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Parabolic Partial Differential Equations

SS 2019

Exercise Sheet 5

Turn in by May 31.

1. *(2P)* For C^1 functions $f : B_1 \subset \mathbb{R}^n \to \mathbb{R}$ the H^1-Sobolev norm is given by

$$\|f\|^2_{H^1(B_1)} = \int_{B_1} |f(x)|^2 + |\nabla f(x)|^2 d\lambda^n(x).$$

Moreover, for $\alpha \in \mathbb{R}$, we define

$$g_\alpha : B_1 \to \mathbb{R}$$
$$x \mapsto |x|^\alpha.$$

For which α and n is $\|g_\alpha\|_{H^1} < \infty$?

Remark. In particular this exercise shows that for $n > 1$ there is no embedding

$$H^1(B_1) \hookrightarrow C^0(B_1).$$

2. *(2P)* Consider the following inhomogeneous heat equation

$$\begin{cases} u_t - u_{xx} = f & 0 < x < \pi, t > 0 \\ u(x,0) = g & 0 < x < \pi \\ u(0,t) = \varphi(t) & t \geq 0 \\ u(\pi,t) = \psi(t) & t \geq 0 \end{cases}$$

where $\varphi : [0,\infty) \to \mathbb{R}$, $\psi(t) : [0,\infty) \to \mathbb{R}$ and f, g as before.

Give a solution formula for this problem!

3. *(2P)* In this exercise, you are asked to study the asymptotic behavior of solutions of the heat equation: given a solution $u(x,t)$ of each of the following problems, describe $u(x,t)$ as $t \to \infty$:

- The homogeneous heat equation with Dirichlet boundary conditions $u(0,t) = u(\pi,t) = 0$
- The homogeneous heat equation with Neumann boundary conditions $u_x(0,t) = u_x(\pi,t) = 0$
- The inhomogeneous heat equation with Dirichlet boundary conditions, where the inhomogeneous term $f(x,t) = f(x)$ depends only on space, but not on time.

4. *(2P)* The most general $1d$ diffusion equation can be written as

$$u_t - u_{xx} = \lambda u + f(x,t), \tag{1}$$

where $\lambda \in \mathbb{R}$ and the terms λu represents the absorption ($\lambda < 0$) or reaction ($\lambda > 0$).

Show how solutions of this equation can be obtained from solutions of an inhomogeneous heat equation.

5. *(2P)* Let u be a solution of the initial value problem

$$\begin{cases} u_t - 23u_{xx} = 0 & -4 < x < 4, t > 0 \\ u(x,0) = -x^3 - 2x^2 + 15x & -4 < x < 4 \\ u(-4,t) = u(4,t) = 0 & t > 0 \end{cases}$$

What is the maximum of $u(x,t)$?

6. *(2P)* Suppose $u : B_R(0) \to \mathbb{R}$ solves the following initial value problem

$$\begin{cases} u_t - \Delta u = f & x \in B_R, t > 0 \\ u(x,0) = u_0(x) & x \in B_R(0) \\ u(x,t) = \lambda & x \in \partial B_R(0), t > 0 \end{cases},$$

where $\lambda \in \mathbb{R}$ and $u_0(x) = u_0(y)$ and $f(x,t) = f(y,t)$ for any $x, y \in B_R$, whenever $|x| = |y|$, i.e. that u_0 and f are radially symmetric.

Hint: Use the fact that if $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is given by $\varphi(x) = Ax$ for some orthogonal matrix $A \in \mathrm{Mat}(n \times n, \mathbb{R})$, then

$$\Delta(u \circ \varphi) = (\Delta u) \circ \varphi$$

and uniqueness of solutions of the heat equation.