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Parabolic Partial Differential Equations

SS 2019

Exercise Sheet 3

Turn in by May 16.

1. *(2P)* Compute the Fourier series for the function

$$f : [0, 2\pi] \to \mathbb{C}$$

$$f(x) = \frac{(\pi - x)^2}{4}$$

2. *(2P)* Find the solution $u : [0, \pi] \times [0, \infty) \to \mathbb{R}$ of the following initial value problem with Neumann boundary conditions:

$$\begin{cases} u_t - u_{xx} = 0 & 0 < x < \pi, t > 0 \\ u(x, 0) = x & 0 < x < \pi \\ u_x(0, t) = u_x(\pi, t) = 0 & t > 0 \end{cases}$$

3. *(2P) (Bounds on Fourier coefficients)* Let $f : [0, 2\pi] \to \mathbb{C}$ be a l times continuously differentiable periodic function. For $k \in \mathbb{Z}$ denote by a_k the Fourier coefficient

$$a_k = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x) e^{-i\pi kx} dx.$$

Show that

$$\sum_{k \in \mathbb{Z}} k^{2l} |a_k|^2 < \infty$$

4. *(2P)* Consider a twice continuously differentiable function $u : [0, L] \times [0, T] \to \mathbb{R}$ satisfying $u(0, t) = u(L, t)$ for all $t \in [0, T]$. Then

$$u_t - u_{xx} = f$$

$$u(0, \cdot) = g$$

for some function $f : [0, T] \times [0, L] \to \mathbb{R}$ and $g : [0, L] \to \mathbb{R}$.
Show that

$$\int_0^L |u_x(x, T)|^2 dx + \int_0^T \int_0^L u_t^2 + |u_{xx}|^2 dx dt = \int_0^T \int_0^L f^2 dx dt + \int_0^L |g_x|^2 dx.$$

Hint: Rewrite $\int_0^L f^2 dx$ using the equation. Then integrate in time.