Periods and Nori Motives

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Introduction (first draft)

Chapter 0

Introduction

The aim of this book is to present the theory of period numbers and their structural properties. The second main theme is the theory of motives and cohomology which is behind these structural properties. Whereas period numbers are quite close to transcendental number theory, motives are rooted in modern algebraic and arithmetic geometry. In combining both viewpoints, we want to present the strong link between formal properties of motives and some features of the exciting algebra of period numbers.

The genesis of this book is involved. Some time ago we were fascinated by a statement of Kontsevich [K1], stating that his algebra of formal periods is a pro-algebraic torsor under the motivic Galois group of motives. He attributed this theorem to Nori, but there was no proof indicated. After realizing this, we started to work out many details in our preprint [HMS] from 2011. For this, we were relying on Nori's lecture notes [N] and [N1], as well as the sketch of the construction by Levine [L1]. The input on period numbers came from Kontsevich [K1] and Kontsevich-Zagier [KZ], in particular elementary definitions of periods and the indication of a connection between period numbers and Nori motives.

Over the years we have come to realize that periods generate a lot of interest, very often by non-specialists who are not familiar with all the techniques going into the story. Hence we thought it would be worthwhile to make these details accessible to a wider audience. We started to write this monograph in a style suited also for non-expert readers by adding several introductory chapters and many examples.

A naive point of view

Period numbers are complex numbers defined as values of integrals

$$\int_\gamma \omega$$

of closed differential forms ω over certain domains of integration γ. One requires restrictive conditions on ω and γ, i.e., that γ is a region given by (semi)algebraic equations with rational coefficients, and the differential form ω is algebraic over $\mathbb{Q}$. The analogous definition can be made for other fields, but we restrict to the main case $k = \mathbb{Q}$ in this introduction.

Many interesting numbers occuring in mathematics can be described in this form.

1. $\log(2)$ is a period because $\int\limits_1^2 \frac{\mathrm{d}x}{x} = \log(2)$.
2. π is a period because $\int\limits_{x^2+y^2\leq 1} \mathrm{d}x\mathrm{d}y = \pi$.
3. The Cauchy integral yields a complex period
$$\int_{|z|=1} \frac{dz}{z} = 2\pi i \ .$$
4. Values of the Riemann zeta function like
$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = \int_{0<x<y<z<1} \frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{(1-x)yz}$$
are periods nubers as well.
5. Indeed, all multiple zeta values (see Chapter 14) are period numbers.
6. A basic observation is that all algebraic numbers are periods, e.g., $\sqrt{5}$ can obtained by integrating the differential form $\mathrm{d}x$ on the algebraic curve $y = x^2$ over the real region where $0 \leq y \leq 5$ and $x \geq 0$.

Period numbers turn up in many parts of mathematics, sometimes in very surprising situations. Of course, they are a traditional object of number theory and have been studied from different points of view. They also generate a lot of interest in mathematical physics because Feynman integrals for rational values of kinematical invariants are period numbers.

It is easy to write down periods. It is much harder to write down numbers which are non-periods. This is surprising, given that the set of all period numbers is a countable algebra containing of $\bar{\mathbb{Q}}$. Indeed, we expect that π^{-1} and the Euler number e are non-periods, but this is not known. We refer to Section 15.5 for an actual, not too explicit example of a non-period.

It is as hard to understand linear or algebraic relations between periods. This aspect of the story starts with Lindemann's 1882 proof of the transcendence of π and the transcendence of $\log(x)$ for $x \in \bar{\mathbb{Q}}$. Grothendieck formulated a conjecture on the transcendence degree of the field generated by the periods of any smooth projective variety. Historical traces of his ideas seem to go back at least to Leibniz, see Chapter 12. The latest development is Kontsevich's formulation of a period conjecture for the algebra of all periods: the only relations are the ones induced from the obvious ones, i.e., functoriality and long exact sequences in cohomology (see Chapter 12). The conjecture is very deep. As a very special case it implies the transcendence of $\zeta(n)$ for n odd. This is wide open, the best available result being the irrationality of $\zeta(3)$!

While this aspect is interesting and important, we really have nothing to say about it. Instead, we aim at explaining a more conceptual interpretation of period number and shed light on some structural properties of the algebra of periods numbers.

As an aside: Periods of integrals are also used in the theory of moduli of algebraic varieties. Given a family of projective varieties, Griffiths defined a map into a period domain by studing the function given by varying period numbers. We are not concerned with this point of view either.

A more conceptual point of view

The period integral $\int_\gamma \omega$ actually only depends on the class of ω in de Rham cohomology and on the class of γ in singular homology. Integration generalizes to the *period pairing* between algebraic de Rham cohomology and singular homology. It has values in $\mathbb{C}$, and the period numbers are precisely the image. Alternatively, one can formulate the relation as a *period isomorphism* between algebraic de Rham cohomology and singular cohomology – after extension of scalars to $\mathbb{C}$. The comparison morphism is then described by a matrix whose entries are periods. The most general situation one can allow here is *relative cohomology* of a possibly singular, possibly non-complete algebraic variety over $\mathbb{Q}$ with respect to a closed subvariety also defined over $\mathbb{Q}$.

In formulas: For a variety X over $\mathbb{Q}$, a closed subvariety Y over $\mathbb{Q}$, and every $i \geq 0$, there is an isomorphism

$$\mathrm{per}: H^i_{\mathrm{dR}}(X,Y) \otimes_{\mathbb{Q}} \mathbb{C} \to H^i_{\mathrm{sing}}(X^{\mathrm{an}}, Y^{\mathrm{an}}; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C},$$

where X^{an} denotes the analytic space attached to X. If X is smooth, X^{an} is simply the complex manifold defined by the same equations as X. The really important thing to point out is the fact that this isomorphism does not respect the $\mathbb{Q}$-structures on both sides. Indeed, consider $X = \mathbb{A}^1 \setminus \{0\} = \mathrm{Spec}\mathbb{Q}[T, T^{-1}]$ and $Y = \emptyset$. The first de Rham cohomology group is one-dimensional and generated by $\frac{dT}{T}$. The first singular cohomology is also one-dimensional, and generated by the dual of the unit circle in $X^{\mathrm{an}} = \mathbb{C}^*$. The comparison factor is the period integral $\int_{S^1} \frac{dT}{T} = 2\pi i$.

Relative cohomology of pairs is a common standard in algebraic topology. The analogue on the de Rham side is much less so, in particular if X and Y are not anymore smooth. Experts have been familiar with very general versions of algebraic de Rham cohomology as by-products of advanced Hodge theory, but no elementary discussion seems to be in the literature. One of our intentions is to provide this here in some detail.

An even more conceptual point of view

An even better language to use is the language of motives. Motives are objects in a universal abelian category attached to the category of algebraic varieties whose most important property is to have cohomology: singular and de Rham cohomology in our case. Every variety has a motive $h(X)$ which should decompose into components $h^i(X)$ for $i = 1, \dots, 2\dim X$. Singular cohomology of $h^i(X)$ is concentrated in degree i and equal to $H^i_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{Q})$ there. Unfortunately, the picture still is largely conjectural. Grothendieck first introduced motives in his approach to Weil conjectures. Pure motives – the ones attached to smooth projective varieties – have an unconditional definition, but their expected properties depend on a choice of equivalence relations and hence on standard conjectures. In the mixed case – all varieties – there are (at least) three candidates for an abelian category of mixed motives (absolute Hodge motives of Deligne and Jannsen; Nori's category; Ayoub's category). There are also a number of constructions of motivic triangulated categories (due to Hanamuara, Levine and Voevodsky) which we think of as derived categories of the true category of mixed motives. They turn out to be equivalent.

All standard properties of cohomology are assumed to be induced by properties of the category of motives. The Künneth formula for the product of two varieties is induced by a tensor structure on motives. Poincaré duality is induced by the existence of strong duals on motives. In fact, every abelian category of motives (conjectural or candidate) is a rigid tensor category. Singular cohomology is (supposed to be) a faithful and exact tensor functor on this tensor category. Hence, we have a Tannaka category. By the main theorem of Tannaka theory, the category has a Tannaka dual: an affine pro-algebraic group scheme whose finite dimensional representations are precisely mixed motives. This group scheme is the *motivic Galois group* G_{mot}.

This viewpoint allows a reinterpretation of the period algebra: singular and de Rham cohomology are two fibre functors on the same Tannaka category, hence there is a torsor of isomorphisms between them. The period isomorphism is nothing but a $\mathbb{C}$-valued point of this torsor.

While the foundations of the theory of motives are still open, the good news is that the definition of the period algebra does not depend on the particular definition chosen. This is in fact one of the main results in the present book, see Chapter 10.5. Indeed, all variants of the definition yield the same set of numbers, as we show in Part III. Among those are versions via cohomology of arbitrary pairs of varieties, or only those of a smooth varieties relative to divisors with normal crossings, or via semialgebraic simplices in $\mathbb{R}^n$, and alternatively, with rational differential forms or only regular ones, and with rational or algebraic coefficients.

Nevertheless, the point of view of Nori's category of motives turns out to be particularly well-suited in order to treat periods. Indeed, the most natural proof of the comparison results mentioned above is done in the language of Nori motives, see Chapter 12. This approach also fits nicely with the formulation of the period conjectures of Grothendieck and Kontsevich.

The period conjecture

Kontsevich in [K1] introduces a *formal period algebra* $\tilde{\mathbb{P}}^{\mathrm{eff}}$ where the $\mathbb{Q}$-linear generators are given by quadruples (X, Y, ω, γ) with X an algebraic variety over $\mathbb{Q}$, Y a closed subvariety, ω a class in $H^n_{\mathrm{dR}}(X, Y)$ and $\gamma \in H_n^{\mathrm{sing}}(X^{\mathrm{an}}, Y^n, \mathbb{Q})$. There are three types of relations:

1. linearity in ω and γ;
2. functoriality with respect to morphisms $f : (X, Y) \to (X', Y)$, i.e.,
$$(X, Y, f^*\omega, \gamma) \sim (X', Y', \omega, f_*\gamma);$$
3. compatibility with respect to connecting morphisms, i.e., for $Z \subset Y \subset X$ and $\partial : H^n_{\mathrm{dR}}(Y, Z) \to H^{n+1}_{\mathrm{dR}}(X, Y)$
$$(Y, Z, \omega, \delta\gamma) \sim (X, Y, \partial\omega, \gamma).$$

This becomes an algebra using the cup-product on cohomology. The relations are defined in a way such that there is a natural evaluation map

$$\tilde{\mathbb{P}}^{\mathrm{eff}} \to \mathbb{C}, \qquad (X, Y, \omega, \gamma) \mapsto \int_\gamma \omega.$$

Actually this is a variant of the original definition, see Chapter 12. In a second step, we localize with respect to the class of $(\mathbb{A}^1 \setminus \{0\}, \{1\}, dT/T, S^1)$, i.e., the formal period giving rise to $2\pi i$. Basically by definition, the image of $\tilde{\mathbb{P}}$ is the period algebra.

Conjecture 0.0.1 (Period Conjecture, Kontsevich [K1]). *The evaluation map is injective.*

Again, we have nothing to say about this conjecture. However, it shows that the elementary object $\tilde{\mathbb{P}}$ is quite natural in our context.

One of the main results in this book is the following result of Nori, which is stated already in [K1]

Theorem 0.0.2 (See Theorem 12.1.3)**.** *The formal period algebra* $\tilde{\mathbb{P}}$ *is a torsor under the motivic Galois group in the sense of Nori, i.e., of the Tannaka dual of Nori's category of motives.*

Under the period conjecture, this should be read as a deep structural result about the period algebra.

Main aim of this book

We want to explain all the notions used above, give complete proofs, and discuss a number of examples of particular interest.

We explain singular cohomology and algebraic de Rham cohomology and the period isomorphism between them. We introduce Nori's abelian category of mixed motives and the necessary generalization of Tannaka theory going into the definition. Various notions of period numbers are introduced and compared. The relation of the formal period algebra to period numbers and the motivic Galois group is explained. We work out examples like periods of curves, multiple zeta-values, Feyman integrals and special values of L-functions.

We strive for a reasonably self-contained presentation aimed also at non-specialists and graduate students.

Relation to the existing literature

Both periods and the theory of motives have a long and rich history. We prefer not to attempt a historical survey, but rather mention the papers closest to the present book.

The definition of the period algebra was folklore for quite some time. The explicit versions we are treating are due to Kontsevich and Zagier in [K1] and [KZ].

Nori's theory of motives became known through a series of talks that he gave, and notes of these talks that started to circulate, see [N], [N1]. Levine's survey article in [L1] sketches the main points.

Finally, we need to mention André's monograph [A2]. Depending on the point of view, one might say that we are looking at similar mathematics in the overlapping parts of both books, and at a completely disjoint part of the theory otherwise.

We recommend that anyone interested in a deeper understanding also study his exposition.

We now turn to a more detailed description of the actual contents of our book.

0.1 Nori motives and Tannaka duality

Motives are supposed to be the universal abelian category over which all cohomology theories factor. In this context, "cohomology theory" means a (mixed) Weil cohomology theory with properties modeled on singular cohomology. A more refined examples of a mixed Weil cohomology theories is the mixed Hodge structure on singular cohomology as defined by Deligne. Another one is ℓ-adic cohomology of the base change of the variety to the algebraic closure of the ground field. It carries a natural operation of the absolute Galois group of the ground field. Key properties are for example a Künneth formula for the product of algebraic varieties. There are other cohomology theories of algebraic varieties which do not follow the same pattern. Examples are algebraic K-theory, Deligne cohomology or étale cohomology over the ground field. In all these cases the Künneth formula fails.

Coming back to theories similar to singular cohomology: they all take values in rigid tensor categories, and this is how the Künneth formula makes sense. We expect the conjectural abelian category of mixed motives also to be a Tannakian category with singular cohomology as a fibre functor, i.e., a faithful exact tensor functor to $\mathbb{Q}$-vector spaces. Nori takes this as the starting point of his definition of his candidate for the category of mixed motives. His category is universal for all cohomology theories comparable to singular cohomology. This is not quite what we hope for, but it does in fact cover all examples we have.

Tannaka duality is built into the very definition. The construction has to main steps.

1. Nori first defines an abelian category which is universal for all cohomology theories compatible with singular cohomology. By construction, it comes with a functor on the category of pairs (X, Y) where X is a variety and Y a closed subvariety. Moreover, it is compatible with the long exact cohomology sequence for triples $X \subset Y \subset Z$.

2. He then introduces a tensor product and establishes rigidity.

The first step is completely formal and rests firmly on representation theory. The same construction can be made for any oriented graph and any representation in a category of modules over a noetherian ring. The abstract construction of this "diagram category" is explained in Chapter 6. Note that neither the tensor product nor rigidity is needed at this point. Nevertheless, Tannaka theory is woven into proving that the diagram category has the necessary universal property: it is initial among all abelian categories over which the representation factors. Looking closely at the arguments in Chapter 6, in particular Section 6.3, we find the same arguments that are used in [DMOS] in order to establish the existence of a Tannaka dual. In the case of a rigid tensor category, by Tannaka duality it is equal to the category of representations of an affine group scheme or equivalently co-representations of a Hopf algebra A. If we do not have rigidity, we do not have the antipodal map. If we do not have a tensor product, we do not have a multiplication. We are left with a coalgebra. Indeed, the diagram category can be described as the co-representations of an explicit coalgebra, if the coefficent ring is a Dedekind ring or a field.

Chapter 7 aims at introducing a rigid tensor structure on the diagram category, or equivalently a Hopf algebra structure on the coalgebra. The product is induced by a product structure on the diagram and multiplicative representations. Rigidity is actually deduced as a property of the diagram category. Nori has a strong criterion for rigidity. Instead of asking for a unit and a counit, we only need one of the two such that it becomes a duality under the representation. This rests on the fact that every algebraic submonoid of an algebraic group is an algebraic group. The argument is analogous to showing that every submonoid of a finite abstract group is a group. Multiplication by an element is injective in these cases, because it is injective on the group. If the monoid is finite, it also has to be surjective. Everything can also be applied to the diagram defined by any Tannaka category. Hence the exposition actually contains a full proof of Tannaka duality.

The second step is of completely different nature. It uses on an insight on algebraic varieties. This is the famous Basic Lemma of Nori, see Section 2.5. As it turned out, Beilinson and also Vilonen had independently found the lemmma before. However, it was Nori who recognized its significance in such motivic situations. Let us explain the problem first. We would like define the tensor product of two motives of the form $H^n(X,Y)$ and $H^{n'}(X',Y')$. The only formula that comes to mind is

$$H^n(X,Y)\otimes H^{n'}(X',Y')=H^N(X\times X',X\times Y'\cup Y\times X')$$

with $N=n+n'$. This is, however, completely false in general. Cup-product will give a map from the left to the right. By the Künneth formula, we get an isomorphism when taking the sum over all n,n' mit $n+n'=N$ on the left, but not for a single summand.

Nori simply defines a pair (X,Y) to be *good*, if its singular cohomology is concentrated in a single degree and, moreover, a free module. In the case of good pairs, the Künneth formula is compatible with the naive tensor product of motives. The Basic Lemma implies that the category of motives is generated by good pairs. The details are explained in Chapter 8, in particular Section 8.2.

We would like to mention an issue that was particularly puzzling to us. How is the graded commutativity of the Künneth formula dealt with in Nori's construction? This is one of the key problems in pure motives because is causes singular cohomology *not* to be compatible with the tensor structure on Chow motives. The signs can be fixed, but only after assumeing the Künneth standard conjecture. Nori's construction does not need to do anything about the problem. So, how does it go away? The answer is the commutative diagram on p. 163: the outer diagrams have signs, but luckily they cancel.

0.2 Cohomology theories

In quite some detail, we cover singular cohomology and algebraic de Rham cohomology of algebraic varieties and the period isomorphism between them.

In Chapter 2 we recall as much of the properties of singular cohomology that is needed in the sequel. We view it primarily as sheaf cohomology of the analytic space associated to a variety over a fixed subfield k of $\mathbb{C}$. In addition to standard properties like Poincaré duality and the Künneth formula, we also discuss more special properties.

One such is Nori's basic lemma: for a given affine variety X there is a closed subvariety $Y \subset$ such that relative cohomology is concentrated in a single degree. As discussed above, this is a crucial input for the construction of the tensor product on Nori motives. We give three proofs, two of them due to Nori, and an earlier one due to Beilinson.

In addition, in order to compare different possible definitions of the set of periods numbers, we need to understand triangulations of algebraic varieties by semi-algebraic simplices defined over $\mathbb{Q}$.

Finally, we give a description of singular cohomology in terms of a Grothendieck topology (the h$'$-topology) on analytic spaces which is used later in order to define the period isomorphism.

Algebraic de Rham cohomology is much less documented in the literature. Through Hodge theory, the specialists have understood for a long time what the correct definition in the singular case should be, but we are not aware of a coherent exposition of algebraic de Rham cohomology by itself. This is what Chapter 3 is providing. We first treat systematically the more standard case of a smooth variety where de Rham cohomology is given as hypercohomology of the de Rham complex. In a second step, starting in Section 3.2, we generalize to the singular case. We choose the approach of the first author and Jörder in [HJ] via the h-cohomology on the category of k-varieties, but also explain the relation to Deligne's approach via hypercovers and Hartshorne's approach via formal completion at the ideal of definition inside a smooth variety.

The final aim is to constract a natural isomorphism between singular cohomology and algebraic de Rham cohomology. This is established via the intermediate step of holomorphic de Rham cohomology. The comparison between singular and holomorphic de Rham cohomology comes from the Poincaré lemma: the de Rham complex is a resolution of the constant sheaf. The comparison between algebraic and holomorphic de Rham cohomology can be reduced to GAGA. This story is fairly well-known for smooth varieties. In our description with the h-topology, the singular case follows easily.

0.3 Periods

We have already periods at some length at the beginning of the introduction. Roughly, a period number is the value of an integral of a differential form over some algebraically defined domain. The definition can be made for any subfield k of $\mathbb{C}$. There are several versions of the definition in the literature and even more folklore versions around. They fall into three classes:

1. "Naive" definitions have as domains of integration semi-algebraic simplices in $\mathbb{R}^N$, over which one integrates rational differential forms defined over k (or over $\bar{k}$), as long as the integral converges, see Chapter 11.

2. In more advanced versions, let X be an algebraic variety, and $Y \subset X$ a subvariety, both defined over k, ω a closed algebraic differential form on X defined over k (or a de relative Rham cohomology class), and consider the period isomorphism between de Rham and singular cohomology. Periods are the numbers coming up as entries of the period matrix. Variants include the cases where X smooth, Y a divisor with normal crossings, or perhaps where X is affine, and smooth outside Y, see Chapter 9.

3. In the most sophisticated versions, take your favourite category of mixed motives and consider the period isomorphism between their de Rham and singular realization. Again, the entries of the period matrix are periods, see Chapter 10.

It is one of the main results of the present book that all these definitions agree. A direct proof of the equivalence of the different versions of cohomological periods is given in Chapter 9. A crucial ingredient of the proof is Nori's description of relative cohomology via the Basic Lemma. The comparison with periods of geometric Voevodsky motives, absolute Hodge motives and Nori motives is discussed in Chapter 10. In Chapter 11, we discuss periods as in 1. and show that they agree with cohomological periods.

The concluding Chapter 12 explains the deeper relation between periods of Nori motives and Kontsevich's period conjecture, as already ementioned earlier in the introduction.

0.4 Recent developments

The ideas of Nori have been taken up by other people in recent years, leading to a rapid development of understanding. We have refrained from trying to incorporate all these results. It is to early to know what the final version of the theory will be. However, we would like to give at least some indication in which direction things are going.

The construction of Nori motives has been generalized to categories over a base S by Arapura in [Ara] and Ivorra [Iv]. Arapura's approach is based on constructible sheaves. His categories allows pull-back and push-forward, the latter being a deep result. Ivorra's approach is based on perverse sheaves. Compatibility under the six functors formalism is open in his setting.

Harrer's thesis [Ha] gives full proofs (based on the sketch of Nori in [N2]) of the construction of the realization functor from Voevodsky's geometric motives to Nori motives.

A comparison result of a different flavour was obtained by Choudhury and Gallauer [CG]: they are able to show that Nori's motivic Galois group agrees with Ayoub's. The latter is defined via the Betti realization functor on triangulated motives over an arbitrary base. This yields formally a Hopf object in a derived category of vector spaces. It is a deep result of Ayoub's that the cohomology of this Hopf object is only concentrated in non-negative degrees. Hence its H^0 is a Hopf algebra, the algebra of functions on Ayoub's motivic Galois group.

The relation between these two objects, whose construction is very different, can be seen as a strong indication that Nori motives are really the true abelian category of mixed motives. One can strengthen this to the conjecture that Voevodsky motives are the derived category of Nori motives.

As usual, the case of 1-motives can be hoped to be more accessible and a very good testing ground for this type of cenjecture. Ayoub and Barbieri-Viale have shown in [AB] that the subcategory of 1-motives in Nori motives agrees with Deligne's 1-motives, and hence also with 1-motives in Voevodsky's category.

There has also been progress on the period aspect of our book. Ayoub, in [Ay], proved a version of the period conjecture in families. There is also independent unpublished work of Nori on a similar question [N3].

0.5 Leitfaden

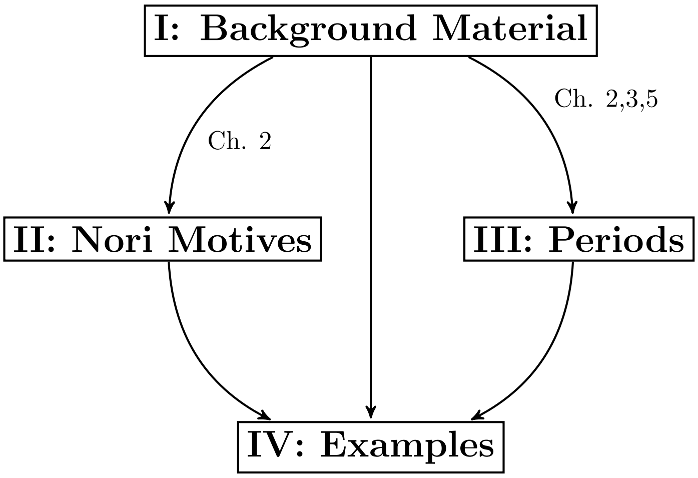
Part I, II, III and IV are supposed to be somewhat independent of each other, whereas the chapters in a given part depend more or less linearly on each other.

Part I is mostly meant as a reference for facts on cohomology that we need in the development of the theory. Most readers will skip this part and only come back to it when needed.

Part II is a self-contained introduction to the theory of Nori motives, where all parts build up on each other. Chapter 8 gives the actual definition. It needs the input from Chapter 2 on singular cohomology.

Part III develops the theory of period numbers. Chapter 9 on cohomological periods needs the period isomorphism of Chapter 5, and of course singular cohomology (Chapter 2) and algebraic de Rham cohomology (Chapter 3). It also develops the linear algebra part of the theory of period numbers needed in the rest of Part III. Chapter 10 has more of a survey character. It uses Nori motives, but should be understandable based just on the survey in Section 8.1. Chapter 11 is mostly self-contained, with some input from Chapter **??**. Finally, Chapter 12 relies on the full force of the theory of Nori motives, in particular on the abstract results on the comparison of fibre functors in Section 7.4.

Part IV has a different flavour: Rather than developing theory, we go through many examples of period numbers. Actually, it may be a good starting point for reading the book or at least a good companion for the more general theory developed in Part III.



0.6 Acknowledgements

This work is fundamentally based on some unpublished work of M. Nori. We thank him for several conversations. The presentation of his work in this book is ours and hence, of course, all mistakes are ours.

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