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Parabolic Partial Differential Equations

SS 2019

Exercise Sheet 4

Turn in by May 23.

1. *(2P) (Parseval's identity)* For any $f \in L^2(-\pi, \pi)$, show that the following identity holds:

$$\|f\|_{L^2(-\pi,\pi)} = 2\pi \sum_{k\in\mathbb{Z}} |c_k|^2$$

where

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx.$$

2. *(2P) (Duhamel's formula)* Show that

$$u(x,t) = (T_t g)(x) + \int_0^t (T_{t-s} f(\cdot, s))(x)\, ds$$

is a solution of the inhomogeneous Dirichlet boundary problem

$$\begin{cases} u_t - u_{xx} = f & 0 < x < \pi, t > 0 \\ u(x,0) = g & 0 < x < \pi \\ u(0,t) = u(\pi,t) = 0 & t > 0 \end{cases}$$

For any $h \in L^2(0,\pi)$ the operator T_t is defined by the formula

$$(T_t h)(x) = \sum_{k=1}^{\infty} b_k e^{-\kappa k^2 t} \sin(kx),$$

where b_k are the sin Fourier coefficients of h.

3. *(2P) (Inhomogeneous heat eqation by Fourier coefficients)* Consider the inhomogeneous Dirichlet boundary problem

$$\begin{cases} u_t - u_{xx} = f & 0 < x < \pi, t > 0 \\ u(x,0) = g & 0 < x < \pi \\ u(0,t) = u(\pi,t) = 0 & t > 0 \end{cases}$$

where $g : [0,\pi] \to \mathbb{R}$ and $f : [0,\pi] \times [0,\infty) \to \mathbb{R}$ are such that $g \in L^2$ and $f(\cdot, t) \in L^2$ for every t.

(a)With b_k the sin Fourier coefficients of g and $f(x,t) = \sum_{k=1}^{\infty} c_k(t) \sin(kx)$, show how a solution u can be found by solving the system

$$\begin{cases} a_k'(t) + \kappa k^2 a_k(t) = c_k(t) \\ \qquad\qquad\quad a_k(0) = b_k \end{cases} \tag{1}$$

(b)By variation of constants, find an explicit formula for $A_k(t) = e^{\kappa k^2 t} a_k(t)$.

4. *(2P)* Find the solution $u : [0, \pi] \times [0, \infty) \to \mathbb{R}$ of the following inhomogeneous initial value problem with Dirichlet boundary conditions:

$$\begin{cases} u_t - u_{xx} = tx & 0 < x < \pi, t > 0 \\ u(x, 0) = x(\pi - x) & 0 < x < \pi \\ u(0, t) = u(\pi, t) = 0 & t > 0 \end{cases}$$