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Parabolic Partial Differential Equations

SS 2019

Exercise Sheet 6

Turn in by Friday, June 21.

1. *(2P)* Prove the strong maximum principle for the heat equation: If $u \in C^{2,1}$ is a solution of

$$u_t - D\Delta u = f \leq 0 \text{ in } Q_T$$

and

$$u(x,t) = \max_{\bar{Q}_T} u = \max_{\partial_P Q_T} u$$

for some $(x,t) \in Q_T$, then u is constant.

2. *(2P)* Prove the following stability result

Let u^1 and u^2 be solutions to

$$\begin{cases} u^1_t - D\Delta u^1 = f^1(x,t) \\ u^2_t - D\Delta u^2 = f^2(x,t), \end{cases} \tag{1}$$

then the following stability estimates holds

$$\max_{\bar{Q}_T} |u^1 - u^2| \leq \max_{\partial_P Q_T} |u^1 - u^2| + T \max_{\bar{Q}_T} |f^1 - f^2|. \tag{2}$$

3. *(2P)* On the cylinder

$$C = \{(x,y,z) : x^2 + y^2 < R^2, 0 < z < H\}$$

of radius $R \in \mathbb{R}_+$ and length $H \in \mathbb{R}_+$, find a solution for the initial value problem

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } C \times [0,\infty) \\ u(\cdot, 0) = g & \text{on } C \times \{0\} \\ u(x,t) = 0 & \text{for } x \in \partial C \end{cases}$$

where g is a continuous, radially symmetric function, i.e. $g(x,y,z)$ only depends on $r(x,y) = \sqrt{x^2+y^2}$ and z.

Hint: The function

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+1)!} \left(\frac{x}{2}\right)^{2m}$$

is a *Bessel function of the first kind.*

The differential equation

$$u_{rr} + \frac{1}{r} u_r = -\lambda u$$

with $u(0) < \infty$ and $u(1) = 0$ has solution, if and only if $\lambda > 0$ and $J_0(\sqrt{\lambda}) = 0$. In that case, the solution is given by

$$u(r) = J_0(\sqrt{\lambda} r).$$

Denote these values by $\lambda_1, \lambda_2, \ldots$ and the corresponding solutions by $u_1, u_2, \ldots$.

The functions $(u_k)_k$ form a basis of $L^2(0,1)$ with respect to the weighted L^2 norm

$$\|f\|^2_{L^2(0,1)} = \int_0^1 t|f(t)|^2 dt.$$

4. Let u be a solution of the initial value problem

$$\begin{cases} u_t - u_{xx} = 0 & 0 < x < 1, t > 0 \\ u(x,0) = x(1-x)x & 0 < x < 1 \\ u(0,t) = u(1,t) = 0 & t > 0 \end{cases}$$

(a)Prove that u is nonnegative and find two positive numbers α, β such that

$$u(x,t) \leq \alpha x(1-x)e^{-\beta t}.$$

(b)Deduce that $u(\cdot, t)$ converges to 0 uniformly as $t \to \infty$.