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Parabolic Partial Differential Equations

SS 2019

Exercise Sheet 1

Turn in by May 2.

1. *(2+1 P)*

Let $u : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$ be defined by

$$u(x,t) = \frac{1}{\sqrt{4\pi t^n}} e^{-\frac{\|x\|^2}{4t}}.$$

This function is also known as the *fundamental solution* of the heat equation. Check that u is indeed a solution of

$$\partial_t u - \Delta u = 0.$$

Bonus question: Show that for any continuous and bounded function $f : \mathbb{R}^n \to \mathbb{R}$

$$\lim_{t \searrow 0} \int u(x,t) f(x) d\lambda^n(x) = f(0).$$

2. *(2P) Superposition principle.* Show that if $u, v \in C^2(\mathbb{R}^n \times (0, \infty))$ solve the heat equation and $\alpha, \beta \in \mathbb{R}$, then $\alpha u + \beta v$ also solves the heat equation.

3. *(2P)* Suppose $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map. Suppose that $u : \mathbb{R}^2 \to \mathbb{R}$ solves the linear parabolic equation

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + F = 0.$$

Show that $\hat{u} = u \circ \varphi$ also satisfies a linear parabolic equation

$$\hat{A}\hat{u}_{xx} + 2\hat{B}\hat{u}_{xy} + \hat{C}\hat{u}_{yy} + \hat{D}\hat{u}_x + \hat{E}\hat{u}_y + \hat{F} = 0.$$

4. *(2P)* Let λ be a real number and $v : \mathbb{R}^2 \to \mathbb{R}$ a solution of the Laplace equation, i.e. $\Delta v = 0$. Find $w : \mathbb{R} \to \mathbb{R}$, such that $u : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ defined by

$$u(x,t) = v(x)w(t)$$

solves

$$\partial_t u(x,t) - \Delta u(x,t) = \lambda u(x,t).$$