Exercise 1
Let $u \in L^1_{loc}(\bar{\mathbb{H}})$, where $\mathbb{H} = \{z = (x, y) \in \mathbb{R}^2;\ y > 0)\}$ and let $\bar{u}$ be its odd extension to $\mathbb{R}^2$, i.e.

$$\bar{u}(x,y) = \begin{cases} u(x,y) & y \geq 0, \\ -u(x,-y) & y < 0. \end{cases}$$

Prove that, if

$$\int_{\mathbb{H}} u \Delta\varphi \, dz = 0, \forall \varphi \in \mathbb{C}^\infty_c(\mathbb{R}^2), \quad \varphi \texttt{ odd},$$

then $\bar{u}$ is smooth harmonic. Show that if u if regular enough, this is equivalent to the following boundary value problem

$$\begin{cases} \Delta u = 0 & \texttt{in } \mathbb{H}, \\ u = 0 & \texttt{on } \{y = 0\}. \end{cases}$$

Exercise 2
Let $u \in C^2(G)$ be a harmonic function and G be a simply connected domain in $\mathbb{R}^2$. Prove the existence and uniqueness (up to constant) of a function $v \in C^2(G)$ harmonic such that $u + iv$ is holomorphic.

Exercise 3
Let F be the revolution surface given by

$$F(t,\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r(t) \\ 0 \\ z(t) \end{pmatrix},$$

where $t \in (a,b) \subset \mathbb{R}$, $\varphi \in \mathbb{R}$ and $r(t) > 0, \forall t \in (a,b)$. Give a conformal reparamerization.

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