rangle := \langle T, f^\sharp\omega\rangle,$$

for every $\omega \in \mathscr{D}^k(\mathbb{R}^d)$.

3.18. Remark.

(i) If $\mathrm{supp}(T)$ is compact, then f does not need to be proper: properness is required to ensure that $f^\sharp\omega$ has compact support, but if $\mathrm{supp}(T)$ is compact, then the action of T is well defined also on forms with non-compact support (by a cut-off outside the support of the current).

(ii) If $\mathrm{supp}(T)$ is compact and $\mathbb{M}(T) < \infty$, then it is sufficient to require that f is of class C^1, because the action of T on all forms is determined by the action on continuous ones by the Riesz Theorem.

3.19. EXERCISE. *Prove that if $\mathbb{M}(T) < \infty$ and therefore $T = \tau\mu$, there holds*

$$\mathbb{M}(f_\sharp T) \leq \int |df|^k\|\tau\|d\mu \leq (\sup_{x\in\mathrm{supp}(T)}\{|df|\})^k\mathbb{M}(T).$$

3.19.1. *Boundary of the push forward.* From the identity $d(f^\sharp\omega) = f^\sharp(d\omega)$ it follows that

$$\partial f^\sharp T = f_\sharp\partial T.$$

3.19.2. *Push-forward of rectifiable currents.* Let $T = [E, \tau, \theta]$ be a rectifiable k-current in $\mathbb{R}^n$ with compact support. Let $f : \mathbb{R}^n \to \mathbb{R}^d$ be of class C^1. Then the rectifiable current $f_\sharp T$ has the form $f_\sharp T = [\tilde{E}, \tilde{\tau}, \tilde{\theta}]$, where

(i) $\tilde{E} = f(E)$ (which is rectifiable);
(ii) $\tilde{\tau}$ is any fixed orientation of $\tilde{E}$;
(iii) $\tilde{\theta}$ is given by the formula

$$\tilde{\theta}(y) = \sum_{x \in f^{-1}(y)} \pm\theta(x),$$

where the sign is positive if the map $df(x)$ preserves the orientation (with respect to the fixed orientations $\tau(x)$ and $\tilde{\tau}(y)$) and negative otherwise.

This formula can be proved via the area formula of Theorem 3.28, and a similar formula holds when f is just Lipschitz (replacing df with $d_\tau f$).

3.20. **Flat norm, homotopy formula, and constancy lemma.**

3.21. Definition. *For every $T \in \mathscr{D}_k(\mathbb{R}^n)$ we define its* flat norm *by*

$$\mathbb{F}(T) := \inf\{\mathbb{M}(R) + \mathbb{M}(S) : T = R + \partial S\}.$$

3.22. EXERCISE. *Let $S_1 = [0,1] \times \{0\}$ and $S_2 = [0,1] \times \{\frac{1}{i}\}$ and let $T_1 = [S_1, e_1, 1]$, $T_2 = [S_2, e_1, 1]$. Prove that $\mathbb{F}(T_1 - T_2) \le \frac{3}{i}$*

3.23. Proposition. *If $\mathbb{F}(T_i - T) \to 0$ as $i \to \infty$, then $T_i \overset{*}{\rightharpoonup} T$.*

Proof. Let $\omega \in \mathscr{D}^k(\mathbb{R}^n)$ and for every i let $T_i = R_i + \partial S_i$. Then

$$|\langle T_i - T, \omega\rangle| = |\langle R_i, \omega\rangle + \langle \partial S_i, \omega\rangle| \le |\langle R_i, \omega\rangle| + |\langle S_i, d\omega\rangle| \le (\mathbb{M}(R_i) + \mathbb{M}(S_i))(\|\omega\|_\infty + \|d\omega\|_\infty).$$

Since this holds for any R_i and S_i, we can conclude

$$|\langle T_i - T, \omega\rangle| \le \mathbb{F}(T_i - T)(\|\omega\|_\infty + \|d\omega\|_\infty)$$

□

3.24. Remark. The reverse also holds in some interesting cases, e.g. if there exists K compact such that $\mathrm{supp}(T_i) \subset K$ for every i and $\sup\{\mathbb{M}(T_i) + \mathbb{M}(\partial T_i)\} < \infty$. This result is not elementary.

3.24.1. *Homotopy formula.* Let $T \in \mathscr{D}^k(\mathbb{R}^n)$ be compactly supported. Let $f_0, f_1 : \mathbb{R}^n \to \mathbb{R}^d$ be *homotopic*, i.e. there exists $F : [0,1] \times \mathbb{R}^n \to \mathbb{R}^d$ smooth such that $F(0,\cdot) = f_0$ and $F(1,\cdot) = f_1$. Denote $T_0 = f_{0\sharp}T$, $T_1 = f_{1\sharp}T \in \mathscr{D}_k(\mathbb{R}^d)$. Denote also $I := [(0,1), e, 1] \in \mathbf{I}_1(\mathbb{R})$ and set $S := F_\sharp(I \times T) \in D_{k+1}(\mathbb{R}^d)$.

3.25. Theorem. *Let F, T, S be as above. Then*

(i) *$T_1 - T_0 = \partial S + R$, where $R := F_\sharp(I \times \partial T)$;*
(ii) *if T is rectifiable, so are R and S, and if T is integer rectifiable, so are R and S;*
(iii) $\mathbb{F}(T_1 - T_0) \le \mathbb{M}(T)\sup_{x \in \mathrm{supp} T}\{|d_t F||d_x F|^k\} + \mathbb{M}(\partial T)\sup_{x \in \mathrm{supp}\partial T}\{|d_t F||d_x F|^{k-1}\}$.

Proof. (i) We compute

$$\partial S = \partial(F_\sharp(I \times T)) = F_\sharp(\partial(I \times T)) \overset{(3.2)}{=} F_\sharp(\partial I \times T - I \times \partial T)$$

$$= F_\sharp(\delta_1 \times T) - F_\sharp(\delta_0 \times T) - R = f_{1\sharp}T - f_{2\sharp}T - R = T_1 - T_0 - R.$$

(ii) The first implication follows from the fact that the product of two rectifiable currents is rectifiable and the push-forward of a rectifiable current is rectifiable. The second implication follows from the formulas for the product and the push-forward of rectifiable currents.

(iii) We assume that T is normal (otherwise the estimate is trivial). We write $T = \tau\mu$ and $\partial T = \tau'\mu'$ with unit τ and τ'. By (i) we have $\mathbb{F}(T_1 - T_0) \leq \mathbb{M}(S) + \mathbb{M}(R)$. We estimate the mass of S, the estimate for $\mathbb{M}(R)$ is analogous. Let $\omega \in \mathscr{D}^{k+1}(\mathbb{R}^d)$, then

$$\langle S, \omega\rangle = \langle F_\sharp(I \times T), \omega\rangle = \langle I \times T, F^\sharp\omega\rangle = \int_0^1 \int_{\mathbb{R}^n} \langle F^\sharp\omega(t,x), e \wedge \tau(x)\rangle d\mu(x)dt$$

$$= \int_0^1 \int_{\mathbb{R}^n} \langle \omega(F(t,x)), (dF(t,x))_\sharp(e \wedge \tau(x))\rangle d\mu(x)dt$$

$$= \int_0^1 \int_{\mathbb{R}^n} \langle \omega(F(t,x)), (d_tF(t,x))_\sharp e \wedge (d_xF(t,x))_\sharp\tau(x))\rangle d\mu(x)dt$$

$$\leq \int_0^1 \int_{\mathbb{R}^n} \|\omega\|_\infty |d_tF(t,x)||d_xF(t,x)|^k d\mu(x)dt.$$

$$\leq \sup_{x\in}\{|d_tF||d_xF|^k\}\mathbb{M}(T)$$

□

3.26. Remark (Linear homotopies). Often one wants to consider *linear homotopies*, i.e. homotopies of the type

$$F(t,x) = tf_1(x) + (1-t)f_0(x).$$

In this case we have:

$$|d_tF(t_0,x)| \leq |f_1(x) - f_0(x)|;$$
$$|d_xF(t,x_0)| \leq |df_1(x_0)| + |df_0(x_0)|.$$

Hence the estimate (iii) of Theorem 3.25 can be replaced with

$$\mathbb{F}(T_1 - T_0) \leq \|f_1 - f_0\|_\infty \int_{\mathbb{R}^n} (|df_0(x)| + |df_1(x)|)^k d\mu, \tag{3.3}$$

and in particular

$$\mathbb{F}(T_1 - T_0) \leq \|f_1 - f_0\|_\infty (L^k\mathbb{M}(T) + L^{k-1}\mathbb{M}(\partial T)),$$

where

$$L := \sup_{x\in \mathrm{supp}(T)} \{|df_0(x)| + |df_1(x)|\}.$$

3.27. Theorem (Constancy lemma). *Let $T \in \mathscr{D}_k(\mathbb{R}^k)$ with $\partial T = 0$. Then $T = [\mathbb{R}^k, e, m]$, where $e = e_1 \wedge \cdots \wedge e_k$ and m is constant.*

Proof. We denote by $\Lambda \in \mathscr{D}_0(\mathbb{R}^k)$ the distribution defined by

$$\langle \Lambda, \phi\rangle := \langle T, \phi dx\rangle, \quad \text{for every } \phi \in \mathscr{D}^0(\mathbb{R}^k),$$

where $dx = dx_1 \wedge \cdots \wedge dx_k$. We claim that the distributional derivative $D\Lambda$ vanishes and we will see later that this implies the conclusion of the theorem. To prove the claim, fix $i \in \{1,\ldots,k\}$ and let

$$\omega := \phi \hat{dx_i} \in \mathscr{D}^{k-1}(R^k),$$

where

$$\hat{dx_i} := dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_k.$$

We can compute

$$0 \overset{\partial T = 0}{=} \langle \partial T, \omega\rangle = \langle T, d\omega\rangle = \langle T, \sum_{j=1}^k \frac{\partial\phi}{\partial x_j} dx_j \wedge \hat{dx_i}\rangle.$$

We observe that $dx_i \wedge \hat{dx_i} = (-1)^{i-1}dx$, while $dx_j \wedge \hat{dx_i} = 0$ for $i \neq j$, hence we can continue the chain of equalities with

$$0 = \langle T, (-1)^{i-1}\frac{\partial\phi}{\partial x_i}dx\rangle = \langle \Lambda, (-1)^{i-1}\frac{\partial\phi}{\partial x_i}\rangle = (-1)^{i-1}\langle D_i\Lambda, \phi\rangle.$$

Lastly we show that the fact that $D\Lambda = 0$ implies that Λ is (represented by) a constant function. We prove it via convolution. Let $\rho \in \mathscr{D}^0(\mathbb{R}^k)$ be positive with $\int_{\mathbb{R}^k} \rho(x)dx = 1$ and let $\rho_\varepsilon(x) =$

$\varepsilon^{-k}\rho(\varepsilon^{-1}x)$. Let $\Lambda_\varepsilon := \Lambda * \rho_\varepsilon$. It is well known that Λ_ε are (represented by) smooth functions, converging in distribution to Λ satisfying

$$\nabla\Lambda_\varepsilon = \nabla(\Lambda * \rho_\varepsilon) = (D\Lambda) * \rho_\varepsilon = 0,$$

hence Λ_ε are constant functions. The limit (in distribution) of constant functions is constant, hence Λ is constant, which implies the claimed representation of T. □

With a similar argument, one can prove the following more general statement

3.28. Proposition. *Let $T \in \mathscr{D}_k(\mathbb{R}^k)$ with $\mathbb{M}(\partial T) < \infty$. Then $T = [\mathbb{R}^k, e, m]$, where $e = e_1 \wedge \cdots \wedge e_k$ and $m \in BV_{loc}(\mathbb{R}^k)$.*

3.29. EXERCISE. *Prove Proposition 3.28.*

3.30. EXERCISE. *Prove the following statement. Let $A \subset V$ be a relatively open subset of an affine k-dimensional subspace V of $\mathbb{R}^n$. Let $T \in \mathbf{N}_k(\mathbb{R}^n)$ with $(\partial T) \llcorner A = 0$, where $T \llcorner A$ is the restriction of the current T to the set A (obtained simply restricting to A the measure associated to T). Then $T \llcorner A = [\mathbb{R}^k, e, m]$, where e is a constant k-vector spanning Tan_V and m is constant.*

3.31. **Proof of Theorem 3.3.** Through this section we will assume $\partial T = 0$. C will denote a dimensional constant, which might change in several steps of the proof. We will use the term *δ-grid* to denote the union of adjacent hypercubes in $\mathbb{R}^n$ of edge-length δ, where the edges are parallel to the coordinate axes and one of the hypercubes has one vertex in the origin. The $(n-1)$ skeleton is obtained as the union of the $(n-1)$-dimensional simplexes supporting the boundary of the hypercubes. Inductively, the *m-skeleton* of the δ-grid is obtained as the union of the (m)-dimensional simplexes supporting the boundary of the simplexes of the $(m+1)$-skeleton.

3.32. Lemma. *Let $T \in \mathscr{D}_k(\mathbb{R}^n)$, with $\partial T = 0$ and let $\delta > 0$. There exists a current $\tilde{T} \in \mathscr{D}^k(\mathbb{R}^n)$, supported on the $(n-1)$-skeleton of the δ-grid, such that $\partial\tilde{T} = 0$, $\mathbb{M}(\tilde{T}) \leq C\mathbb{M}(T)$, $\tilde{T} - T = \partial S$, with $\mathbb{M}(S) \leq C\delta\mathbb{M}(T)$, hence $\mathbb{F}(\tilde{T} - T) \leq C\delta\mathbb{M}(T)$. Moreover, if T is rectifiable, so are $\tilde{T}$ and S, and if T is integer rectifiable, so are $\tilde{T}$ and S.*

Proof. For every hypercube Q in the grid, we pick a point x_Q (which will be chosen wisely later). Let $P_Q : Q \setminus \{x_Q\} \to \partial Q$ be the radial projection. Denote f_δ the map

$$f_\delta : \mathbb{R}^n \setminus (\bigcup_Q)\{x_Q\} \to \bigcup_Q \partial Q.$$

Note that f_δ is only locally Lipschitz on $\mathbb{R}^n \setminus (\bigcup_Q)\{x_Q\}$, and more precisely

$$|df_\delta(x)| \leq C\frac{\delta}{|x - x_Q|}, \quad \text{for every } x \in Q \setminus \{x_Q\}. \tag{3.4}$$

We will continue with the proof as if f_δ was a map of class C^1. The correct constructions, indeed, would replace f_δ with a smoothed approximation. This construction will be described in Remark 3.34.
Set $\tilde{T} := f_{\delta\sharp}T$. Obviously $\tilde{T}$ is supported on the $(n-1)$-skeleton. By Exercise 3.19, we have

$$\mathbb{M}(\tilde{T}) = \int_{\mathbb{R}^n} |df_\delta(x)|^k d\mu(x) \leq C\int_{\mathbb{R}^n} g^k d\mu(x), \tag{3.5}$$

where we denoted $g(x)$ the a function which coincides with $\delta(|x - x_Q|)^{-1}$ on the interior of each cube Q[1]. We claim that a suitable choice of the points x_Q yields, for every cube Q

$$\int_Q \frac{\delta^k}{|x - x_Q|^k} d\mu(x) \leq C\mu(Q) \tag{3.6}$$

[1]we can assume, possibly by translating the grid, that μ gives zero measure to the $(n-1)$-skeleton, hence the values of g on the $(n-1)$-skeleton are not relevant.

which, combined with (3.5), gives $\mathbb{M}(\tilde{T}) \le C\mathbb{M}(T)$. Next we apply homotopy formula, with a linear homotopy between $f_0 := Id$ and $f_1 := f_\delta$, being Id the identity map. By (3.3), we get

$$\mathbb{M}(S) \le \sup\{|f_0 - f_1|\} \int (1 + |df_\delta|)^k d\mu \le C\delta \int 1 + |df|^k d\mu \le C\delta\mathbb{M}(T).$$

Now we prove the claim (3.6). To do so, we average the left hand side over all $x_Q \in Q$

$$\begin{aligned}
\frac{1}{|Q|}\int_Q \left(\int_Q \frac{\delta^k}{|x-x_Q|^k} d\mu(x)\right) dx_Q &= \delta^{k-n}\int_Q \left(\int_Q \frac{1}{|x-x_Q|^k} dx_Q\right) d\mu(x) \\
&\le \delta^{k-n}\int_Q \left(\int_{B(x,\sqrt{n}\delta)} \frac{1}{|x-x_Q|^k} dx_Q\right) d\mu(x) \\
&= \delta^{k-n}\int_Q \left(\int_0^{\sqrt{n}\delta} \frac{1}{r^k} C r^{n-1} r^k dr\right) d\mu(x) = C\mu(Q),
\end{aligned}$$

where the inequality follows from the fact that $Q \subset B(x, \sqrt{n}\delta)$, for every $x \in Q$. The last part of the statement follows from the fact that both properties are stable under product and push forward, which are the only operations which we employed to get $\tilde{T}$ and S. □

Analogously one can prove the following generalization of Lemma 3.32.

3.33. Lemma. *Let $T \in \mathscr{D}_k(\mathbb{R}^n)$, with $\partial T = 0$ be a current which is supported on the h-skeleton $(h > k)$ and let $\delta > 0$. There exists a current $\tilde{T} \in \mathscr{D}^k(\mathbb{R}^n)$, supported on the $(h-1)$-skeleton of the δ-grid, such that $\partial\tilde{T} = 0$, $\mathbb{M}(\tilde{T}) \le C\mathbb{M}(T)$, $\tilde{T} - T = \partial S$, with $\mathbb{M}(S) \le C\delta\mathbb{M}(T)$, hence $\mathbb{F}(\tilde{T} - T) \le C\delta\mathbb{M}(T)$. Moreover, if T is rectifiable, so are $\tilde{T}$ and S, and if T is integer rectifiable, so are $\tilde{T}$ and S.*

Using Lemma 3.33 iteratively, we easily obtain the proof of Theorem 3.3. Apply Lemma 3.33 with $T_0 := T$ obtaining a current $T_1 := \tilde{T}_0$, and iteratively with $T_1, \dots, T_{n-k+1}$ obtaining at the end of the iteration a current $T_{n-k} \in \mathscr{D}_k(\mathbb{R}^n)$ which is supported on the k-skeleton. Clearly we have $\mathbb{M}(T_{n-k}) \le C\mathbb{M}(T)$. Moreover we have

$$T - T_{n-k} = (T - T_1) + (T_1 - T_2) + \dots (T_{n-k+1} - T_{n-k}) = \partial(S_1 + \dots + S_{n-k}),$$

with $\mathbb{M}(S_1 + \dots + S_{n-k}) \le C\delta\mathbb{M}(T)$. It remains to show that T_{n-k} is polyhedral. Let A be a face of the k-skeleton of the δ-grid. The fact that T_{n-k} has constant multiplicity on A follows from Exercise 3.30.

3.34. Remark (The correct construction of f_δ). As we observed, f_δ does not have the minimal requirements which allow to define the push-forward of a current according to it. The correct construction requires to approximate each projection P_Q with a smoothing $P_{Q,\varepsilon}$ in an ε-neighbourhood of the singularity x_Q. More precisely we set $P_{Q,\varepsilon}(x_Q) = x_Q$ and for every $x \in Q \setminus \{x_Q\}$

$$P_{Q,\varepsilon}(x) = x_Q + (P_Q(x) - x_Q)\sigma(\frac{x - x_Q}{\varepsilon}),$$

where $\sigma : \mathbb{R} \to [0, 1]$ is a smooth function which is constantly equal to 0 for $t \le 0$ and to 1 for $t \ge 1$. We can define, as in the proof of Lemma 3.32, the corresponding map[2] $f_{\delta,\varepsilon}$ and the current $\tilde{T}_\varepsilon = {f_{\delta,\varepsilon}}_\sharp T$. The final current $\tilde{T}$ is the limit as $\varepsilon \to 0$ of the currents $\tilde{T}_\varepsilon$.

3.35. EXERCISE. *Prove that the currents $\tilde{T}_\varepsilon$ defined in Remark 3.34 actually converge in mass to a current $\tilde{T}$.*

Let us summarize the main consequence of Lemma 3.32 an Lemma 3.33 in the following proposition. Actually we proved only the version with $\partial T = 0$, nevertheless we state it in the general case.

[2] note that the singularity at x_Q is eliminated, but $f_{\delta,\varepsilon}$ fails to be of class C^1 at the boundary of the cubes. However it is easy to see that at least $f_{\delta,\varepsilon}$ is Lipschitz.

3.36. Theorem (Polyhedral deformation). *Let $T \in \mathbf{N}_k(\mathbb{R}^n)$ be a normal k-current with compact support and let $\delta > 0$. There exists a polyhedral current $P_\delta \in \mathscr{P}^k(\mathbb{R}^n)$, supported on the k-skeleton of the δ-grid, such that*

$$\mathbb{M}(P_\delta) \leq C\mathbb{M}(T), \quad \mathbb{M}(\partial P_\delta) \leq C\mathbb{M}(T),$$

$P_\delta - T = R + \partial S$, with

$$\mathbb{M}(S) \leq C\delta\mathbb{M}(T), \quad \mathbb{M}(R) \leq C\delta\mathbb{M}(\partial T),$$

hence $\mathbb{F}(P_\delta - T) \leq C\delta(\mathbb{M}(T) + \mathbb{M}(\partial T))$. Moreover, if T is rectifiable, so are P_δ, R and S, and if T is integer rectifiable, so are P_δ, R and S.

A curious application of Theorem 3.36, namely with a choice of a "large" δ, gives the following generalization of the isoperimetric inequality.

3.37. Theorem (Generalized isoperimetric inequality). *Let $T \in \mathbf{I}_k(\mathbb{R}^n)$ with compact support and $\partial T = 0$. Then there exists $S \in \mathbf{I}_{k+1}(\mathbb{R}^n)$ with compact support, such that $\partial S = T$ and*

$$\mathbb{M}(S) \leq C(n)[\mathbb{M}(T)]^{1+\frac{1}{k}}.$$

Proof. Let $L := C\mathbb{M}(T)$, where C is the constant in Theorem 3.36. Fix δ such that $L < \delta^k \leq 2L$ and apply Theorem 3.36 to obtain a polyhedral current P_δ. We claim that $P_\delta = 0$. Indeed, since P_δ is an integer polyhedral k-current with constant multiplicities on each k-cell of the δ-grid, then $\mathbb{M}(P_\delta)$ is an integer multiple of δ^k. Since $\mathbb{M}(P_\delta) \leq L < \delta^k$, it follows that $\mathbb{M}(P_\delta) = 0$, hence $P_\delta = 0$. This implies that $\partial S = T$, where S is the integral $(k+1)$-current given by Theorem 3.36. The estimate on the mass of S easily follows:

$$\mathbb{M}(S) \leq C\delta\mathbb{M}(T) \leq C(2L)^{\frac{1}{k}}\mathbb{M}(T) \leq C[\mathbb{M}(T)]^{1+\frac{1}{k}}.$$

□

Another consequence of the Polyhedral deformation theorem is the boundary rectifiability theorem in the case $k = 1$.

3.38. Theorem (Boundary rectifiability ($k = 1$)). *Let T be an integer rectifiable 1-current, with $\mathbb{M}(\partial T) < \infty$. Then T is integral.*

Proof. Let $\delta_i \searrow 0$ and $P_i := P_{\delta_i}$ be the polyhedral 1-currents (with integer coefficients!) given by Theorem 3.36. Since $\mathbb{M}(\partial P_i) \leq C\mathbb{M}(\partial T)$, then for every i, ∂P_i is supported on a set of at most $C\mathbb{M}(T)$ points (see Exercise 3.2). The class of such measures is compact, hence $\partial(T)$ has the same structure, which implies that T is integral. □

3.39. Remark. In general dimension k, the boundary rectifiability theorem is a consequence of the polyhedral deformation theorem and the $(k-1)$-dimensional version of the closure theorem (Theorem 2.15), the proof being identical to that presented above. In particular, in a possible proof by induction of the closure theorem, the boundary rectifiability theorem can be given for granted. The proof of the closure theorem that we will present here is not by induction, but we will reduce to a lower dimensional problem with a technique which we will introduce in the next section.

4. Slicing

We begin recalling a classical result in Real Analysis.

4.1. Theorem (Sard). *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be of class C^k, for some $k \geq \max\{n - m, 1\}$. Denote by*

$$C_f := \{x \in \mathbb{R}^n : \mathrm{rk}(Df(x)) < m\}$$

the critical set of f. Then $\mathscr{L}^m(f(C_f)) = 0$.

4.2. Remark. The result holds also if $\mathbb{R}^n$ and $\mathbb{R}^m$ are replaced by smooth manifolds N and M of dimension n and m, respectively, with the obvious necessary changes (e.g. $\mathscr{L}^m$ is replaced by $\mathscr{H}^m \llcorner M$).

4.3. Corollary. *Let $0 < m \leq k \leq n$. Let M be a smooth k-surface in $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ be smooth. Denote $M_y := M \cap f^{-1}(y)$. Then for $\mathscr{L}^m$-a.e. y, M_y is a smooth surface of dimension $k - m$ (or it is empty).*

We want to extend the previous corollary, when M is replaced by a rectifiable set and f is only Lipschitz. We need the coarea formula

4.4. Theorem (Coarea formula). *Let $E \subset \mathbb{R}^n$ be k-rectifiable and $f\mathbb{R}^n \to \mathbb{R}^m$ Lipschitz. For $y \in \mathbb{R}^m$, denote $E_y := E \cap f^{-1}(y)$. For $\mathscr{H}^k$-a.e. $x \in E$ denote $\nabla_\tau f(x)$ the tangential gradient of f at x. Moreover, let*

$$J_\tau f(x) := |\nabla_\tau f_1(x) \wedge \cdots \wedge \nabla_\tau f_m(x)|$$

denote the tangential Jacobian. Then for every Borel function $g : E \to [0, +\infty]$, it holds

$$\int_{y\in\mathbb{R}^m} \left(\int_{x\in E_y} g(x) d\mathscr{H}^{k-m}(x) \right) d\mathscr{L}^m(y) = \int_{x\in E} g(x) J_\tau f(x) d\mathscr{H}^k(x). \tag{4.1}$$

We are ready to prove the main proposition of this section.

4.5. Proposition (Slicing of rectifiable currents). *Let $T = [E, \tau, \theta]$ be a rectifiable k-current in $\mathbb{R}^n$, with $\mathbb{M}(T) < \infty$. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz, with $0 < m \leq k \leq n$. Denote*

$$\tilde{E} := \{x \in E : \nabla_\tau f(x) \text{ is defined and has rank } m\}.$$

For $y \in \mathbb{R}^m$, denote $E_y := E \cap f^{-1}(y)$. Then the following facts hold.

(i) $\mathscr{H}^{k-m}(E_y \setminus \tilde{E}) = 0$, *for $\mathscr{L}^m$-a.e. y.*

(ii) *E_y is $(k - m)$ rectifiable for $\mathscr{L}^m$-a.e. y.*

(iii) *Denoting*

$$\eta(x) := \nabla_\tau f_1(x) \wedge \cdots \wedge \nabla_\tau f_m(x),$$

we have that

$$\mathrm{Tan}(E, x) = \mathrm{Tan}(E_y, x) \oplus \mathrm{span}\{\eta(x)\},$$

for $\mathscr{L}^m$-a.e. y and for $\mathscr{H}^{k-m}$-a.e. $x \in E_y$. Hence, for $\mathscr{L}^m$-a.e. y, we can define the orientation on E_y as the $(k - m)$-vector $\tilde{\tau}$ such that

$$\frac{\eta(x)}{|\eta(x)|} \wedge \tilde{\tau}(x) = \tau(x), \quad \text{for } \mathscr{H}^{k-m}\text{-a.e. } y \in E_y.$$

(iv) *we have*

$$\int_{y\in\mathbb{R}^m} \left(\int_{x\in E_y} |\theta(x)| d\mathscr{H}^{k-m}(x) \right) d\mathscr{L}^m(y) = \int_{x\in E} |\theta(x)| J_\tau f(x) d\mathscr{H}^k(x).$$

(v) *For $\mathscr{L}^m$-a.e. y, it is well defined the $(k - m)$-rectifiable current $T_y := [E_y, \tilde{\tau}, \theta \llcorner E_y]$, which is called the* slice of T at y according to f. *Moreover*

$$\int_{y\in\mathbb{R}^m} \mathbb{M}(T_y) d\mathscr{L}^m(y) = \int_{x\in E} |\theta(x)| J_\tau f(x) d\mathscr{H}^k(x) \leq [\mathrm{Lip}(f)]^m \mathbb{M}(T).$$

Proof. (i) By (4.1), with $g := \mathbf{1}_{E\setminus\tilde{E}}$, we get

$$\int_{y\in\mathbb{R}^m} \mathscr{H}^{k-m}(E_y \setminus \tilde{E}) d\mathscr{L}^m(y) = \int_{E\setminus\tilde{E}} J_\tau f(x) d\mathscr{H}^k(x) = 0,$$

where the last equality follows from the fact that $J_\tau f = 0$ $\mathscr{H}^k$-a.e. on $E \setminus \tilde{E}$. Hence $\mathscr{H}^{k-m}(E_y \setminus \tilde{E}) = 0$ for $\mathscr{L}^m$-a.e. y.

(ii) We can reduce to the case that f is of class C^1 (via the Lusin type approximation of Lipschitz functions with C^1 functions) and E is contained in a unique C^1-surface S. Since $\nabla_\tau f$ has maximal rank on $\tilde{E}$, then $\tilde{E} \cap f^{-1}(y)$ is contained in a $(k - m)$-dimensional surface of class C^1, for every y. The fact that E_y is $(k - m)$ rectifiable for $\mathscr{L}^m$-a.e. y follows from (i), writing

$$E_y = (E_y \setminus \tilde{E}) \cup (\tilde{E} \cap f^{-1}(y)).$$

(iii) The fact is trivial when E is contained in a C^1 surface. The general case follows from the properties of the weak tangent field to a rectifiable set.
(iv) This is the coarea formula (4.1) with $g = |\theta|$.
(v) This is just a summary of points (i)-(iv).

□

In general it is not possible to reconstruct the action of a current on a form knowing only the slices of the current according to a map f. The next proposition shows that it is possible to reconstruct such an action, when the form is "tangent" to the level sets of f.

4.6. Proposition (First operative definition of slicing for rectifiable currents). *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz and of class C^1 and let T and $\{T_y\}_y$ be as in Proposition 4.5. Then for every $\omega \in \mathscr{D}^{k-m}(\mathbb{R}^n)$ it holds*

$$\int_{y\in\mathbb{R}^m} \langle T_y, \omega\rangle d\mathscr{L}^m(y) = \langle T, df_1 \wedge \cdots \wedge df_m \wedge \omega\rangle.$$

the proof uses two simple fact of multilinear algebra, the proof of which is left as an exercise.

4.7. Lemma. *Let v and $\tilde{v}$ be simple m- and $\tilde{m}$-vectors, respectively, with $v \wedge \tilde{v} = 0$. Let α and $\tilde{\alpha}$ be m- and $\tilde{m}$-covectors such that $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_m$ satisfies $\alpha_i = 0$ on span$\{\tilde{v}\}$, for every $i = 1, \dots, m$. Then*

$$\langle \alpha \wedge \tilde{\alpha}, v \wedge \tilde{v}\rangle = \langle \alpha, v\rangle\langle\tilde{\alpha}, \tilde{v}\rangle.$$

4.8. Lemma. *Let $(V, \cdot)$ be any vector space endowed with a scalar product. Let $v_1, \dots, v_m \in V$, and for every $i = 1, \dots, m$, consider the covector v_i^0 defined by*

$$\langle v_i^0; w\rangle := v_i \cdot w.$$

Then we have

$$\langle v_1^0 \wedge \cdots \wedge v_m^0, v_1 \wedge \cdots \wedge v_m\rangle = |v_1 \wedge \cdots \wedge v_m|^2.$$

Proof of Proposition 4.6. Take a point x, in which

$$\eta(x) := \nabla_\tau f_1(x) \wedge \cdots \wedge \nabla_\tau f_m(x) \neq 0.$$

By Proposition 4.5 (iii), we have that $df_i = 0$ on span($\{\tilde{\tau}\}$), hence, by Lemma 4.7, for every form $\omega \in \mathscr{D}^{k-m}(\mathbb{R}^n)$, we have

$$\langle df_1 \wedge \cdots \wedge df_m \wedge \omega, \tau\rangle = \langle df_1 \wedge \cdots \wedge df_m \wedge \omega, \frac{\eta}{|\eta|} \wedge \tilde{\tau}\rangle = \langle df_1 \wedge \cdots \wedge df_m, \frac{\eta}{|\eta|}\rangle\langle\omega, \tilde{\tau}\rangle. \tag{4.2}$$

On the other hand,

$$\begin{aligned}\langle df_1 \wedge \cdots \wedge df_m, \eta\rangle &= \langle df_1 \wedge \cdots \wedge df_m, \nabla_\tau f_1 \wedge \cdots \wedge \nabla_\tau f_m\rangle \\ &= \langle d_\tau f_1 \wedge \cdots \wedge d_\tau f_m, \nabla_\tau f_1 \wedge \cdots \wedge \nabla_\tau f_m\rangle \\ &= |\nabla_\tau f_1 \wedge \cdots \wedge \nabla_\tau f_m|^2 = |\eta|^2\end{aligned} \tag{4.3}$$

Combining (4.2) and (4.3), we have

$$\langle T, df_1 \wedge \cdots \wedge df_m \wedge \omega = \int_E \langle df_1 \wedge \cdots \wedge df_m \wedge \omega, \tau\rangle\theta d\mathscr{H}^k = \int_E \langle\omega, \tilde{\tau}\rangle\theta J_\tau f d\mathscr{H}^k.$$

By the coarea formula (4.1) this is equal to

$$\int_{y\in\mathbb{R}^m}\left(\int_{E_y}\langle\omega, \tau\rangle\theta d\mathscr{H}^{k-m}\right)d\mathscr{L}^m(y) = \int_{\mathbb{R}^m}\langle T_y, \omega\rangle d\mathscr{L}^m(y).$$

□

4.9. Proposition (Second operative definition of slicing for rectifiable currents). *Let T be a rectifiable k-current in $\mathbb{R}^n$, with $\mathbb{M}(\partial T) < infty$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz and of class C^1. Let $\{T_y\}_y$ be as in Proposition 4.5. Then for $\mathscr{L}^1$-a.e. $y \in \mathbb{R}$ it holds*

$$T_y = \partial(T \llcorner \{f \leq y\}) - \partial T \llcorner \{f \leq y\}.$$

Proof. Let $\rho : \mathbb{R} \to \mathbb{R}$ be a positive smooth function which is supported on $[0, 1]$ and with $\int_0^1 \rho = 1$. Fix $\bar{y} \in \mathbb{R}$. For $\varepsilon > 0$, let $\rho_\varepsilon(y) := \frac{1}{\varepsilon}\rho\left(\frac{y-\bar{y}}{\varepsilon}\right)$. Let $R_\varepsilon : \mathbb{R} \to \mathbb{R}$ be the positive and smooth function such that $R_\varepsilon(\bar{y} + \varepsilon) = 0$ and $R'_\varepsilon = -\rho_\varepsilon$. For $\omega \in \mathscr{D}^{k-1}(\mathbb{R}^n)$, we compute

$$\begin{aligned}\int_{y\in\mathbb{R}} \langle T_y, \omega\rangle \rho_\varepsilon(y) dy = \int_{y\in\mathbb{R}} \langle T_y, (\rho_\varepsilon \circ f)\omega\rangle dy &\overset{\text{Prop.4.6}}{=} \langle T, (\rho_\varepsilon \circ f) df \wedge \omega\rangle = -\langle T, d(R_\varepsilon \circ f) \wedge \omega\rangle \\ = \langle T, (R_\varepsilon \circ f) d\omega\rangle - \langle T, d(R_\varepsilon \circ f)\omega\rangle &= \langle T, (R_\varepsilon \circ f) d\omega\rangle - \langle \partial T, (R_\varepsilon \circ f)\omega\rangle \\ &= \langle (R_\varepsilon \circ f)T, d\omega\rangle - \langle (R_\varepsilon \circ f)\partial T, \omega\rangle.\end{aligned}$$

where the last equality in the first line follows from the identity

$$d(R_\varepsilon \circ f) = -(\rho_\varepsilon \circ f) df$$

and the first equality in the second line follows from the identity

$$d((R_\varepsilon \circ f)\omega) = d(R_\varepsilon \circ f) \wedge \omega + (R_\varepsilon \circ f) d\omega.$$

Observe that when $\varepsilon \to 0$, we have that

$$(R_\varepsilon \circ f) \overset{*}{\rightharpoonup} T \llcorner \{f \leq \bar{y}\}$$

and

$$(R_\varepsilon \circ f)\partial T \overset{*}{\rightharpoonup} \partial T \llcorner \{f \leq \bar{y}\}.$$

Moreover, if $\bar{y}$ is a point of Lebesgue continuity of the map $y \mapsto \langle T_y, \omega\rangle$, then

$$\int_{y\in\mathbb{R}} \langle T_y \omega\rangle \rho_\varepsilon(y) dy \to \langle T_{\bar{y}}, \omega\rangle.$$

Take now a countable dense set of $\omega_j \in \mathscr{D}^{k-1}(\mathbb{R}^n)$. Then $\mathscr{L}^1$-a.e. $\bar{y}$ is a point of Lebesgue continuity of every map $y \mapsto \langle T_y, \omega_j\rangle$. To show this, we observe that all such maps are integrable, indeed

$$\int |\langle T_y, \omega_j\rangle| dy \leq \|\omega_j\|_\infty \int \mathbb{M}(T_y) dy \leq \|\omega_j\|_\infty [\mathrm{Lip}(f)]^k \mathbb{M}(t) < \infty.$$

$\square$

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