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Parabolic Partial Differential Equations

SS 2019

Exercise Sheet 2

Turn in by May 9.

1. *(2P)* Find an explicit solution $u : [0, \pi] \times [0, \infty) \to \mathbb{R}$ for the following initial value problem.

$$u_t(x,t) = u_{xx}(x,t)$$

$$u(0,t) = u(\pi,t)$$

$$u(x,0) = 2\sin(3x) + 4\frac{\tan(5x)}{\sqrt{1+\tan(5x)^2}} + 8\cos\left(\frac{\pi}{2} - 7x\right)$$

2. *(2P) Orthonormality relations.* For every $k \in \mathbb{N}$ define

$$e_k : [0, L] \to \mathbb{R}$$

$$e_k(t) = \sqrt{\frac{2}{L}} \sin\left(k\frac{\pi}{L}x\right)$$

Check that

$$\langle e_k, e_m \rangle_{L^2} = \int_0^L e_k(x) e_m(x) dx = \begin{cases} 1, & \text{if } k = m \\ 0, & \text{if } k \neq m \end{cases}$$

3. *(2P) Energy equality.* Let $u : [0, L] \to \mathbb{R}$ be a continuously differentiable function. The *Dirichlet energy* of u is defined to be

$$E(u) = \frac{1}{2} \int_0^L |u_x(x)|^2 dx.$$

Show that if $u : [0, L] \times [0, \infty) \to \mathbb{R}$ is twice continuously differentiable and solves the heat equation

$$u_t = u_{xx}$$

with periodic boundary conditions

$$u(0,t) = u(L,t),$$

then

$$\frac{d}{dt} E(u(\cdot, t)) = -\int_0^L |u_{xx}(x,t)|^2 dx.$$

4. *(2P) Uniqueness.* Suppose that $u, v : [0, L] \times [0, \infty) \to \mathbb{R}$ are twice continuously differentiable and solve the heat equation with periodic boundary conditions.

Show that if $u(x, 0) = v(x, 0)$ for $x \in [0, L]$, then $u(x,t) = v(x,t)$ for all $(x,t) \in [0, L] \times [0, \infty)$.

Hint: Consider the difference $u - v$ and use the energy equality.