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Parabolic Partial Differential Equations

SS 2019

Exercise Sheet 8

Turn in by Thursday, July 11.

1. *(2P) [Weak Maximum Principle]* Suppose $u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ is bounded, twice continuously differentiable in $x \in \mathbb{R}^n$ and continously differentiable in $t \in [0,\infty)$ and satisfies

$$\begin{cases} \partial_t u - D\Delta u \leq 0 & \text{on } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^n \end{cases},$$

where $u_0 \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$.

Then

$$\sup_{\mathbb{R}^n \times (0,\infty)} u = \sup_{\mathbb{R}^n} u_0.$$

Hint: Use the maximum principle in bounded domains!

2. *(2P) [Strong Maximum Principle]* Suppose $u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ is bounded, twice continuously differentiable in $x \in \mathbb{R}^n$ and continously differentiable in $t \in [0,\infty)$ and satisfies

$$\begin{cases} \partial_t u - D\Delta u \leq 0 & \text{on } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^n \end{cases},$$

where $u_0 \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$.

If u attains a maximum at (x,t) with $t > 0$, then u is constant.

Hint: Use the fundamental solution!

3. *(2P)* Prove there is at most one solution $u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$, which is bounded, twice continuously differentiable in $x \in \mathbb{R}^n$ and continously differentiable in $t \in [0,\infty)$ satisfying

$$\begin{cases} \partial_t u - D\Delta u = 0 & \text{on } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^n \end{cases},$$

where $u_0 \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$.

4. *(2P)* Find the solution to the initial value problem

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{on } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = \exp(-|x|^2) & \text{for } x \in \mathbb{R}^n \end{cases}$$