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Parabolic Partial Differential Equations

SS 2019

Exercise Sheet 8

Turn in by Thursday, July 4.

1. Find all smooth, radially symmetric solutions $u : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ of

$$\Delta u = 0.$$

2. Find the solution of the problem

$$\partial_t u - D\Delta u = cu$$

$$u(x,0) = g(x)$$

on $\mathbb{R}^n$.

3. The *Heaviside function* is the function

$$H : \mathbb{R} \to \mathbb{R}$$

$$x \mapsto \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Show that the Dirac δ-function is a weak derivative of H in the following sense:

$$\int_{-\infty}^{\infty} H(x)(\partial_x \phi)(x)dx = \phi(0) = \int_{\mathbb{R}} \phi\delta$$

for any continuously differentiable function ϕ with compact support.

4. Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain and let $f \in L^2(\Omega)$. A weak solution of

$$\Delta u = f \text{ in } \Omega,$$

$$u = 0 \text{ on } \partial\Omega,$$

is a function $u \in H_0^1(\Omega)$, satisfying

$$B[u,v] = (f,v)_{L^2(\Omega)}$$

for all $v \in H_0^1(\Omega)$.

The space $H_0^1(\Omega)$ is the closure of the space $\mathcal{C}_{cpt}^{\infty}(\Omega)$ in $H^1(\Omega)$ with respect to the norm

$$\|u\|_{H^1(\Omega)}^2 = \|u\|_{L^2(\Omega)}^2 + \sum_{k=1}^{n} \|\partial_k u\|_{L^2(\Omega)}^2.$$

The bilinear form $B : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is defined by

$$B[u, v] = \int_\Omega \sum_{i=1}^n (\partial_i u(x))(\partial_i v(x)) dx.$$

Given $f \in L^2(\Omega)$ find a weak solution.

Hint: Apply Galerkin's method as follows. You may assume that there is an orthonormal basis $w_1, w_2, \dots$ of $L^2(\Omega)$, which is also an orthogonal basis of $H_0^1(\Omega)$.

Step 1: Construct $u_m = \sum_{k=1}^m d_m^k w_k$, solving the following set of equations:

$$B[u_m, w_k] = (f, w_k)_{L^2(\Omega)}, \qquad k = 1, \dots, m.$$

Step 2: Show that there is a uniform bound

$$\|u_m\|_{H_0^1(\Omega)} \leq C.$$

Step 3: Use weak compactness of H_0^1 to find a subsequence of u_m converging to a weak solution.