Periods and Nori Motives

Annette Huber und Stefan Müller-Stach,
with contributions by Benjamin Friedrich, Jonas von Wangenheim

September 22, 2015

Contents

Part

Introduction

Chapter 0

Introduction

The aim of this book is to present the theory of period numbers and their structural properties. The second main theme is the theory of motives and cohomology which is behind these structural properties.

The genesis of this book is involved. Some time ago we were fascinated by a statement of Kontsevich [K1], stating that his algebra of formal periods is a pro-algebraic torsor under the motivic Galois group of motives. He attributed this theorem to Nori, but no proof was indicated.

We came to understand that it would indeed follow more or less directly from Nori's unpublished description of an abelian category of motives. After realizing this, we started to work out many details in our preprint [HMS] from 2011.

Over the years we have also realized that periods themselves generate a lot of interest, very often from non-specialists who are not familiar with all the techniques going into the story. Hence we thought it would be worthwhile to make this background accessible to a wider audience.

We started to write this monograph in a style suited also for non-expert readers by adding several introductory chapters and many examples.

0.1 General introduction

So what are periods?

A naive point of view

Period numbers are complex numbers defined as values of integrals

$$\int_\gamma \omega$$

of closed differential forms ω over certain domains of integration γ. One requires restrictive conditions on ω and γ, i.e., that γ is a region given by (semi)algebraic equations with rational coefficients, and the differential form ω is algebraic over $\mathbb{Q}$. The analogous definition can be made for other fields, but we restrict to the main case $k = \mathbb{Q}$ in this introduction.

Many interesting numbers occuring in mathematics can be described in this form.

1. $\log(2)$ is a period because $\int_1^2 \frac{\mathrm{d}x}{x} = \log(2)$.
2. π is a period because $\int_{x^2+y^2\leq 1} \mathrm{d}x\mathrm{d}y = \pi$.
3. The Cauchy integral yields a complex period
$$\int_{|z|=1} \frac{dz}{z} = 2\pi i \ .$$
4. Values of the Riemann zeta function like
$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = \int_{0<x<y<z<1} \frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{(1-x)yz}$$
are periods nubers as well.
5. More generally, all multiple zeta values (see Chapter 14) are period numbers.
6. A basic observation is that all algebraic numbers are periods, e.g., $\sqrt{5}$ can obtained by integrating the differential form $\mathrm{d}x$ on the algebraic curve $y = x^2$ over the real region where $0 \leq y \leq 5$ and $x \geq 0$.

Period numbers turn up in many parts of mathematics, sometimes in very surprising situations. Of course, they are a traditional object of number theory and have been studied from different points of view. They also generate a lot of interest in mathematical physics because Feynman integrals for rational values of kinematical invariants are period numbers.

It is easy to write down periods. It is much harder to write down numbers which are non-periods. This is surprising, given that the set of all period numbers is a countable algebra containing of $\bar{\mathbb{Q}}$. Indeed, we expect that π^{-1} and the Euler number e are non-periods, but this is not known. We refer to Section 15.5 for an actual, not too explicit example of a non-period.

It is as hard to understand linear or algebraic relations between periods. This aspect of the story starts with Lindemann's 1882 proof of the transcendence of π and the transcendence of $\log(x)$ for $x \in \bar{\mathbb{Q}} \setminus \{0, 1\}$. Grothendieck formulated a conjecture on the transcendence degree of the field generated by the periods of any smooth projective variety. Historical traces of his ideas seem to go back at least to Leibniz, see Chapter 12. The latest development is Kontsevich's formulation of a period conjecture for the algebra of all periods: the only relations are the ones induced from the obvious ones, i.e., functoriality and long exact sequences in cohomology (see Chapter 12). The conjecture is very deep. As a very special case it implies the transcendence of $\zeta(n)$ for n odd. This is wide open, the best available result being the irrationality of $\zeta(3)$!

While this aspect is interesting and important, we really have almost nothing to say about it. Instead, we aim at explaining a more conceptual interpretation of period numbers and shed light on some structural properties of the algebra of periods numbers.

As an aside: Periods of integrals are also used in the theory of moduli of algebraic varieties. Given a family of projective varieties, Griffiths defined a map into a period domain by studing the function given by varying period numbers. We are not concerned with this point of view either.

A more conceptual point of view

The period integral $\int_\gamma \omega$ actually only depends on the class of ω in de Rham cohomology and on the class of γ in singular homology. Integration generalizes to the *period pairing* between algebraic de Rham cohomology and singular homology. It has values in $\mathbb{C}$, and the period numbers are precisely the image. Alternatively, one can formulate the relation as a *period isomorphism* between algebraic de Rham cohomology and singular cohomology – after extension of scalars to $\mathbb{C}$. The comparison morphism is then described by a matrix whose entries are periods. The most general situation one can allow here is *relative cohomology* of a possibly singular, possibly non-complete algebraic variety over $\mathbb{Q}$ with respect to a closed subvariety also defined over $\mathbb{Q}$.

In formulas: For a variety X over $\mathbb{Q}$, a closed subvariety Y over $\mathbb{Q}$, and every $i \geq 0$, there is an isomorphism

$$\mathrm{per} : H^i_{\mathrm{dR}}(X, Y) \otimes_{\mathbb{Q}} \mathbb{C} \to H^i_{\mathrm{sing}}(X^{\mathrm{an}}, Y^{\mathrm{an}}; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C},$$

where X^{an} denotes the analytic space attached to X. If X is smooth, X^{an} is simply the complex manifold defined by the same equations as X. The really important thing to point out is the fact that this isomorphism does not respect the $\mathbb{Q}$-structures on both sides. Indeed, consider $X = \mathbb{A}^1 \setminus \{0\} = \mathrm{Spec}\mathbb{Q}[T, T^{-1}]$ and $Y = \emptyset$. The first de Rham cohomology group is one-dimensional and generated by $\frac{dT}{T}$. The first singular cohomology is also one-dimensional, and generated by the dual of the unit circle in $X^{\mathrm{an}} = \mathbb{C}^*$. The comparison factor is the period integral $\int_{S^1} \frac{dT}{T} = 2\pi i$.

Relative cohomology of pairs is a common standard in algebraic topology. The analogue on the de Rham side is much less so, in particular if X and Y are not anymore smooth. Experts have been familiar with very general versions of algebraic de Rham cohomology as by-products of advanced Hodge theory, but no elementary discussion seems to be in the literature. One of our intentions is to provide this here in some detail.

An even more conceptual point of view

An even better language to use is the language of motives. The concept was introduced by Grothendieck in his approach to Weil conjectures.

Motives are objects in a universal abelian category attached to the category of algebraic varieties whose most important property is to have cohomology: singular and de Rham cohomology in our case. Every variety has a motive $h(X)$ which should decompose into components $h^i(X)$ for $i = 1, \dots, 2\dim X$. Singular cohomology of $h^i(X)$ is concentrated in degree i and equal to $H^i_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{Q})$ there.

Unfortunately, the picture still is largely conjectural. Pure motives – the ones attached to smooth projective varieties – have an unconditional definition due to Grothendieck, but their expected properties depend on a choice of equivalence relations and hence on standard conjectures. In the mixed case – all varieties – there are (at least) three candidates for an abelian category of mixed motives (absolute Hodge motives of Deligne and Jannsen; Nori's category; Ayoub's category). There are also a number of constructions of motivic triangulated categories (due to Hanamuara, Levine and Voevodsky) which we think of as derived categories of the true category of mixed motives. They turn out to be equivalent.

All standard properties of cohomology are assumed to be induced by properties of the category of motives: the Künneth formula for the product of two varieties is induced by a tensor structure on motives; Poincaré duality is induced by the existence of strong duals on motives. In fact, every abelian category of motives (conjectural or candidate) is a rigid tensor category. Singular cohomology is (supposed to be) a faithful and exact tensor functor on this tensor category. Hence, we have a Tannaka category. By the main theorem of Tannaka theory, the category has a Tannaka dual: an affine pro-algebraic group scheme whose finite dimensional representations are precisely mixed motives. This group scheme is the *motivic Galois group* G_{mot}.

This viewpoint allows a reinterpretation of the period algebra: singular and de Rham cohomology are two fibre functors on the same Tannaka category, hence there is a torsor of isomorphisms between them. The period isomorphism is nothing but a $\mathbb{C}$-valued point of this torsor.

While the foundations of the theory of motives are still open, the good news is that at least the definition of the period algebra does not depend on the particular definition chosen. This is in fact one of the main results in the present book, see Chapter 10.5. Indeed, all variants of the definition yield the same set of numbers, as we show in Part III. Among those are versions via cohomology of arbitrary pairs of varieties, or only those of smooth varieties relative to divisors with normal crossings, or via semialgebraic simplices in $\mathbb{R}^n$, and alternatively, with rational differential forms or only regular ones, and with rational or algebraic coefficients.

Nevertheless, the point of view of Nori's category of motives turns out to be particularly well-suited in order to treat periods. Indeed, the most natural proof of the comparison results mentioned above is done in the language of Nori motives, see Chapter 12. This approach also fits nicely with the formulation of the period conjectures of Grothendieck and Kontsevich.

The period conjecture

Kontsevich in [K1] introduces a *formal period algebra* $\tilde{\mathbb{P}}^{\mathrm{eff}}$ where the $\mathbb{Q}$-linear generators are given by quadruples (X, Y, ω, γ) with X an algebraic variety over $\mathbb{Q}$, Y a closed subvariety, ω a class in $H^n_{\mathrm{dR}}(X, Y)$ and $\gamma \in H_n^{\mathrm{sing}}(X^{\mathrm{an}}, Y^n, \mathbb{Q})$. There are three types of relations:

1. linearity in ω and γ;
2. functoriality with respect to morphisms $f : (X, Y) \to (X', Y)$, i.e.,
$$(X, Y, f^*\omega, \gamma) \sim (X', Y', \omega, f_*\gamma);$$
3. compatibility with respect to connecting morphisms, i.e., for $Z \subset Y \subset X$ and $\delta : H^{n-1}_{\mathrm{dR}}(Y, Z) \to H^n_{\mathrm{dR}}(X, Y)$
$$(Y, Z, \omega, \partial\gamma) \sim (X, Y, \delta\omega, \gamma).$$

The set $\tilde{\mathbb{P}}^{\mathrm{eff}}$ becomes an algebra using the cup-product on cohomology. The relations are defined in a way such that there is a natural evaluation map
$$\tilde{\mathbb{P}}^{\mathrm{eff}} \to \mathbb{C}, \qquad (X, Y, \omega, \gamma) \mapsto \int_\gamma \omega.$$

(Actually this is a variant of the original definition, see Chapter 12.) In a second step, we localize with respect to the class of $(\mathbb{A}^1 \setminus \{0\}, \{1\}, dT/T, S^1)$, i.e., the formal period giving rise to $2\pi i$. Basically by definition, the image of $\tilde{\mathbb{P}}$ is the period algebra.

Conjecture 0.1.1 (Period Conjecture, Kontsevich [K1])**.** *The evaluation map is injective.*

Again, we have nothing to say about this conjecture. However, it shows that the elementary object $\tilde{\mathbb{P}}$ is quite natural in our context.

One of the main results in this book is the following result of Nori, which is stated already in [K1]

Theorem 0.1.2 (See Theorem 12.1.3). *The formal period algebra $\tilde{\mathbb{P}}$ is a torsor under the motivic Galois group in the sense of Nori, i.e., of the Tannaka dual of Nori's category of motives.*

Under the period conjecture, this should be read as a deep structural result about the period algebra.

Main aim of this book

We want to explain all the notions used above, give complete proofs, and discuss a number of examples of particular interest.

- We explain singular cohomology and algebraic de Rham cohomology and the period isomorphism between them.
- We introduce Nori's abelian category of mixed motives and the necessary generalization of Tannaka theory going into the definition.
- Various notions of period numbers are introduced and compared.
- The relation of the formal period algebra to period numbers and the motivic Galois group is explained.
- We work out examples like periods of curves, multiple zeta-values, Feynman integrals and special values of L-functions.

We strive for a reasonably self-contained presentation aimed also at non-specialists and graduate students.

Relation to the existing literature

Both periods and the theory of motives have a long and rich history. We prefer not to attempt a historical survey, but rather mention the papers closest to the present book.

The definition of the period algebra was folklore for quite some time. The explicit versions we are treating are due to Kontsevich and Zagier in [K1] and [KZ].

Nori's theory of motives became known through a series of talks that he gave, and notes of these talks that started to circulate, see [N], [N1]. Levine's survey article in [L1] sketches the main points.

The relation between (Nori) motives and formal periods is formulated by Kontsevich [K1].

Finally, we would like to mention André's monograph [A2]. Superficially, there is a lot of overlap (motives, Tannaka theory, periods). However, as our perspective is very different, we end up covering a lot of disjoint material as well.

We recommend that anyone interested in a deeper understanding also study his exposition.

0.2 Recent developments

The ideas of Nori have been taken up by other people in recent years, leading to a rapid development of understanding. We have refrained from trying to incorporate all these results. It is too early to know what the final version of the theory will be. However, we would like to give at least some indication in which direction things are going.

The construction of Nori motives has been generalized to categories over a base S by Arapura in [Ara] and Ivorra [Iv]. Arapura's approach is based on constructible sheaves. His categories allows pull-back and push-forward, the latter being a deep result. Ivorra's approach is based on perverse sheaves. Compatibility under the six functors formalism is open in his setting.

Harrer's thesis [Ha] gives full proofs (based on the sketch of Nori in [N2]) of the construction of the realization functor from Voevodsky's geometric motives to Nori motives.

A comparison result of a different flavour was obtained by Choudhury and Gallauer [CG]: they are able to show that Nori's motivic Galois group agrees with Ayoub's. The latter is defined via the Betti realization functor on triangulated motives over an arbitrary base. This yields formally a Hopf object in a derived category of vector spaces. It is a deep result of Ayoub's that the cohomology of this Hopf object is only concentrated in non-negative degrees. Hence its H^0 is a Hopf algebra, the algebra of functions on Ayoub's motivic Galois group.

The relation between these two objects, whose construction is very different, can be seen as a strong indication that Nori motives are really the true abelian category of mixed motives. One can strengthen this to the conjecture that Voevodsky motives are the derived category of Nori motives.

In the same way as for other questions about motives, the case of 1-motives can be hoped to be more accessible and a very good testing ground for this type of cenjecture. Ayoub and Barbieri-Viale have shown in [AB] that the subcategory of 1-motives in Nori motives agrees with Deligne's 1-motives, and hence also with 1-motives in Voevodsky's category.

There has also been progress on the period aspect of our book. Ayoub, in [Ay], proved a version of the period conjecture in families. There is also independent unpublished work of Nori on a similar question [N3].

We now turn to a more detailed description of the actual contents of our book.

0.3 Nori motives and Tannaka duality

Motives are supposed to be the universal abelian category over which all cohomology theories factor. In this context, "cohomology theory" means a (mixed) Weil cohomology theory with properties modeled on singular cohomology. A more refined example of a mixed Weil cohomology theories is the mixed Hodge structure on singular cohomology as defined by Deligne. Another one is ℓ-adic cohomology of the base change of the variety to the algebraic closure of the ground field. It carries a natural operation of the absolute Galois group of the ground field. Key properties are for example a Künneth formula for the product of algebraic varieties. There are other cohomology theories of algebraic varieties which do not follow the same pattern. Examples are algebraic K-theory, Deligne cohomology or étale cohomology over the ground field. In all these cases the Künneth formula fails.

Coming back to theories similar to singular cohomology: they all take values in rigid tensor categories, and this is how the Künneth formula makes sense. We expect the conjectural abelian category of mixed motives also to be a Tannakian category with singular cohomology as a fibre functor, i.e., a faithful exact tensor functor to $\mathbb{Q}$-vector spaces. Nori takes this as the starting point of his definition of his candidate for the category of mixed motives. His category is universal for all cohomology theories comparable to singular cohomology. This is not quite what we hope for, but it does in fact cover all examples we have.

Tannaka duality is built into the very definition. The construction has two main steps:

1. Nori first defines an abelian category which is universal for all cohomology theories compatible with singular cohomology. By construction, it comes with a functor on the category of pairs (X, Y) where X is a variety and Y a closed subvariety. Moreover, it is compatible with the long exact cohomology sequence for triples $X \subset Y \subset Z$.

2. He then introduces a tensor product and establishes rigidity.

The first step is completely formal and rests firmly on representation theory. The same construction can be made for any oriented graph and any representation in a category of modules over a noetherian ring. The abstract construction of this "diagram category" is explained in Chapter 6. Note that neither the tensor product nor rigidity is needed at this point. Nevertheless, Tannaka theory is woven into proving that the diagram category has the necessary universal property: it is initial among all abelian categories over which the representation factors. Looking closely at the arguments in Chapter 6, in particular Section 6.3, we find the same arguments that are used in [DMOS] in order to establish the existence of a Tannaka dual. In the case of a rigid tensor category, by Tannaka duality it is equal to the category of representations of an affine group scheme or equivalently co-representations of a Hopf algebra A. If we do not have rigidity, we do not have the antipodal map. We are left with a bialgebra. If we do not have a tensor product, we do not have a multiplication. We are left with a coalgebra. Indeed, the diagram category can be described as the co-representations of an explicit coalgebra, if the coefficent ring is a Dedekind ring or a field.

Chapter 7 aims at introducing a rigid tensor structure on the diagram category, or equivalently a Hopf algebra structure on the coalgebra. The product is induced by a product structure on the diagram and multiplicative representations. Rigidity is actually deduced as a property of the diagram category. Nori has a strong criterion for rigidity. Instead of asking for a unit and a counit, we only need one of the two such that it becomes a duality under the representation. This rests on the fact that an algebraic submonoid of an algebraic group is an algebraic group. The argument is analogous to showing that a submonoid of a finite abstract group is a group. Multiplication by an element is injective in these cases, because it is injective on the group. If the monoid is finite, it also has to be surjective. Everything can also be applied to the diagram defined by any Tannaka category. Hence the exposition actually contains a full proof of Tannaka duality.

The second step is of completely different nature. It uses an insight on algebraic varieties. This is the famous Basic Lemma of Nori, see Section 2.5. As it turned out, Beilinson and also Vilonen had independently found the lemmma before. However, it was Nori who recognized its significance in such motivic situations. Let us explain the problem first. We would like to define the tensor product of two motives of the form $H^n(X,Y)$ and $H^{n'}(X',Y')$. The only formula that comes to mind is

$$H^n(X,Y)\otimes H^{n'}(X',Y') = H^N(X\times X', X\times Y'\cup Y\times X')$$

with $N=n+n'$. This is, however, completely false in general. Cup-product will give a map from the left to the right. By the Künneth formula, we get an isomorphism when taking the sum over all n,n' mit $n+n'=N$ on the left, but not for a single summand.

Nori simply defines a pair (X,Y) to be *good*, if its singular cohomology is concentrated in a single degree and, moreover, a free module. In the case of good pairs, the Künneth formula is compatible with the naive tensor product of motives. The Basic Lemma implies that the category of motives is generated by good pairs. The details are explained in Chapter 8, in particular Section 8.2.

We would like to mention an issue that was particularly puzzling to us. How is the graded commutativity of the Künneth formula dealt with in Nori's construction? This is one of the key problems in pure motives because it causes singular cohomology *not* to be compatible with the tensor structure on Chow motives. The signs can be fixed, but only after assuming the Künneth standard conjecture. Nori's construction does not need to do anything about the problem. So, how does it go away? The answer is the commutative diagram on p. 165: the outer diagrams have signs, but luckily they cancel.

0.4 Cohomology theories

In Part I, we develop singular cohomology and algebraic de Rham cohomology of algebraic varieties and the period isomorphism between them in some detail.

In Chapter 2, we recall as much of the properties of singular cohomology that is needed in the sequel. We view it primarily as sheaf cohomology of the analytic space associated to a variety over a fixed subfield k of $\mathbb{C}$. In addition to standard properties like Poincaré duality and the Künneth formula, we also discuss more special properties.

One such is Nori's Basic Lemma: for a given affine variety X there is a closed subvariety $Y \subset$ such that relative cohomology is concentrated in a single degree. As discussed above, this is a crucial input for the construction of the tensor product on Nori motives. We give three proofs, two of them due to Nori, and an earlier one due to Beilinson.

In addition, in order to compare different possible definitions of the set of periods numbers, we need to understand triangulations of algebraic varieties by semi-algebraic simplices defined over $\mathbb{Q}$.

Finally, we give a description of singular cohomology in terms of a Grothendieck topology (the h′-topology) on analytic spaces which is used later in order to define the period isomorphism.

Algebraic de Rham cohomology is much less documented in the literature. Through Hodge theory, the specialists have understood for a long time what the correct definition in the singular case are, but we are not aware of a coherent exposition of algebraic de Rham cohomology by itself. This is what Chapter 3 is providing. We first treat systematically the more standard case of a smooth variety where de Rham cohomology is given as hypercohomology of the de Rham complex. In a second step, starting in Section 3.2, we generalize to the singular case. We choose the approach of the first author and Jörder in [HJ] via the h-cohomology on the category of k-varieties, but also explain the relation to Deligne's approach via hypercovers and Hartshorne's approach via formal completion at the ideal of definition inside a smooth variety.

The final aim is to constract a natural isomorphism between singular cohomology and algebraic de Rham cohomology. This is established via the intermediate step of holomorphic de Rham cohomology. The comparison between singular and holomorphic de Rham cohomology comes from the Poincaré lemma: the de Rham complex is a resolution of the constant sheaf. The comparison between algebraic and holomorphic de Rham cohomology can be reduced to GAGA. This story is fairly well-known for smooth varieties. In our description with the h-topology, the singular case follows easily.

0.5 Periods

We have already discussed periods at some length at the beginning of the introduction. Roughly, a period number is the value of an integral of a differential form over some algebraically defined domain. The definition can be made for any subfield k of $\mathbb{C}$. There are several versions of the definition in the literature and even more folklore versions around. They fall into three classes:

1. "Naive" definitions have as domains of integration semi-algebraic simplices in $\mathbb{R}^N$, over which one integrates rational differential forms defined over k (or over $\bar{k}$), as long as the integral converges, see Chapter 11.

2. In more advanced versions, let X be an algebraic variety, and $Y \subset X$ a subvariety, both defined over k, ω a closed algebraic differential form on X defined over k (or a de relative Rham cohomology class), and consider the period isomorphism between de Rham and singular cohomology. Periods are the numbers coming up as entries of the period matrix. Variants include the cases where X smooth, Y a divisor with normal crossings, or perhaps where X is affine, and smooth outside Y, see Chapter 9.

3. In the most sophisticated versions, take your favourite category of mixed motives and consider the period isomorphism between their de Rham and singular realization. Again, the entries of the period matrix are periods, see Chapter 10.

It is one of the main results of the present book that all these definitions agree. A direct proof of the equivalence of the different versions of cohomological periods is given in Chapter 9. A crucial ingredient of the proof is Nori's description of relative cohomology via the Basic Lemma. The comparison with periods of geometric Voevodsky motives, absolute Hodge motives and Nori motives is discussed in Chapter 10. In Chapter 11, we discuss periods as in 1. and show that they agree with cohomological periods.

The concluding Chapter 12 explains the deeper relation between periods of Nori motives and Kontsevich's period conjecture, as already ementioned earlier in the introduction. We also discuss the period conjecture itself.

0.6 Leitfaden

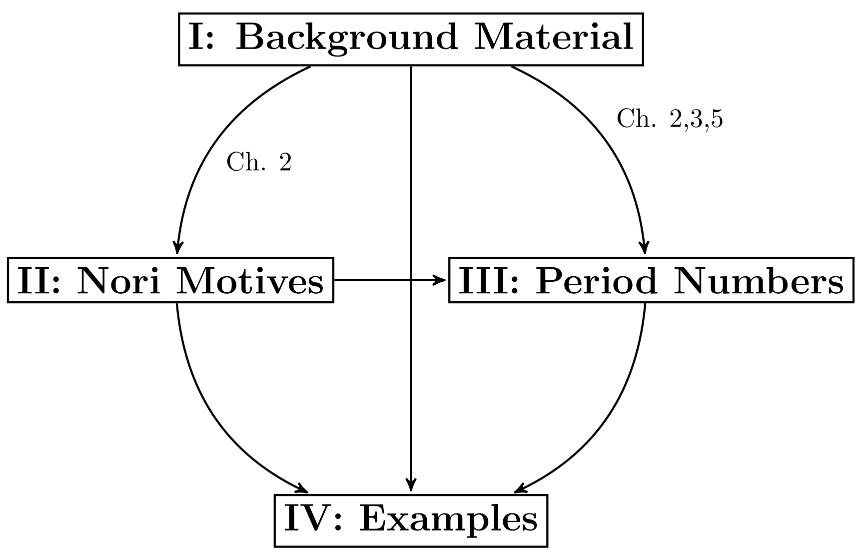
Part I, II, III and IV are supposed to be somewhat independent of each other, whereas the chapters in a each part depend more or less linearly on each other.

Part I is mostly meant as a reference for facts on cohomology that we need in the development of the theory. Most readers will skip this part and only come back to it when needed.

Part II is a self-contained introduction to the theory of Nori motives, where all parts build up on each other. Chapter 8 gives the actual definition. It needs the input from Chapter 2 on singular cohomology.

Part III develops the theory of period numbers. Chapter 9 on cohomological periods needs the period isomorphism of Chapter 5, and of course singular cohomology (Chapter 2) and algebraic de Rham cohomology (Chapter 3). It also develops the linear algebra part of the theory of period numbers needed in the rest of Part III. Chapter 10 has more of a survey character. It uses Nori motives, but should be understandable based just on the survey in Section 8.1. Chapter 11 is mostly self-contained, with some input from Chapter 9. Finally, Chapter 12 relies on the full force of the theory of Nori motives, in particular on the abstract results on the comparison of fibre functors in Section 7.4.

Part IV has a different flavour: Rather than developing theory, we go through many examples of period numbers. Actually, it may be a good starting point for reading the book or at least a good companion for the more general theory developed in Part III.



0.7 Acknowledgements

This work is fundamentally based on some unpublished work of M. Nori. We thank him for several conversations. The presentation of his work in this book is ours and hence, of course, all mistakes are ours.

Besides the preprint [HMS] of the main authors, this book is built on the work of B. Friedrich [Fr] on periods and J. von Wangenheim [vW] on diagram categories.

We are very grateful to B. Friedrich and J. von Wangenheim for allowing us to use their work in this book. The material of Friedrich's preprint is contained in Section 2.6, Chapters 9, 11, 13, and also 14. The diploma thesis of Wangenheim is basically Chapter 6.

Special thanks go to J. Ayoub, G. Wüstholz for organizing with us the Alpbach Workshop "Motives, periods and transcendence" on [HMS] and related topics in 2011. We thank all participants for their careful reading and subsequent corrections. In particular, we would like to mention M. Gallauer, who found a severe flaw in Chapter 7 – and fixed it.

We have taken advantage from discussions and comments of Y. André, M. Gallauer, D. Harrer, F. Hörmann, P. Jossen, S. Kebekus, D. van Straten, K. Völkel, and M. Wendt.

We are very grateful to W. Soergel for his continuous support in the Tannakian aspects of Part II. His insights greatly improved the exposition. We thank J. Schürmann for his patient explanations on weakly constructible sheaves in Section 2.5. We would not have been able to find all these references without him.

We also thank all participants of the lectures on the topic held in Mainz and in Freiburg in 2014 and 2015.

Part I

Background Material

Chapter 1

General Set-up

In this chapter we collect some standard notation used throughout the book.

1.1 Varieties

Let k be field. It will almost always be of characteristic zero or even a subfield of $\mathbb{C}$.

By a *scheme* over k we mean a separated scheme of finite type over k. Let Sch be the category of k-schemes. By a *variety* over k we mean a quasi-projective reduced scheme of finite type over k. Let Var be the category of varieties over k. Let Sm and Aff be the full subcategories of smooth varieties and affine varieties, respectively.

1.1.1 Linearizing the category of varieties

We also need the additive categories generated by these categories of varieties. More precisely:

Definition 1.1.1. Let $\mathbb{Z}[\mathrm{Var}]$ be the category with objects the objects of Var. If $X = X_1 \cup \cdots \cup X_n$, $Y = Y_1 \cup \cdots \cup Y_m$ are varieties with connected components X_i, Y_j, we put

$$\mathrm{Mor}_{\mathbb{Z}[\mathrm{Var}]}(X,Y) = \left\{ \sum_{i,j} a_{ij} f_{ij} | a_{ij} \in \mathbb{Z},\ f_{ij} \in \mathrm{Mor}_{\mathrm{Var}}(X_i, Y_j) \right\}$$

with the addition of formal linear combinations. Composition of morphisms is defined by extending composition of morphisms of varieties $\mathbb{Z}$-linearly.

Analogously, we define $\mathbb{Z}[\mathrm{Sm}]$, $\mathbb{Z}[\mathrm{Aff}]$ from Sm and Aff. Moreover, let $\mathbb{Q}[\mathrm{Var}]$,

$\mathbb{Q}$[Sm] and $\mathbb{Q}$[Aff] be the analogous $\mathbb{Q}$-linear additive categories where morphisms are formal $\mathbb{Q}$-linear combinations of morphisms of varieties.

Let $F = \sum a_i f_i : X \to Y$ be a morphism in $\mathbb{Z}$[Var]. The *support* of F is the set of f_i with $a_i \neq 0$.

$\mathbb{Z}$[Var] is an additive category with direct sum given by the disjoint union of varieties. The zero object corresponds to the empty variety, or, if you prefer, has to be added formally.

We are also going to need the category of *smooth correspondences* SmCor. It has the same objects as Sm and as morphisms *finite correspondences*

$$\mathrm{Mor}_{\mathrm{SmCor}}(X, Y) = \mathrm{Cor}(X, Y),$$

where $\mathrm{Cor}(X, Y)$ is the free $\mathbb{Z}$-module with generators integral subschemes $\Gamma \subset X \times Y$ such that $\Gamma \to X$ is finite and dominant over a component of X.

1.1.2 Divisors with normal crossings

Definition 1.1.2. Let X be a smooth variety of dimension n and $D \subset X$ a closed subvariety. D is called *divisor with normal crossings* if for every point of D there is an affine neighbourhood U of x in X which is étale over $\mathbb{A}^n$ via coordinates $t_1, \ldots, t_n$ and such that $D|_U$ has the form

$$D|_U = V(t_1 t_2 \cdots t_r)$$

for some $1 \leq r \leq n$.

D is called a *simple divisor with normal crossings* if in addition the irreducible components of D are smooth.

In other words, D looks étale locally like an intersection of coordinate hyperplanes.

Example 1.1.3. Let $D \subset \mathbb{A}^2$ be the nodal curve, given by the equation $y^2 = x^2(x-1)$. It is smooth in all points different from $(0,0)$ and looks étale locally like $xy = 0$ in the origin. Hence it is a divisor with normal crossings but not a simple normal crossings divisor.

1.2 Complex analytic spaces

A classical reference for complex analytic spaces is the book of Grauert and Remmert [GR].

Definition 1.2.1. A *complex analytic space* is a locally ringed space $(X, \mathcal{O}_X^{\mathrm{hol}})$ with X paracompact and Hausdorff, and such that $(X, \mathcal{O}_X^{\mathrm{hol}})$ is locally isomorphic to the vanishing locus Z of a set S of holomorphic functions in some open

$U \subset \mathbb{C}^n$ and $\mathcal{O}_Z^{\text{hol}} = \mathcal{O}_U^{\text{hol}}/\langle S \rangle$, where $\mathcal{O}_U^{\text{hol}}$ is the sheaf of holomorphic functions on U.

A *morphism* of complex analytic spaces is a morphism $f : (X, \mathcal{O}_X^{\text{hol}}) \to (Y, \mathcal{O}_Y^{\text{hol}})$ of locally ringed spaces, which is given by a morphism of sheaves $\tilde{f} : \mathcal{O}_Y^{\text{hol}} \to f_*\mathcal{O}_X^{\text{hol}}$ that sends a germ $h \in \mathcal{O}_{Y,y}^{\text{hol}}$ of a holomorphic function h near y to the germs $h \circ f$, which are holomorphic near x for all x with $f(x) = y$. A morphism will sometimes simply be called a holomorphic map, and will be denoted in short form as $f : X \to Y$.

Let An be the category of complex analytic spaces.

Example 1.2.2. Let X be a complex manifold. Then it can be viewed as a complex analytic space. The structure sheaf is defined via the charts.

Definition 1.2.3. A morphism $X \to Y$ between complex analytic spaces is called *proper* if the preimage of any compact subset in Y is compact.

1.2.1 Analytification

Polynomials over $\mathbb{C}$ can be viewed as holomorphic functions. Hence an affine variety immediately defines a complex analytic space. If X is smooth, it is even a complex submanifold. This assignment is well-behaved under gluing and hence it globalizes. A general reference for this is [SGA1], exposé XII by M. Raynaud.

Proposition 1.2.4. *There is a functor*

$$\cdot^{\text{an}} : \text{Sch}_{\mathbb{C}} \to \text{An}$$

which assigns to a scheme of finite type over $\mathbb{C}$ its analytification. There is a natural morphism of locally ringed spaces

$$\alpha : (X^{\text{an}}, \mathcal{O}_{X^{\text{an}}}^{\text{hol}}) \to (X, \mathcal{O}_X)$$

and $\cdot^{\text{an}}$ is universal with this property. Moreover, α is the identity on points.

If X is smooth, then X^{an} is a complex manifold. If $f : X \to Y$ is proper, the so ist f^{an}.

Proof. By the universal property it suffices to consider the affine case where the obvious construction works. Note that X^{an} is Hausdorff because X is separated, and it is paracompact because it has a finite cover by closed subsets of some $\mathbb{C}^n$. If X is smooth then X^{an} is smooth by [SGA1], Prop. 2.1 in exposé XII, or simply by the Jacobi criterion. The fact that f^{an} is proper if f is proper is shown in [SGA1], Prop. 3.2 in exposé XII. □

1.3 Complexes

1.3.1 Basic definitions

Let $\mathcal{A}$ be an additive category. If not specified otherwise, a complex will always mean a cohomological complex, i.e., a sequence A^i for $i \in \mathbb{Z}$ of objects of $\mathcal{A}$ with *ascending* differential $d^i : A^i \to A^{i+1}$ such that $d^{i+1}d^i = 0$ for all $i \in \mathbb{Z}$. The category of complexes is denoted by $C(\mathcal{A})$. We denote $C^+(\mathcal{A})$, $C^-(\mathcal{A})$ and $C^b(\mathcal{A})$ the full subcategories of complexes bounded below, bounded above and bounded, respectively.

If $K^\bullet \in C(\mathcal{A})$ is a complex, we define the *shifted* complex $K^\bullet[1]$ with

$$(K^\bullet[1])^i = K^{i+1} \ , \quad d^i_{K^\bullet[1]} = -d^{i+1}_{K^\bullet} \ .$$

If $f : K^\bullet \to L^\bullet$ is a morphism of complexes, its *cone* is the complex $\mathrm{Cone}(f)^\bullet$ with

$$\mathrm{Cone}(f)^i = K^{i+1} \oplus L^i, d^i_{\mathrm{Cone}(f)} = (-d^{i+1}_K, f^{i+1} + d^i_L) \ .$$

By construction there are morphisms

$$L^\bullet \to \mathrm{Cone}(f) \to K^\bullet[1] \ ,$$

Let $K(\mathcal{A})$, $K^+(\mathcal{A})$, $K^-(\mathcal{A})$ and $K^b(\mathcal{A})$ be the corresponding homotopy categories where the objects are the same and morphisms are homotopy classes of morphisms of complexes. Note that these categories are always triangulated with the above shift functor and the class of distinguished triangles are those homotopy equivalent to

$$K^\bullet \xrightarrow{f} L^\bullet \to \mathrm{Cone}(f) \to K^\bullet[1]$$

for some morphism of complexes f.

Recall:

Definition 1.3.1. Let $\mathcal{A}$ be an abelian category. A morphism $f^\bullet : K^\bullet \to L^\bullet$ of complexes in $\mathcal{A}$ is called *quasi-isomorphism* if

$$H^i(f) : H^i(K^\bullet) \to H^i(L^\bullet)$$

is an isomorphism for all $i \in \mathbb{Z}$.

We will always assume that an abelian category has enough injectives, or is essentially small, in order to avoid set-theoretic problems. If $\mathcal{A}$ is abelian, let $D(\mathcal{A})$, $D^+(\mathcal{A})$, $D^-(\mathcal{A})$ and $D^b(\mathcal{A})$ the induced derived categories where the objects are the same as in $K^?(\mathcal{A})$ and morphisms are obtained by localizing $K^?(\mathcal{A})$ with respect to the class of quasi-isomorphisms. A triangle is distinguished if it is isomorphic in $D^?(\mathcal{A})$ to a distinguished triangle in $K^?(\mathcal{A})$.

Remark 1.3.2. Let $\mathcal{A}$ be abelian. If $f : K^\bullet \to L^\bullet$ is a morphism of complexes, then

$$0 \to L^\bullet \to \mathrm{Cone}(f) \to K^\bullet[1] \to 0$$

is an exact sequence of complexes. Indeed, it is degreewise split-exact.

1.3.2 Filtrations

Filtrations on complexes are used in order to construct spectral sequences. We mostly need two standard cases.

Definition 1.3.3. 1. Let $\mathcal{A}$ be an additive category, $K^\bullet$ a complex in $\mathcal{A}$. The *stupid filtration* $F^{\geq p}K^\bullet$ on $K^\bullet$ is given by

$$F^{\geq p}K^\bullet = \begin{cases} K^i & i \geq p, \\ 0 & i < p. \end{cases}$$

The quotient $K^\bullet / F^{\geq p}K^\bullet$ is given by

$$F^{<p}K^\bullet = \begin{cases} 0 & i \geq p, \\ K^i & i < p. \end{cases}$$

2. Let $\mathcal{A}$ be an abelian category, $K^\bullet$ a complex in $\mathcal{A}$. The *canonical filtration* $\tau_{\leq p}K^\bullet$ on $K^\bullet$ is given by

$$F^{\leq p}K^\bullet = \begin{cases} K^i & i < p, \\ \mathrm{Ker}(d^p) & i = p, \\ 0 & i > p. \end{cases}$$

The quotient $K^\bullet / F^{\leq p}K^\bullet$ is given by

$$\tau_{>p}K^\bullet = \begin{cases} 0 & i < p, \\ K^p/\mathrm{Ker}(d^p) & i = p, \\ K^i & i > p. \end{cases}$$

The associated graded pieces of the stupid filtration are given by

$$F^{\geq p}K^\bullet / F^{\geq p+1}K^\bullet = K^p \ .$$

The associated graded pieces of the canonical filtration are given by

$$\tau_{\leq p}K^\bullet / \tau_{\leq p-1}K^\bullet = H^p(K^\bullet) \ .$$

1.3.3 Total complexes and signs

We return to the more general case of an additive category $\mathcal{A}$. We consider complexes in $K^{\bullet,\bullet} \in C(\mathcal{A})$, i.e., double complexes consisting of a set of objects $K^{i,j} \in \mathcal{A}$ for $i, j \in \mathbb{Z}$ with differentials

$$d_1^{i,j} : K^{i,j} \to K^{i,j+1} \ , \quad d_2^{i,j} : K^{i,j} \to K^{i+1,j}$$

such that $(K^{i,\bullet}, d_2^{i,\bullet})$ and $(K^{\bullet,j}, d_1^{\bullet,j})$ are complexes and the diagrams

$$\begin{array}{ccc}
K^{i,j+1} & \xrightarrow{d_2^{i,j+1}} & K^{i+1,j+1} \\
{\scriptstyle d_1^{i,j}}\uparrow & & \uparrow{\scriptstyle d_1^{i+1,j}} \\
K^{i,j} & \xrightarrow{d_2^{i,j}} & K^{i+1,j}
\end{array}$$

commute for all $i, j \in \mathbb{Z}$. The *associated simple complex* or *total complex* $\mathrm{Tot}(K^{\bullet,\bullet})$ is defined as

$$\mathrm{Tot}(K^{\bullet,\bullet})^n = \bigoplus_{i+j=n} K^{i,j} \ , \quad d^n_{\mathrm{Tot}(K^{\bullet,\bullet})} = \sum_{i+j=n} (d_1^{i,j} + (-1)^j d_2^{i,j}) \ .$$

In order to take the direct sum, either the category has to allow infinite direct sums or we have to assume boundedness conditions to make sure that only finite direct sums occur. This is the case if $K^{i,j} = 0$ unless $i, j \geq 0$.

Examples 1.3.4. 1. Our definition of the cone is a special case: for $f : K^\bullet \to L^\bullet$

$$\mathrm{Cone}(f) = \mathrm{Tot}(\tilde{K}^{\bullet,\bullet}) \ , \qquad \tilde{K}^{\bullet,-1} = K^\bullet, \tilde{K}^{\bullet,0} = L^\bullet \ .$$

2. Another example is given by the tensor product. Given two complexes $(F^\bullet, d_F)$ and $(G^\bullet, d_G)$, the tensor product

$$(F^\bullet \otimes G^\bullet)^n = \bigoplus_{i+j=n} F^i \otimes G^j$$

has a natural structure of a double complex with $K^{i,j} = F^i \otimes G^j$, and the differential is given by $d = \mathrm{id}_F \otimes d_G + (-1)^i d_F \otimes \mathrm{id}_F$.

Remark 1.3.5. There is a choice of signs in the definition of the total complex. See for example [Hu1] §2.2 for a discussion. We use the convention opposite to the one of loc. cit. For most formulae it does matter which choice is used, as long as it is used consistently. However, it does have an asymmetric effect on the formula for the compatibility of cup-products with boundary maps. We spell out the source of this assymmetry.

Lemma 1.3.6. *Let $F^\bullet$, $G^\bullet$ be complexes in an additive tensor category. Then:*

1. *$F^\bullet \otimes (G^\bullet[1]) = (F^\bullet \otimes G^\bullet)[1]$.*
2. *$\epsilon : (F^\bullet[1] \otimes G^\bullet) \to (F^\bullet \otimes G^\bullet)[1]$ with $\epsilon = (-1)^j$ on $F^i \otimes G^j$ (in degree $i + j - 1$) is an isomorphism of complexes.*

Proof. We compute the differential on $F^i \otimes G^i$ in all three complexes. Note that

$$F^i \otimes G^j = (F[1])^{i-1} \otimes G^j = F^i \otimes (G[1])^{j-1}.$$

For better readability, we drop $\otimes$id and id$\otimes$ and $|_{F^i\otimes G^j}$ everywhere. Hence we have

$$\begin{aligned}
d^{i+j-1}_{(F^\bullet\otimes G^\bullet)[1]} &= -d^{i+j}_{F^\bullet\otimes G^\bullet}\\
&= -\left(d^j_{G^\bullet} + (-1)^j d^i_{F^\bullet}\right)\\
&= -d^j_{G^\bullet} + (-1)^{j-1} d^i_{F^\bullet}\\
d^{i+j-1}_{F^\bullet\otimes(G^\bullet[1])} &= d^{j-1}_{G^\bullet[1]} + (-1)^{j-1} d^i_{F^\bullet}\\
&= -d^j_{G^\bullet} + (-1)^{j-1} d^i_{F^\bullet}\\
d^{i+j-1}_{(F^\bullet[1])\otimes G^\bullet} &= d^j_{G^\bullet} + (-1)^j d^{i-1}_{F^\bullet[1]}\\
&= d^j_{G^\bullet} + (-1)^{j-1} d^i_{F^\bullet}
\end{aligned}$$

We see that the first two complexes agree, whereas the differential of the third is different. Multiplication by $(-1)^j$ on the summand $F^i \otimes G^j$ is a morphism of complexes. □

1.4 Hypercohomology

Let X be a topological space and $\mathrm{Sh}(X)$ the abelian category of sheaves of abelian groups on X.

We want to extend the definition of sheaf cohomology on X, as explained in [Ha2], Chap. III, to complexes of sheaves.

1.4.1 Definition

Definition 1.4.1. Let $\mathcal{F}^\bullet$ be a bounded below complex of sheaves of abelian groups on X. An *injective resolution* of $\mathcal{F}^\bullet$ is a quasi-isomorphism

$$\mathcal{F}^\bullet \to \mathcal{I}^\bullet$$

where $\mathcal{I}^\bullet$ is a bounded below complex with $\mathcal{I}^n$ *injective* for all n, i.e., $\mathrm{Hom}(-,\mathcal{I}^n)$ is exact.

Sheaf cohomology of X with coefficients in $\mathcal{F}^\bullet$ is defined as

$$H^i(X,\mathcal{F}^\bullet) = H^i\left(\Gamma(X,\mathcal{I}^\bullet)\right) \qquad i \in \mathbb{Z}$$

where $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ is an injective resolution.

Remark 1.4.2. In the older literature, it is customary to write $\mathbb{H}^i(X,\mathcal{F}^\bullet)$ instead of $H^i(X,\mathcal{F}^\bullet)$ and call it *hypercohomology*. We do not see any need to distinguish. Note that in the special case $\mathcal{F}^\bullet = \mathcal{F}[0]$ a sheaf viewed as a complex concentrated in degree 0, the notion of an injective resolution in the above sense agrees with the ordinary one in homological algebra.

Remark 1.4.3. In the language of derived categories, we have

$$H^i(X, \mathcal{F}^\bullet) = \mathrm{Hom}_{D^+(\mathrm{Sh}(X))}(\mathbb{Z}, \mathcal{F}^\bullet[i])$$

because $\Gamma(X, \cdot) = \mathrm{Hom}_{\mathrm{Sh}(X)}(\mathbb{Z}, \cdot)$.

Lemma 1.4.4. *$H^i(X, \mathcal{F}^\bullet)$ is well-defined and functorial in $\mathcal{F}^\bullet$.*

Proof. We first need existence of injective resolutions. Recall that the category $\mathrm{Sh}(X)$ has enough injectives. Hence every sheaf has an injective resolution. This extends to bounded below complexes in $\mathcal{A}$ by [We] Lemma 5.7.2 (or rather, its analogue for injective rather than projective objects).

Let $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ and $\mathcal{G}^\bullet \to \mathcal{J}^\bullet$ be injective resolutions. By loc.cit. Theorem 10.4.8

$$\mathrm{Hom}_{D^+(\mathrm{Sh}(X))}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \mathrm{Hom}_{K^+(\mathrm{Sh}(X))}(\mathcal{I}^\bullet, \mathcal{J}^\bullet).$$

This means in particular that every morphism of complexes lifts to a morphism of injective resolutions and that the lift is unique up to homotopy of complexes. Hence the induced maps

$$H^i(\Gamma(X, \mathcal{I}^\bullet)) \to H^i(\Gamma(X, \mathcal{J}^\bullet))$$

agree. This implies that $H^i(X, \mathcal{F}^\bullet)$ is well-defined and a functor. □

Remark 1.4.5. Injective sheaves are abundant (by our general assumption that there are enough injectives), but not suitable for computations. Every injective sheaf $\mathcal{F}$ is *flasque* [Ha1, III. Lemma 2.4], i.e., the restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$ between non-empty open sets $V \subset U$ are always surjective. Below we will introduce the canonical flasque Godemont resolution for any sheaf $\mathcal{F}$. More generally, every flasque sheaf $\mathcal{F}$ is *acyclic*, i.e., $H^i(X, \mathcal{F}) = 0$ for $i > 0$. One may compute sheaf cohomology of $\mathcal{F}$ using any acyclic resolution $F^\bullet$. This follows from the hypercohomology spectral sequence

$$E_2^{p,q} = H^p(H^q(F^\bullet)) \Rightarrow H^{p+q}(X, \mathcal{F})$$

which is supported entirely on the $q = 0$-line.

Special acylic resolutions on X are the so-called *fine* resolutions. See [Wa, pg. 170] for a definition of fine sheaves involving partitions of unity. Their importance comes from the fact that sheaves of $\mathcal{C}^\infty$-functions and sheaves of $\mathcal{C}^\infty$-differential forms on X are fine sheaves.

1.4.2 Godement resolutions

For many purposes, it is useful to have functorial resolutions of sheaves. One such is given by the Godement resolution introduced in [God] chapter II §3.

Let X be a topological space. Recall that a sheaf on X vanishes if and only the stalks at all $x \in X$ vanish. For all $x \in X$ we denote $i_x : x \to X$ the natural inclusion.

Definition 1.4.6. Let $\mathcal{F} \in \mathrm{Sh}(X)$. Put

$$I(\mathcal{F}) = \prod_{x \in X} i_{x*}\mathcal{F}_x \ .$$

Inductively, we define the *Godement resolution* $Gd^\bullet(\mathcal{F})$ of $\mathcal{F}$ by

$$\begin{aligned} Gd^0(\mathcal{F}) &= I(\mathcal{F}) \ , \\ Gd^1(\mathcal{F}) &= I(\mathrm{Coker}(\mathcal{F} \to Gd^0(\mathcal{F}))) \ , \\ Gd^{n+1}(\mathcal{F}) &= I(\mathrm{Coker}(Gd^{n-1}(\mathcal{F}) \to Gd^n(\mathcal{F}))) \quad n > 0. \end{aligned}$$

Lemma 1.4.7. *1. Gd is an exact functor with values in* $C^+(\mathrm{Sh}(X))$.

2. The natural map $\mathcal{F} \to Gd^\bullet(\mathcal{F})$ *is a flasque resolution.*

Proof. Functoriality is obvious. The sheaf $I(\mathcal{F})$ is given by

$$U \mapsto \prod_{x \in U} i_{x*}\mathcal{F}_x \ .$$

All the sheaves involved are flasque, hence acyclic. In particular, taking the direct products is exact (it is not in general), turning $I(\mathcal{F})$ into an exact functor. $\mathcal{F} \to I(\mathcal{F})$ is injective, and hence by construction, $Gd^\bullet(\mathcal{F})$ is then a flasque resolution. □

Definition 1.4.8. Let $\mathcal{F}^\bullet \in C^+(\mathrm{Sh}(X))$ be a complex of sheaves. We call

$$Gd(\mathcal{F}^\bullet) := \mathrm{Tot}(Gd^\bullet(\mathcal{F}^\bullet))$$

the *Godement resolution* of $\mathcal{F}^\bullet$.

Corollary 1.4.9. *The natural map*

$$\mathcal{F} \to Gd(\mathcal{F}^\bullet)$$

is a quasi-isomorphism and

$$H^i(X, \mathcal{F}^\bullet) = H^i\left(\Gamma(X, Gd(\mathcal{F}^\bullet))\right) \ .$$

Proof. By Lemma 1.4.7, the first assertion holds if $\mathcal{F}^\bullet$ is concentrated in a single degree. The general case follows by the hypercohomology spectral sequence or by induction on the length of the complex using the stupid filtration.

All terms in $Gd(\mathcal{F}^\bullet)$ are flasque, hence acyclic for $\Gamma(X, \cdot)$. □

We now study functoriality of the Godement resolution. For a continuous map $f : X \to Y$ be denote f^{-1} the pull-back functor on sheaves of abelian groups. Recall that it is exact.

Lemma 1.4.10. *Let $f : X \to Y$ be a continuous map between topological spaces, $\mathcal{F}^\bullet \in C^+(\mathrm{Sh}(Y))$. Then there is a natural quasi-isomorphism*

$$f^{-1}Gd_Y(\mathcal{F}^\bullet) \to Gd_X(f^{-1}\mathcal{F}^\bullet)\ .$$

Proof. Consider a sheaf $\mathcal{F}$ on Y. We want to construct

$$f^{-1}I(\mathcal{F}) \to I(f^{-1}\mathcal{F}) = \prod_{x\in X} i_{x*}(f^{-1}\mathcal{F})_x = \prod_{x\in X} i_{x*}\mathcal{F}_{f(x)}\ .$$

By the universal property of the direct product and adjunction for f^{-1}, this is equivalent to specifying for all $x \in X$

$$\prod_{y\in Y} i_{y*}\mathcal{F}_y = I(\mathcal{F}) \to f_* i_{x*}\mathcal{F}_{f(x)} = i_{f(x)*}\mathcal{F}_{f(x)}\ .$$

We use the natural projection map. By construction, we have a natural commutative diagram

$$\begin{array}{ccccc} f^{-1}\mathcal{F} & \longrightarrow & f^{-1}I(\mathcal{F}) & \longrightarrow & \mathrm{Coker}\left(f^{-1}\mathcal{F} \to f^{-1}I(\mathcal{F})\right) \\ {\scriptstyle =}\downarrow & & \downarrow & & \\ f^{-1}\mathcal{F} & \longrightarrow & I(f^{-1}\mathcal{F}) & \longrightarrow & \mathrm{Coker}\left(f^{-1}\mathcal{F} \to I(f^{-1}\mathcal{F})\right) \end{array}$$

It induces a map between the cokernels. Proceeding inductively, we obtain a morphism of complexes

$$f^{-1}Gd^\bullet_Y(\mathcal{F}) \to Gd^\bullet_X(f^{-1}\mathcal{F})\ .$$

It is a quasi-isomorphism because both are resolutions of $f^{-1}\mathcal{F}$. This transformation of functors extends to double complexes and hence defines a transformation of functors on $C^+(\mathrm{Sh}(Y))$. □

Remark 1.4.11. We are going to apply the theory of Godement resolutions in the case where X is a variety over a field k, a complex manifold or more generally a complex analytic space. The continuous maps that we need to consider are those in these categories, but also the maps of schemes $X_K \to X_k$ for the change of base field K/k and a variety over k; and the continuous map $X^{\mathrm{an}} \to X$ for an algebraic variety over $\mathbb{C}$ and its analytification.

1.4.3 Čech cohomology

Neither the definition of sheaf cohomology via injective resolutions nor Godement resolutions are convenient for concrete computations. We introduce Čech cohomology for this task. We follow [Ha2], Chap. III §4, but extend to hypercohomology.

Let k be a field. We work in the category of varieties over k. Let $I = \{1, \dots, n\}$ as ordered set and $\mathfrak{U} = \{U_i | i \in I\}$ an affine open cover of X. For any subset $J \subset \{1, \dots, n\}$ we denote

$$U_J = \bigcap_{j \in J} U_j \ .$$

As X is separated, they are all affine.

Definition 1.4.12. Let X and $\mathfrak{U}$ be as above. Let $\mathcal{F} \in \mathrm{Sh}(X)$. We define the *Čech complex* of $\mathcal{F}$ as

$$C^p(\mathfrak{U}, \mathcal{F}) = \prod_{J \subset I, |J| = p+1} \mathcal{F}(U_J) \qquad p \geq 0$$

with differential $\delta^p : C^p(\mathfrak{U}, \mathcal{F}) \to C^{p+1}(\mathfrak{U}, \mathcal{F})$

$$(\delta^p \alpha)_{i_0 < i_1 < \dots < i_p} = \sum_{j=0}^{p+1} (-1)^p \alpha_{i_0 \dots < \hat{i}_j < \dots < i_{p+1}} |_{U_{i_0 \dots i_{p+1}}} \ ,$$

where, as usual, $i_0 \dots < \hat{i}_j < \dots < i_{p+1}$ means the tuple with $\hat{i}_j$ removed.

We define the p-th *Čech cohomology* of X with coefficients in $\mathcal{F}$ as

$$\check{H}^p(\mathfrak{U}, \mathcal{F}) = H^p(C^\bullet(\mathfrak{U}, \mathcal{F}), \delta) \ .$$

Proposition 1.4.13 ([Ha2], chap. III Theorem 4.5)**.** *Let X be a variety, $\mathfrak{U}$ an affine open cover as before. Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules on X. Then there is a natural isomorphism*

$$H^p(X, \mathcal{F}) = \check{H}^p(\mathfrak{U}, \mathcal{F}) \ .$$

We now extend to complexes. We can apply the functor $C^\bullet(\mathfrak{U}, \cdot)$ to all terms in a complex $\mathcal{F}^\bullet$ and obtain a double complex $C^\bullet(\mathfrak{U}, \mathcal{F}^\bullet)$.

Definition 1.4.14. Let X and $\mathfrak{U}$ as before. Let $\mathcal{F}^\bullet \in C^+(\mathrm{Sh}(X))$. We define the *Čech complex* of $\mathfrak{U}$ with coefficients in $\mathcal{F}^\bullet$ as

$$C^\bullet(\mathfrak{U}, \mathcal{F}^\bullet) = \mathrm{Tot}\,(C^\bullet(\mathfrak{U}, \mathcal{F}^\bullet))$$

and *Čech cohomology* as

$$\check{H}^p(\mathfrak{U}, \mathcal{F}) = H^p(C^\bullet(\mathfrak{U}, \mathcal{F}^\bullet)) \ .$$

Proposition 1.4.15. *Let X be a variety, $\mathfrak{U}$ as before an open affine cover of X. Let $\mathcal{F}^\bullet \in C^+(\mathrm{Sh}(X))$ be complex such that all $\mathcal{F}^n$ are coherent sheaves of $\mathcal{O}_X$-modules. Then there is a natural isomorphism*

$$H^p(X, \mathcal{F}) \to \check{H}^p(\mathfrak{U}, \mathcal{F}^\bullet) \ .$$

Proof. The essential step is to define the map. We first consider a single sheaf $\mathcal{G}$. Let $\mathcal{C}^\bullet(\mathfrak{U},\mathcal{G})$ be a sheafified version of the Čech complex for a sheaf $\mathcal{G}$. By [Ha2], chap. III Lemma 4.2, it is a resolution of $\mathcal{G}$. We apply the Godement resolution and obtain a flasque resolution of $\mathcal{G}$ by

$$\mathcal{G} \to \mathcal{C}^\bullet(\mathfrak{U},\mathcal{G}) \to Gd\,(\mathcal{C}^\bullet(\mathfrak{U},\mathcal{G}))\ .$$

By Proposition 1.4.13, the induced map

$$C^\bullet(\mathfrak{U},\mathcal{G}) \to \Gamma(X, Gd\,(\mathcal{C}^\bullet(\mathfrak{U},\mathcal{G}))$$

is a quasi-isomorphism as both compute $H^i(X,\mathcal{G})$.

The construction is functorial in $\mathcal{G}$, hence we can apply it to all components of a complex $\mathcal{F}^\bullet$ and obtain double complexes. We use the previous results for all $\mathcal{F}^n$ and take total complexes. Hence

$$\mathcal{F}^\bullet \to \mathrm{Tot}\mathcal{C}^\bullet(\mathfrak{U},\mathcal{F}^\bullet) \to Gd\,(\mathcal{C}^\bullet(\mathfrak{U},\mathcal{F}^\bullet))$$

are quasi-isomorphisms. Taking global sections we get a quasi-isomorphism

$$\mathrm{Tot}C^\bullet(\mathfrak{U},\mathcal{F}^\bullet) \to \mathrm{Tot}\Gamma(X, Gd\,(\mathcal{C}^\bullet(\mathfrak{U},\mathcal{F}^\bullet)))\ .$$

By definition, the complex on the left computes Čech cohomology of $\mathcal{F}^\bullet$ and the complex on right computes hypercohomology of $\mathcal{F}^\bullet$. □

Corollary 1.4.16. *Let X be an affine variety and $\mathcal{F}^\bullet \in C^+(\mathrm{Sh}(X))$ such that all $\mathcal{F}^n$ are coherent sheaves of $\mathcal{O}_X$-modules. Then*

$$H^i(\Gamma(X,\mathcal{F}^\bullet)) = H^i(X,\mathcal{F}^\bullet)\ .$$

Proof. We use the affine covering $\mathfrak{U} = \{X\}$ and apply Proposition 1.4.15. □

1.5 Simplicial objects

We introduce simplicial varieties in order to carry out some of the constructions in [D5]. Good general references on the topic are [May] or [We] Ch. 8.

Definition 1.5.1. Let Δ be the category whose objects are finite ordered sets

$$[n] = \{0, 1, \dots, n\} \quad n \in \mathbb{N}_0$$

with morphisms nondecreasing monotone maps. Let Δ_N be the full subcategory with objects the $[n]$ with $n \le N$.

If $\mathcal{C}$ is a category, we denote by $\mathcal{C}^\Delta$ the *category of simplicial objects* in $\mathcal{C}$ defined as contravariant functors

$$X_\bullet : \Delta \to \mathcal{C}$$

with transformation of functors as morphisms. We denote by $\mathcal{C}^{\Delta^\circ}$ the *category of cosimplicial objects* in $\mathcal{C}$ defined as covariant functors

$$X^\bullet : \Delta \to \mathcal{C} \ .$$

Similarly, we defined the categories $\mathcal{C}^{\Delta_N}$ and $\mathcal{C}^{\Delta_N^\circ}$ of *N-truncated simplicial and cosimplicial objects.*

Example 1.5.2. Let X be an object of $\mathcal{C}$. The constant functor

$$\Delta^\circ \to \mathcal{C}$$

which maps all objects to X and all morphism to the identity morphism is a simplicial object. It is called the *constant simplicial object* associated to X.

In Δ, we have in particular the *face maps*

$$\epsilon_i : [n-1] \to [n] \quad i = 0, \ldots, n,$$

the unique injective map leaving out the index i, and the *degeneracy maps*

$$\eta_i : [n+1] \to [n] \quad i = 0, \ldots, n,$$

the unique surjective map with two elements mapping to i. More generally, a map in Δ is called *face* or *degeneracy* if it is a composition of ϵ_i or η_i, respectively. Any morphism in Δ can be decomposed into a degeneracy followed by a face ([We] Lemma 8.12).

For all $m \geq n$, we denote $S_{m,n}$ the set of all degeneracy maps $[m] \to [n]$.

A simplicial object $X_\bullet$ is determined by a sequence of objects

$$X_0, X_1, \ldots$$

and face and degeneracy morphisms between them. In particular, we write

$$\partial_i : X_n \to X_{n-1}$$

for the image of ϵ_i and

$$s_i : X_n \to X_{n+1}$$

for the image of η_i.

Example 1.5.3. For every n, there is a simplicial set $\Delta[n]$ with

$$\Delta[n]_m = \mathrm{Mor}_\Delta([m],[n])$$

and the natural face and degeneracy maps. It is called the *simplicial n-simplex.* For $n = 0$, this is the *simplicial point*, and for $n = 1$ the *simplicial interval.* Functoriality in the first argument induces maps of simplicial sets. In particular, there are

$$\delta_0 = \epsilon_0^*, \delta_1 = \epsilon_1^* : \Delta[1] \to \Delta[0] \ .$$

Definition 1.5.4. Let $\mathcal{C}$ be a category with finite products and coproducts. Let $\star$ be the final object. Let $X_\bullet$, $Y_\bullet$ simplicial objects in $\mathcal{C}$ and $S_\bullet$ a simplicial set

1. $X_\bullet \times Y_\bullet$ is the simplicial object with
$$(X_\bullet \times Y_\bullet)_n = X_n \times Y_n$$
with face and degeneracy maps induced from $X_\bullet$ and $Y_\bullet$.

2. $X_\bullet \times S_\bullet$ is the simplicial object with
$$(X_\bullet \times S_\bullet)_n = \coprod_{s \in S_n} X_n$$
with face and degeneracy maps induced from $X_\bullet$ and $S_\bullet$.

3. Let $f, g : X_\bullet \to Y_\bullet$ be morphisms of simplicial objects. Then f is called *homotopic* to g if there is a morphism
$$h : X_\bullet \times \Delta[1] \to Y_\bullet$$
such that $h \circ \delta_0 = f$ and $h \circ \delta_1 = g$.

The inclusion $\Delta_N \to \Delta$ induces a natural restriction functor
$$\mathrm{sq}_N : \mathcal{C}^\Delta \to \mathcal{C}^{\Delta_N} .$$
For a simplicial object $X_\bullet$, we call $\mathrm{sq}_N X_\bullet$ its N*-skeleton.* If $Y_\bullet$ is a fixed simplicial objects, we also denote sq_N the restriction functor from simplicial objects over $Y_\bullet$ to simplicial objects over $\mathrm{sq}_N Y_\bullet$.

Remark 1.5.5. The skeleta $\mathrm{sq}_k X_\bullet$ define the *skeleton filtration*, i.e., a chain of maps
$$\mathrm{sq}_0 X_\bullet \to \mathrm{sq}_1 X_\bullet \to \cdots \to \mathrm{sq}_N X_\bullet = X_\bullet.$$
Later, in section 2.3, we will define the topological realization $|X_\bullet|$ of a simplicial set $X_\bullet$. The skeleton filtration then defines a filtration of $|X_\bullet|$ by closed subspaces.

An important example in this book is the case when the simplicial set $X_\bullet$ is a finite set, i.e., all X_n are finite sets, and empty for $n > N$ sufficiently large. See section 2.3.

Lemma 1.5.6. *Let* $\mathcal{C}$ *be a category with finite limits. Then the functor* sq_N *has a right adjoint*
$$\mathrm{cosq}_N : \mathcal{C}^{\Delta_N} \to \mathcal{C}^\Delta .$$
If $Y_\bullet$ *is a fixed simplicial object, then*
$$\mathrm{cosq}_N^{Y_\bullet}(X_\bullet) = \mathrm{cosq}_N X_\times \times_{\mathrm{cosq}_N \mathrm{sq}_N Y_\bullet} Y_\bullet$$
is the right adjoint of the relative version of sq_N*.*

Proof. The existence of cosq_N is an instance of a Kan extension. We refer to [ML, chap. X] or [AM, chap. 2] for its existence. The relative case follows from the universal properties of fibre products. □

If $X_\bullet$ is an N-truncated simplicial object, we call $\mathrm{cosq}_N X_\bullet$ its *coskeleton*.

Remark 1.5.7. We apply this in particular to the case where $\mathcal{C}$ is one of the categories Var, Sm or Aff over a fixed field k. The disjoint union of varieties is a coproduct in these categories and the direct product a product.

Definition 1.5.8. Let S be a class of covering maps of varieties containing all identity morphisms. A morphism $f : X_\bullet \to Y_\bullet$ of simplicial varieties (or the simplicial variety $X_\bullet$ itself) is called an *S-hypercovering* if the adjunction morphisms

$$X_n \to (\mathrm{cosq}_{n-1}^{Y_\bullet} \mathrm{sq}_{n-1} X_\bullet)_n$$

are in S.

If S is the class of proper, surjective morphisms, we call $X_\bullet$ a *proper hypercover* of $Y_\bullet$.

Definition 1.5.9. Let $X_\bullet$ be a simplical variety. It is called *split* if for all $n \in \mathbb{N}_0$

$$N(X_n) = X_n \smallsetminus \bigcup_{i=0}^{n-1} s_i(X_{n-1})$$

is an open and closed subvariety of X_n.

We call $N(X_n)$ the non-degenerate part of X_n. If $X_\bullet$ is a split simplicial variety, we have a decomposition as varieties

$$X_n = N(X_n) \amalg \coprod_{m<n} \coprod_{s \in S_{m,n}} sN(X_m)$$

where $S_{m,n}$ is the set of degeneracy maps from X_m to X_n.

Theorem 1.5.10 (Deligne)**.** *Let k be a field and Y a variety over k. Then there is a split simplicial variety $X_\bullet$ with all X_n smooth and a proper hypercover $X_\bullet \to Y$.*

Proof. The construction is given in [D5] Section (6.2.5). It depends on the existence of resolutions of singularities. In positive characteristic, we may use de Jong's result on alterations instead. □

The other case we are going to need is the case of additive categories.

Definition 1.5.11. Let $\mathcal{A}$ be an additive category. We define a functor

$$C : \mathcal{A}^\Delta \to C^-(\mathcal{A})$$

by mapping a simplicial object $X_\bullet$ to the cohomological complex

$$\dots X_{-n} \xrightarrow{d^{-n}} X_{-(n-1)} \to \dots \to X_0 \to 0$$

with differential

$$d^{-n} = \sum_{i=0}^{n} (-1)^i \partial_i \ .$$

We define a functor

$$C : \mathcal{A}^{\Delta^\circ} \to C^+(\mathcal{A})$$

by mapping a cosimplicial object $X^\bullet$ to the cohomological complex

$$0 \to X^0 \to \dots X^n \xrightarrow{d^n} X_{n+1} \to \dots$$

with differential

$$d^n = \sum_{i=0}^{n} (-1)^i \partial_i \ .$$

Let $\mathcal{A}$ be an abelian category. We define a functor

$$N : \mathcal{A}^{\Delta^\circ} \to C^+(\mathcal{A})$$

by mapping a cosimplicial object $X^\bullet$ to the *normalized complex* $N(X^\bullet)$ with

$$N(X^\bullet)_n = \bigcap_{i=0}^{n-1} \mathrm{Ker}(s_i : X^n \to X^{n-1})$$

and differential $d^n|_{N(X^\bullet)}$.

Proposition 1.5.12 (Dold-Kan correspondence)**.** *Let $\mathcal{A}$ be an abelian category, $X^\bullet \in \mathcal{A}^{\Delta^\circ}$ a cosimplicial object. Then the natural map*

$$N(X^\bullet) \to C(X^\bullet)$$

is a quasi-isomorphism.

Proof. This is the dual result of [We], Theorem 8.3.8. □

Remark 1.5.13. We are going to apply this in the case of cosimplicial complexes, i.e., to $C(\mathcal{A})^{\Delta^\circ}$, where $\mathcal{A}$ is abelian, e.g., a category of vector spaces.

1.6 Grothendieck topologies

Grothendieck topologies generalize the notion of open subsets in topological spaces. Using them one can define cohomology theories in more abstract settings. To define a Grothendieck topology, we need the notion of a *site, or situs.* Let $\mathcal{C}$ be a category. A basis for a Grothendieck topology on $\mathcal{C}$ is given by *covering families* in the category $\mathcal{C}$ satisfying the following definition.

Definition 1.6.1. A *site/situs* is a category $\mathcal{C}$ together with a collection of morphism in $\mathcal{C}$

$$(\varphi_i : V_i \longrightarrow U)_{i\in I}\,,$$

the *covering families.*

The covering families satisfy the following axioms:

- An isomorphism $\varphi : V \to U$ is a covering family with an index set containing only one element.
- If $(\varphi_i : V_i \longrightarrow U)_{i\in I}$ is a covering family, and $f : V \to U$ a morphism in $\mathcal{C}$, then for each $i \in I$ there exists the pullback diagram

$$\begin{array}{ccc} V \times_U V_i & \xrightarrow{F_i} & V_i \\ {\scriptstyle \Phi_i}\big\downarrow & & \big\downarrow{\scriptstyle \varphi_i} \\ V & \xrightarrow{f} & U \end{array}$$

  in $\mathcal{C}$, and $(\Phi_i : V \times_U V_i \to V)_{i\in I}$ is a covering family of V.

- If $(\varphi_i : V_i \longrightarrow U)_{i\in I}$ is a covering family of U, and for each V_i there is given a covering family $\left(\varphi^i_j : V^i_j \to V_i\right)_{j\in J(i)}$, then

$$\left(\varphi_i \circ \varphi^i_j : V^i_j \to U\right)_{i\in I, j\in J(i)}$$

  is a covering family of U.

Example 1.6.2. Let X be a topological space. Then the category of open sets in X together with inclusions as morphisms form a site, where the covering maps are the families $(U_i)_{i\in I}$ of open subsets of U such that $\cup_{i\in I} U_i = U$. Thus each topological space defines a canonical site. For the Zariski open subsets of a scheme X this is called the *(small) Zariski site* of X.

Definition 1.6.3. A *presheaf* $\mathcal{F}$ of abelian groups on a situs $\mathcal{C}$ is a contravariant functor

$$\mathcal{F} : \mathcal{C} \to \mathrm{Ab}, U \mapsto \mathcal{F}(U).$$

A presheaf $\mathcal{F}$ is a *sheaf*, if for each covering family $(\varphi_i : V_i \longrightarrow U)_{i\in I}$, the difference kernel sequence

$$0 \to \mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(V_i) \rightrightarrows \prod_{(i,j)\in I\times I} \mathcal{F}(V_i \times_U V_j)$$

is exact. This means that a section $s \in \mathcal{F}(U)$ is determined by its restrictions to each V_i, and a tuple $(s_i)_{i\in I}$ of sections comes from a section on U, if one has $s_i = s_j$ on pullbacks to the fiber product $V_i \times_U V_j$.

Once we have a notion of sheaves in a certain Grothendieck topology, then we may define cohomology groups $H^*(X, \mathcal{F})$ by using injective resolutions as in section 1.4 as the right derived functor of the left-exact global section functor $X \mapsto \mathcal{F}(X) = H^0(X, \mathcal{F})$ in the presence of enough injectives.

Example 1.6.4. The *(small) étale site* over a smooth variety X consists of the category of all étale morphisms $\varphi : U \to X$ from a smooth variety U to X. See [Ha2, Chap. III] for the notion of étale maps. We just note here that étale maps are quasi-finite, i.e., have finite fibers, are unramified, and the image $\varphi(U) \subset X$ is a Zariski open subset.

A morphism in this site is given by a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & U \\ \downarrow & & \downarrow \\ X & \xrightarrow{\text{id}} & X. \end{array}$$

Let U be étale over X. A family $(\varphi_i : V_i \longrightarrow U)_{i \in I}$ of étale maps over X is called a covering family of U, if $\bigcup_{i \in I} \varphi_i(V_i) = U$, i.e., the images form a Zariski open covering of U.

This category has enough injectives by Grothendieck [SGA4.2], and thus we can define étale cohomology $H^*_{\text{et}}(X, \mathcal{F})$ for étale sheaves $\mathcal{F}$.

Example 1.6.5. In Section 2.7 we are going to introduce the h′-topology on the category of analytic spaces.

Definition 1.6.6. Let $\mathcal{C}$ and $\mathcal{C}'$ be sites. A *morphism of sites* $f : \mathcal{C} \to \mathcal{C}'$ consists of a functor $F : \mathcal{C}' \to \mathcal{C}$ (sic) which preserves fibre products and such that the F applied to a covering family of $\mathcal{C}'$ yields a covering family of $\mathcal{C}$.

A morphism of sites induces an adjoint pair of functors (f^*, f_*) between sheaves of sets on $\mathcal{C}$ and $\mathcal{C}'$.

Example 1.6.7.

1. Let $f : X \to Y$ be continuous map of topological spaces. As in Example 1.6.2 we view them as sites. Then the functor F mapping an open subset of Y to its preimage $f^{-1}(U)$.
2. Let X be a scheme. Then there is morphism of sites from the small étale site of X to the Zariki-site of X. The functor views an open subscheme $U \subset X$ as an étale X-scheme. Open covers are in particular étale covers.

Definition 1.6.8. Let $\mathcal{C}$ be a site. A $\mathcal{C}$-hypercover is an S-hypercover in the sense of Definition 1.5.8 with S the class of morphism

$$\coprod_{i \in I} U_i \to U$$

for all covering families $\{\phi_i : U_i \to U\}_{i \in I}$ in the site.

If $X_\bullet$ is a simplical object and $\mathcal{F}$ is a presheaf of abelian groups, then $\mathcal{F}(X_\bullet)$ is a cosimplicial abelian group. By applying the total complex functor C of Definition 1.5.11, we get a complex of abelian groups.

Proposition 1.6.9. *Let $\mathcal{C}$ be a site closed under finite products and fibre products, $\mathcal{F}$ a sheaf of abelian groups on $\mathcal{C}$, $X \in \mathcal{C}$. Then*

$$H^i(X, \mathcal{F}) = \lim_{X_\bullet \to X} H^i\left(C(\mathcal{F}(X_\bullet))\right)$$

where the direct limit runs through the system of all $\mathcal{C}$-hypercovers of X.

Proof. This is [SGA4V, Théorème 7.4.1] □

1.7 Torsors

Informally, a torsor is a group without a unit. The standard notion in algebraic geometry is sheaf theoretic: a torsor under a sheaf of groups G is a sheaf of sets X with an operation $G \times X \to X$ such that there is a cover over which X becomes isomorphic to G and the action becomes the group operation. This makes sense in any site.

In this section, we are going to discuss a variant of this idea which does not involve sites or topologies but rather schemes. This approach fits well with the Tannaka formalism that will be discussed in Chapters 7.4 and 12.

It is used by Kontsevich in [K1]. This notion at least goes back to a paper of R. Baer [Ba] from 1929, see the footnote on page 202 of loc. cit. where Baer explains how the notion of a torsor comes up in the context of earlier work of H. Prüfer [Pr]. In yet another context, ternary operations satisfying these axioms are called associative Malcev operations, see [Joh] for a short account.

1.7.1 Sheaf theoretic definition

Definition 1.7.1. Let $\mathcal{C}$ be a category equipped with a Grothendieck topology t. Assume S is a final object of $\mathcal{C}$. Let G be a group object in $\mathcal{C}$. A *(left) G-torsor* is an object X with a (left) operation

$$\mu : G \times X \to X$$

such that there is a t-covering $U \to S$ such that restriction of G and X to U is the trivial torsor, i.e., $X(U)$ is non-empty, and the choice $x \in X(U)$ induces a natural isomorphism

$$\begin{aligned} \cdot x : G(U') &\to X(U') \\ g &\mapsto \mu(g, x). \end{aligned}$$

for all $U' \to U$.

The condition can also be formulated as an isomorphism

$$\begin{aligned} G \times U &\to X \times U \\ (g,u) &\mapsto g(u), u) \end{aligned}$$

Remark 1.7.2. 1. As μ is an operation, the isomorphism of the definition is compatible with the operation as well, i.e., the diagram

$$\begin{array}{ccc} G(U') \times X(U') & \xrightarrow{\mu} & X(U') \\ \uparrow{\scriptstyle (\mathrm{id}, \cdot x)} & & \uparrow{\scriptstyle \cdot x} \\ G(U') \times G(U') & \longrightarrow & G(U') \end{array}$$

commutes.

2. If, moreover, $X \to S$ is a t-cover, then $X(X)$ is always non-empty and we recover a version of the definition that often appears in the literature, namely that
$$G \times X \to X \times X$$
has to be an isomorphism.

We are interested in the topology that is in use in Tannaka theory. It is basically the flat topology, but we have to be careful what we mean by this because the schemes involved are not of finite type over the base.

Definition 1.7.3. Let S be an affine scheme and $\mathcal{C}$ the category of affine S-schemes. The *fpqc*-topology on $\mathcal{C}$ is generated by covers of the form $X \to Y$ with $\mathcal{O}(X)$ faithfully flat over $\mathcal{O}(Y)$.

The letters *fpqc* stand for fidèlement plat quasi-compact. Recall that Spec$A$ is quasi-compact for all rings A.

We do not discuss the non-affine case at all, but see the survey [Vis] by Vistoli for the general case. The topology is discussed in loc. cit. Section 2.3.2. The above formulation follows from loc. cit. Lemma 2.60.

Remark 1.7.4. If, moreover, $S = \mathrm{Spec}(k)$ is the spectrum of a field, then any non-trivial Spec$A \to Spec(k)$ is an *fpqc*-cover. Hence, we are in the situation of Remark 1.7.2. Note that X still has to be non-empty!

The importance of the *fpqc*-topology is that all representable presheaves are *fpqc*-sheaves, see [Vis, Theorem 2.55].

1.7.2 Torsors in the category of sets

Definition 1.7.5 ([Ba] p. 202, [K1] p. 61, [Fr] Definition 7.2.1)**.** A *torsor* is a set X together with a map

$$(\cdot,\cdot,\cdot) : X \times X \times X \to X$$

satisfying:

1. $(x, y, y) = (y, y, x) = x$ for all $x, y \in X$
2. $((x, y, z), u, v) = (x, (u, z, y), v) = (x, y, (z, u, v))$ for all $x, y, z, u, v \in X$.

Morphisms are defined in the obvious way, i.e., maps $X \to X'$ of sets commuting with the torsor structure.

Lemma 1.7.6. *Let G be a group. Then $(g, h, k) = gh^{-1}k$ defines a torsor structure on G.*

Proof. This is a direct computation:

$$\begin{aligned}(x, y, y) &= xy^{-1}y = x = yy^{-1}x = (y, y, x),\\ ((x, y, z), u, v) &= (xy^{-1}z, u, v) = xy^{-1}zu^{-1}v = (x, y, zu^{-1}v) = (x, y, (z, u, v)),\\ (x, (u, z, y), v) &= (x, uz^{-1}y, v) = x(uz^{-1}y)^{-1}v) = xy^{-1}zu^{-1}v.\end{aligned}$$

□

Lemma 1.7.7 ([Ba] page 202)**.** *Let X be a torsor, $e \in X$ an element. Then $G_e := X$ carries a group structure via*

$$gh := (g, e, h), \quad g^{-1} := (e, g, e).$$

Moreover, the torsor structure on X is given by the formula $(g, h, k) = gh^{-1}k$ in G_e.

Proof. First we show associativity:

$$(gh)k = (g, e, h)k = ((g, e, h), e, k) = (g, e, (h, e, k)) = g(h, e, k) = g(hk).$$

e becomes the neutral element:

$$eg = (e, e, g) = g; ge = (g, e, e) = g.$$

We also have to show that g^{-1} is indeed the inverse element:

$$gg^{-1} = g(e, g, e) = (g, e, (e, g, e)) = ((g, e, e), g, e) = (g, g, e) = e.$$

Similarly one shows that $g^{-1}g = e$. One gets the torsor structure back, since

$$\begin{aligned}gh^{-1}k &= g(e, h, e)k = (g, e, (e, h, e))k = ((g, e, (e, h, e)), e, k)\\ &= (g, (e, (e, h, e), e), k) = (g, ((e, e, h), e, e), k)\\ &= (g, (h, e, e), k) = (g, h, k).\end{aligned}$$

□

Proposition 1.7.8. *Let $\mu_l : X^2 \times X^2 \to X^2$ be given by*

$$\mu_l\left((a,b),(c,d)\right) = ((a,b,c),d).$$

Then μ_l is associative and has (x,x) for $x \in X$ as left-neutral elements. Let $G^l = X^2/\sim_l$ where $(a,b) \sim_l (a,b)(x,x)$ for all $x \in X$ is an equivalence relation. Then μ_l is well-defined on G^l and turns G^l into a group. Moreover, the torsor structure map factors via a simply transitive left G^l-operation on X which is defined by

$$(a,b)x := (a,b,x).$$

Let $e \in X$. Then

$$i_e : G_e \to G^l, \qquad x \mapsto (x,e)$$

is group isomorphism inverse to $(a,b) \mapsto (a,b,e)$.
In a similar way, using $\mu_r\left((a,b),(c,d)\right) := (a,(b,c,d))$ we obtain a group G^r with analogous properties acting transitively on the right on X and such that μ_r factors through the action $X \times G^r \to X$.

Proof. First we check associativity of μ_l:

$$(a,b)[(c,d)(e,f)] = (a,b)((c,d,e),f) = ((a,b,(c,d,e)),f) = (((a,b,c),d,e),f)$$
$$[(a,b)(c,d)](e,f) = ((a,b,c),d)(e,f) = (((a,b,c),d,e),f)$$

(x,x) is a left neutral element for every $x \in X$:

$$(x,x)(a,b) = ((x,x,a),b) = (a,b)$$

We also need to check that $\sim_l$ is an equivalence relation: $\sim_l$ is reflexive, since one has $(a,b) = ((a,b,b),b) = (a,b)(b,b)$ by the first torsor axiom and the definition of μ. For symmetry, assume $(c,d) = (a,b)(x,x)$. Then

$$\begin{aligned}(a,b) &= ((a,b,b),b) = ((a,b,(x,x,b)),b) = (((a,b,x),x,b),b)\\ &= ((a,b,x),x)(b,b) = (a,b)(x,x)(b,b) = (c,d)(b,b)\end{aligned}$$

again by the torsor axioms and the definition of μ_l. For transitivity observe that

$$(a,b)(x,x)(y,y) = (a,b)((x,x,y),y) = (a,b)(y,y).$$

Now we show that μ_l is well-defined on G^l:

$$[(a,b)(x,x)][(c,d)(y,y)] = (a,b)[(x,x)(c,d)](y,y) = (a,b)(c,d)(y,y).$$

The inverse element to (a,b) in G^l is given by (b,a), since

$$(a,b)(b,a) = ((a,b,b),a) = (a,a).$$

Define the left G^l-operation on X by $(a,b)x := (a,b,x)$. This is compatible with μ_l, since

$$[(a,b)(c,d)]x = ((a,b,c),d)x = ((a,b,c),d,x),$$
$$(a,b)[(c,d)x] = (a,b)(c,d,x) = ((a,b,(c,d,x))$$

are equal by the second torsor axiom. The left G^l-operation is well-defined with respect to $\sim_l$:

$$[(a,b)(x,x)]y = ((a,b,x),x)y = ((a,b,x),x,y) = (a,(x,x,b),y) = (a,b,y) = (a,b)y.$$

Now we show that i_e is a group homomorphism:

$$ab = (a,e,b) \mapsto ((a,e,b),e) = (a,e)(b,e)$$

The inverse group homomorphism is given by

$$(a,b)(c,d) = ((a,b,c),d) \mapsto ((a,b,c),d,e).$$

On the other hand in G_e one has:

$$(a,b,e)(c,d,e) = ((a,b,e),e,(c,d,e)) = (a,b,(e,e,(c,d,e))) = (a,b,(c,d,e)).$$

This shows that i_e is an isomorphism. The fact that G_e is a group implies that the operation of G^l on X is simply transitive. Indeed the group structure on $G_e = X$ is the one induced by the operation of G^l. The analogous group G^r is constructed using μ_r and an equivalence relation $\sim_r$ with opposite order, i.e., $(a,b) \sim_r (x,x)(a,b)$ for all $x \in X$. The properties of G^r can be verified in the same way as for G^l and are left to the reader. □

1.7.3 Torsors in the category of schemes (without groups)

Definition 1.7.9. Let S be a scheme. A *torsor* in the category of S-schemes is a non-empty scheme X and a morphism

$$X \times X \times X \to X$$

which on T-valued points is a torsor in the sense of Definition 1.7.5 for all T over S.

This simply means that the diagrams of the previous definition commute as morphisms of schemes. The following is the scheme theoretic version of Lemma 1.7.8.

Recall the *fpqc*-topology of Definition 1.7.3.

Proposition 1.7.10. *Let S be affine. Let X be a torsor in the category of affine schemes. Assume that X/S is faithfully flat. Then there are affine group* *schemes G^l and G^r operating from the left and right on X, respectively, such that the natural maps*

$$\begin{aligned} G^l \times X \to X \times X & \quad (g,x) \mapsto (gx, x) \\ X \times G^r \to X \times X & \quad (x,g') \mapsto (x, xg') \end{aligned}$$

are isomorphisms.

Moreover, X is a left G^l- and right G^r-torsor with respect to the fpqc-topology on the category of affine schemes.

Proof. We consider G^l. The arguments for G^r are the same. We define G^l as the *fpqc*-sheafification of the presheaf

$$T \mapsto X^2(T)/\sim_l$$

We are going to see below that it is representable by an affine scheme. The map of presheaves μ_l defines a multiplication on G^l. It is associative as it is associative on the presheaf level.

We construct the neutral element. Recall that $X \to S$ is an *fpqc*-cover. The diagonal $\Delta : X \to X^2/\sim_l$ induces a section $e \in G(X)$. It satisfies descent for the cover X/S by the definition of the equivalence relation $\sim_l$. Hence it defines an element $e \in G(S)$. We claim that it is the neutral element of G. This can be tested *fpqc*-locally, e.g., after base change to X. For T/X the set $X(T)$ is non-empty, hence $X^2/\sim_l (T)$ is a group with neutral element e by Proposition 1.7.8.

The inversion map ι exists on $X^2(T)/\sim_l$ for T/X, hence it also exists and is the inverse on $G(T)$ for T/X. By the sheaf condition this gives a well-defined map with the correct properties on G.

By the same arguments, the action homomorphism $X^2(T)/\sim_l \times X(T) \to X(T)$ defines a left action $G^l \times X \to X$. The induced map $G^l \times X \to X \times X$ is an isomorphism because it as an isomorphism on the presheaf level for T/X. In particular, X is a left G^l-torsor.

We now turn to representability.

We are going to construct G^l by flat descent with respect to the faithfully flat cover $X \to S$ following [BLR, Section 6.1]. In order to avoid confusion, put $T = X$ and $Y = X \times X$ viewed as T-scheme over the second factor. A descent datum on $Y \to T$ consists of the choice of an isomorphism

$$\phi : p_1^*Y \to p_2^*Y$$

subject to the coycle condition

$$p_{13}^*\phi = p_{23}^*\phi \circ p_{12}^*\phi$$

with the obvious notation. We have $p_1^*Y = Y \times T = X^2 \times X$ and $p_2^*Y = T \times Y = X \times X^2$ and use

$$\phi(x_1, x_2, x_3) = (x_2, \rho(x_1, x_2, x_3), x_3)$$

where $\rho : X^2 \to X$ is the structural morphism of X. We have $p_{12}^* p_1^* Y = X^2 \times X \times X$ etc. and

$$\begin{aligned}
p_{12}^*\phi(x_1, x_2, x_3, x_4) &= (x_2, \rho(x_1, x_2, x_3), x_3, x_4) \\
p_{23}^*\phi(x_1, x_2, x_3, x_4) &= (x_1, x_3, \rho(x_2, x_3, x_4), x_4) \\
p_{13}^*\phi(x_1, x_2, x_3, x_4) &= (x_2, x_3\rho(x_1, x_3, x_4), x_4)
\end{aligned}$$

and the cocyle condition is equivalent to

$$\rho(\rho(x_1, x_2, x_3), x_3, x_4) = \rho(x_1, x_2, x_4),$$

which is an immediate consequence of the properties of a torsor. In the affine case (that we are in) any descent datum is effective, i.e., induced from a uniquely determined S-scheme $\tilde{G}^l$. In other words, it represents the *fpqc*-sheaf defined as the coequalizer of

$$X^2 \times X \rightrightarrows X^2$$

with respect to the projection p_1 mapping (x_1, x_2, x_3) to (x_1, x_2) and $p_2 \circ \phi : X^2 \times X \to X \times X^2 \to X^2$ mapping

$$(x_1, x_2, x_3) \mapsto (x_2, \rho(x_1, x_2, x_3), x_3) \mapsto (\rho(x_1, x_2, x_3), x_3)$$

This is precisely the equivalence relation $\sim_l$. Hence

$$\tilde{G}^l = X^2 / \sim_l$$

as *fpqc*-sheaves. □

Remark 1.7.11. If S is the spectrum of a field, then the flatness assumption is always satisfied. In general, some kind of assumption is needed, as the following example shows. Let S be the spectrum of a discrete valuation ring with closed point s. Let G be an algebraic group over s and $X = G$ the trivial torsor defined by G. In particular, we have the structure map

$$X \times_s X \times_s X \to X.$$

We now view X as an S-scheme. Note that

$$X \times_S X \times_S X = X \times_s X \times_s X$$

hence X is also a torsor over S in the sense of Definition 1.7.9. However, it is not a torsor with respect to the *fpqc* (or any other reasonable Grothendieck topology) as $X(T)$ is empty for all $T \to S$ surjective.

Chapter 2

Singular Cohomology

In this chapter we give a short introduction to singular cohomology. Many properties are only sketched, as this theory is considerably easier than de Rham cohomology for example.

2.1 Sheaf cohomology

Let X be a topological space. Sometimes, if indicated, X will be the underlying topological space of an analytic or algebraic variety also denoted by X. To avoid technicalities, X will always be assumed to be a paracompact space, i.e., locally compact, Hausdorff, and satisfying the second countability axiom.

From now on, let $\mathcal{F}$ be a sheaf of abelian groups on X and consider sheaf cohomology $H^i(X, \mathcal{F})$ from Section 1.4. Mostly, we will consider the case of the constant sheaf $\mathcal{F} = \mathbb{Z}$. Later we will also consider other constant coefficients $R \supset \mathbb{Z}$, but this will not change the following topological statements.

Definition 2.1.1 (Relative Cohomology)**.** Let $A \subset X$ be a closed subset, $U = X \setminus A$ the open complement, $i : A \hookrightarrow X$ and $j : U \hookrightarrow X$ be the inclusion maps. We define *relative cohomology* as

$$H^i(X, A; \mathbb{Z}) := H^i(X, j_! \mathbb{Z}),$$

where $j_!$ is the extension by zero, i.e., the sheafification of the presheaf $V \mapsto \mathbb{Z}$ for $V \subset U$ and $V \mapsto 0$ else.

Remark 2.1.2 (Functoriality and homotopy invariance)**.** The association

$$(X, A) \mapsto H^i(X, A; \mathbb{Z})$$

is a contravariant functor from pairs of topological spaces to abelian groups. In particular, for every continuous map $f : (X, A) \to (X', A')$ of pairs, i.e., satisfying $f(A) \subset A'$, one has a homomorphism $f^* : H^i(X', A'; \mathbb{Z}) \to H^i(X, A; \mathbb{Z})$.

Given two homotopic maps f and g, then the homomorphisms f^*, g^* are equal. As a consequence, if two pairs (X, A) and (X', A') are homotopy equivalent, then the cohomology groups $H^i(X', A'; \mathbb{Z})$ and $H^i(X, A; \mathbb{Z})$ are isomorphic.

Proposition 2.1.3. *There is a long exact sequence*

$$\cdots \to H^i(X, A; \mathbb{Z}) \to H^i(X, \mathbb{Z}) \to H^i(A, \mathbb{Z}) \xrightarrow{\delta} H^{i+1}(X, A; \mathbb{Z}) \to \cdots$$

Proof. This follows from the exact sequence of sheaves

$$0 \to j_!\mathbb{Z} \to \mathbb{Z} \to i_*\mathbb{Z} \to 0.$$

□

Note that by our definition of cones, see section 1.3, one has a quasi-isomorphism $j_!\mathbb{Z} = \mathrm{Cone}(\mathbb{Z} \to i_*\mathbb{Z})[-1]$. For Nori motives we need a version for triples, which can be proved using iterated cones by the same method:

Corollary 2.1.4. *Let $X \supset A \supset B$ be a triple of relative closed subsets. Then there is a long exact sequence*

$$\cdots \to H^i(X, A; \mathbb{Z}) \to H^i(X, B; \mathbb{Z}) \to H^i(A, B; \mathbb{Z}) \xrightarrow{\delta} H^{i+1}(X, A; \mathbb{Z}) \to \cdots$$

Here, δ is the connecting homomorphism, which in the cone picture is explained in Section 1.3.

Proposition 2.1.5 (Mayer-Vietoris)**.** *Let $\{U, V\}$ be an open cover of X. Let $A \subset X$ be closed. Then there is a natural long exact sequence*

$$\cdots \to H^i(X, A; \mathbb{Z}) \to H^i_{\mathrm{dR}}(U, U \cap A; \mathbb{Z}) \oplus H^i(V, V \cap A; \mathbb{Z})$$
$$\to H^i(U \cap V, U \cap V \cap A; \mathbb{Z}) \to H^{i+1}(X, A; \mathbb{Z}) \to \cdots$$

Proof. Pairs (U, V) of open subsets form an excisive couple in the sense of [Spa, pg. 188], and therefore the Mayer-Vietoris property holds by [Spa, pg. 189-190]. □

Theorem 2.1.6 (Proper base change)**.** *Let $\pi : X \to Y$ be proper (i.e., the preimage of a compact subset is compact). Let $\mathcal{F}$ be a sheaf on X. Then the stalk in $y \in Y$ is computed as*

$$(R^i\pi_*\mathcal{F})_y = H^i(X_y, \mathcal{F}|_{X_y}).$$

Proof. See [KS] Proposition 2.6.7. As π is proper, we have $R\pi_* = R\pi_!$. □

Now we list some properties of the sheaf cohomology of algebraic varieties over a field $k \hookrightarrow \mathbb{C}$. As usual, we will not distinguish in notation between a variety X and the topological space $X(\mathbb{C})$. The first property is:

Proposition 2.1.7 (Excision, or abstract blow-up)**.** *Let $f : (X', D') \to (X, D)$ be a proper, surjective morphism of algebraic varieties over $\mathbb{C}$, which induces an isomorphism $F : X' \setminus D' \to X \setminus D$. Then*

$$f^* : H^*(X, D; \mathbb{Z}) \cong H^*(X', D'; \mathbb{Z}).$$

Proof. This fact goes back to A. Aeppli [Ae]. It is a special case of proper-base change: Let $j : U \to X$ be the complement of D and $j' : U \to X'$ its inclusion into X'. For all $x \in X$, we have

$$R^i\pi_* j'_!\mathbb{Z} = H^i(X_x, j'_!\mathbb{Z}|_{X'_x}).$$

For $x \in U$, the fibre is one point. It has no higher cohomology. For $x \in D$, the restriction of $j'_!\mathbb{Z}$ to X'_x is zero. Together this means

$$R\pi_* j'_!\mathbb{Z} = j_!\mathbb{Z}.$$

The statement follows from the Leray spectral sequence. □

We will later prove a slightly more general proper base change theorem for singular cohomology, see Theorem 2.5.12.

The second property is:

Proposition 2.1.8 (Gysin isomorphism, localization, weak purity)**.** *Let X be an irreducible variety of dimension n over k, and Z a closed subvariety of pure codimension r. Then there is an exact sequence*

$$\cdots \to H^i_Z(X, \mathbb{Z}) \to H^i(X, \mathbb{Z}) \to H^i(X \setminus Z, \mathbb{Z}) \to H^{i+1}_Z(X, \mathbb{Z}) \to \cdots$$

where $H^i_Z(X, \mathbb{Z})$ is cohomology with supports in Z, defined as the hypercohomology of $\mathrm{Cone}(\mathbb{Z}_X \to \mathbb{Z}_{X\setminus U})[-1]$.

If, moreover, X and Z are both smooth, then one has the Gysin isomorphism

$$H^i_Z(X, \mathbb{Z}) \cong H^{i-2r}(Z, \mathbb{Z}).$$

In particular, one has weak purity:

$$H^i_Z(X, \mathbb{Z}) = 0 \textit{ for } i < 2r,$$

and $H^{2r}_Z(X, \mathbb{Z}) = H^0(Z, \mathbb{Z})$ is free of rank the number of components of Z.

Proof. See [Pa, Sect. 2] for this statement and an axiomatic treatment with more general properties and examples of cohomology theories. □

2.2 Singular (co)homology

Let X be a topological space (same general assumptions as in section 2.1). The definition of singular homology and cohomology uses topological simplexes.

Definition 2.2.1. The topological *n-simplex* Δ_n is defined as

$$\Delta_n := \{(t_0, ..., t_n) \mid \sum_{i=0}^{n} t_i = 1,\ t_i \geq 0\}\ .$$

Δ_n has natural codimension one faces defined by $t_i = 0$.

Singular (co)homology is defined by looking at all possible continuous maps from simplices to X.

Definition 2.2.2. A singular *n-simplex* σ is a continuous map

$$f\colon \Delta_n \to X.$$

In the case where X is a differentiable manifold, a singular simplex σ is called *differentiable*, if the map f can be extended to a $\mathcal{C}^\infty$-map from a neighbourhood of $\Delta_n \subset \mathbb{R}^{n+1}$ to X. The *group of singular n-chains* is the free abelian group

$$S_n(X) := \mathbb{Z}[f\colon \Delta_n \to X \mid f \text{ singular chain }].$$

In a similar way, we denote by $S_n^\infty(X)$ the free abelian *group of differentiable chains*. The boundary map $\partial_n : S_n(X) \to S_{n-1}(X)$ is defined as

$$\partial_n(f) := \sum_{i=0}^{n} (-1)^i f|_{t_i=0}.$$

The *group of singular n-cochains* is the free abelian group

$$S^n(X) := \mathrm{Hom}_{\mathbb{Z}}(S_n(X), \mathbb{Z}).$$

The *group of differentiable singular n-cochains* is the free abelian group

$$S^n(X) := \mathrm{Hom}_{\mathbb{Z}}(S_n^\infty(X), \mathbb{Z}).$$

The adjoint of ∂_{n+1} defines the boundary map

$$d_n : S^n(X) \to S^{n+1}(X).$$

Lemma 2.2.3. *One has $\partial_{n-1}\partial_n = 0$ and $d_{n+1}d_n = 0$, i.e., the groups $S_\bullet(X)$ and $S^\bullet(X)$ define complexes of abelian groups.*

The proof is left to the reader as an exercise.

Definition 2.2.4. Singular homology and cohomology with values in $\mathbb{Z}$ is defined as

$$H^i_{\rm sing}(X,\mathbb{Z}) := H^i(S^\bullet(X), d_\bullet),\ H_i^{\rm sing}(X,\mathbb{Z}) := H_i(S_\bullet(X), \partial_\bullet)\ .$$

In a similar way, we define (for X a manifold) the differentiable singular (co)homology as

$$H^i_{{\rm sing},\infty}(X,\mathbb{Z}) := H^i(S^\bullet_\infty(X), d_\bullet),\ H_i^{{\rm sing},\infty}(X,\mathbb{Z}) := H_i(S^\infty_\bullet(X), \partial_\bullet)\ .$$

Theorem 2.2.5. *Assume that X is a locally contractible topological space, i.e., every point has an open contractible neighborhood. In this case, singular cohomology $H^i_{\rm sing}(X,\mathbb{Z})$ agrees with sheaf cohomology $H^i(X,\mathbb{Z})$ with coefficients in $\mathbb{Z}$. If Y is a differentiable manifold, differentiable singular (co)homology agrees with singular (co)homology.*

Proof. Let $\mathcal{S}^n$ be the sheaf associated to the presheaf $U \mapsto S^n(U)$. One shows that $\mathbb{Z} \to \mathcal{S}^\bullet$ is a fine resolution of the constant sheaf $\mathbb{Z}$ [Wa, pg. 196]. In the proof it is used that X is locally contractible, see [Wa, pg. 194]. If X is a manifold, differentiable cochains also define a fine resolution [Wa, pg. 196]. Therefore, the inclusion of complexes $S^\infty_\bullet(X) \hookrightarrow S_\bullet(X)$ induces isomorphisms

$$H^i_{{\rm sing},\infty}(X,\mathbb{Z}) \cong H^i_{\rm sing}(X,\mathbb{Z}) \text{ and } H_i^{{\rm sing},\infty}(X,\mathbb{Z}) \cong H_i^{\rm sing}(X,\mathbb{Z})\,.$$

□

Of course, topological manifolds satisfy the assumption of the theorem.

2.3 Simplicial cohomology

In this section we want to introduce simplicial (co)homology and its relation to singular (co)homology. Simplicial (co)homology can be defined for topological spaces with an underlying combinatorial structure.

In the literature there are various notions of such spaces. In increasing order of generality, these are: (geometric) simplicial complexes and topological realizations of abstract simplicial complexes, of Δ-complexes (sometimes also called semi-simplicial complexes), and of simplicial sets. A good reference with a discussion of various definitions is the book by Hatcher [Hat], or the introductory paper [Fri] by Friedman.

By construction, such spaces are built from topological simplices Δ_n in various dimensions n, and the faces of each simplex are of the same type. Particularly nice examples are polyhedra, for example a tetrahedron, where the simplicial structure is obvious.

Geometric simplicial complexes come up more generally in geometric situations in the triangulations of manifolds with certain conditions. An example is the case of an analytic space X^{an} where X is an algebraic variety defined over $\mathbb{R}$. There one can always find a semi-algebraic triangulation by a result of Lojasiewicz, cf. Hironaka [Hi2, p. 170] and Prop. 2.6.8.

In this section, we will think of a simplicial space as the topological realization of a finite simplicial set:

Definition 2.3.1. Let $X_\bullet$ be a finite simplicial set in the sense of Remark 1.5.5. One has the face maps

$$\partial_i : X_n \to X_{n-1} \quad i = 0, \dots, n,$$

and the degeneracy maps

$$s_i : X_n \to X_{n+1} \quad i = 0, \dots, n.$$

The topological realization $|X_\bullet|$ of $X_\bullet$ is defined as

$$|X_\bullet| := \coprod_{n=0}^{\infty} X_n \times \Delta_n / \sim,$$

where each X_n carries the discrete topology, Δ_n is the topological n-simplex, and the equivalence relation is given by the two relations

$$(x, \partial_i(y)) \sim (\partial_i(x), y), \quad (x, s_i(y)) \sim (s_i(x), y), \quad x \in X_{n-1},\ y \in \Delta_n.$$

(Note that we denote the face and degeneracy maps for the n-simplex by the same letters ∂_i, s_i.)

In this way, every finite simplicial set gives rise to a topological space $|X_\bullet|$. It is known that $|X_\bullet|$ is a compactly generated CW-complex [Hat, Appendix]. In fact, every finite CW-complex is homotopy equivalent to a finite simplical complex of the same dimension by [Hat, Thm. 2C.5]. Thus, our restriction to realizations of finite simplicial sets is not a severe restriction.

The skeleton filtration from Remark 1.5.5 defines a filtration of $|X_\bullet|$

$$|\mathrm{sq}_0 X_\bullet| \subseteq |\mathrm{sq}_1 X_\bullet| \subseteq \cdots \subseteq |\mathrm{sq}_N X_\bullet| = |X_\bullet|$$

by closed subspaces, if X_n is empty for $n > N$.

There is finite number of simplices in each degree n. Associated to each of them is a continuous map $\sigma : \Delta_n \to |X_\bullet|$. We denote the free abelian group of all such σ of degree n by $C_n^\Delta(X_\bullet)$

$$\partial_n : C_n^\Delta(X_\bullet) \to C_{n-1}^\Delta(X_\bullet)$$

are given by alternating restriction maps to faces, as in the case of singular homology. Note that the vertices of each simplex are ordered, so that this is well-defined.

Definition 2.3.2. Simplicial homology of the topological space $X = |X_\bullet|$ is defined as

$$H_n^{\mathrm{simpl}}(X, \mathbb{Z}) := H_n(C_*^\Delta(X_\bullet), \partial_*),$$

and simplicial cohomology as

$$H^n_{\mathrm{simpl}}(X; \mathbb{Z}) := H^n(C^*_\Delta(X_\bullet), d_*),$$

where $C^n_\Delta(X_\bullet) = \mathrm{Hom}(C_n^\Delta(X_\bullet), \mathbb{Z})$ and d_n is adjoint to ∂_n.

Example 2.3.3. A tetrahedron arises from a simplicial set with four vertices (0-simplices), six edges (1-simplices), and four faces (2-simplices). A computation shows that $H_n = \mathbb{Z}$ for $i = 0, 2$ and zero otherwise (this was a priori clear, since it is topologically a sphere).

A torus T^2 can be obtained from a square by identifying opposite sides, called a and b. If we look at the diagonal of the square, we see that there is a simplicial complex with one vertex (!), three edges, and two faces. A computation shows that $H_1(T^2, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$ as expected, and $H_0(T^2, \mathbb{Z}) = H_2(T^2, \mathbb{Z}) = \mathbb{Z}$.

This definition does not depend on the representation of a topological space X as the topological realization of a simplicial set, since one can prove that simplicial (co)homology coincides with singular (co)homology:

Theorem 2.3.4. *Singular and simplicial (co)homology of X are equal.*

Proof. (For homology only.) The chain of closed subsets

$$|\mathrm{sq}_0 X_\bullet| \subseteq |\mathrm{sq}_1 X_\bullet| \subseteq \cdots \subseteq |\mathrm{sq}_N X_\bullet| = |X_\bullet|$$

gives rise to long exact sequences of simplicial homology groups

$$\cdots \to H_n^{\mathrm{simpl}}(|\mathrm{sq}_{n-1} X_\bullet|, \mathbb{Z}) \to H_n^{\mathrm{simpl}}(|\mathrm{sq}_n X_\bullet|, \mathbb{Z}) \to H_n^{\mathrm{simpl}}(|\mathrm{sq}_{n-1} X_\bullet|, |\mathrm{sq}_n X_\bullet|; \mathbb{Z}) \to \cdots$$

A similar sequence holds for singular homology, and there is a canonical map $C_n^\Delta(X) \to C_n(X)$ from simplicial to singular chains. The result is then proved by induction on n. We use that the relative complex $C_n(|\mathrm{sq}_{n-1} X_\bullet|, |\mathrm{sq}_n X_\bullet|)$ has zero differential and is a free abelian group of rank equal to the cardinality of X_n. Therefore, one concludes by observing a computation of the singular (co)homology of Δ_n, i.e., $H^i(\Delta_n, \mathbb{Z}) = \mathbb{Z}$ for $i = 0$ and zero otherwise. □

In a similar way, one can define the simplicial (co)homology of a pair $(X, D) = (|X_\bullet|, |D_\bullet|)$, where $D_\bullet \subset X_\bullet$ is a simplicial subobject. The associated chain complex is given by the quotient complex $C_*(X_\bullet)/C_*(D_\bullet)$. The same proof will then show that the singular and simplicial (co)homology of pairs coincide.

From the definition of the topological realization, we see that X is a CW-complex. In the special case, when X is the topological space underlying an affine algebraic variety X over $\mathbb{C}$, or more generally a Stein manifold, then one can show:

Theorem 2.3.5 (Artin vanishing)**.** *Let X be an affine variety over $\mathbb{C}$ of dimension n. Then $H^q(X^{\mathrm{an}},\mathbb{Z})=0$ for $q>n$. In fact, X^{an} is homotopy equivalent to a finite simplicial complex where all cells are of dimension $\le n$.*

Proof. The proof was first given by Andreotti and Fraenkel [AF] for Stein manifolds. For Stein spaces, i.e., allowing singularities, this is a theorem of Kaup, Narasimhan and Hamm, see [Ham1, Satz 1] and the correction in [Ham2]. An algebraic proof was given by M. Artin [A, Cor. 3.5, tome 3]. □

Corollary 2.3.6 (Good topological filtration)**.** *Let X be an affine variety over $\mathbb{C}$ of dimension n. Then the skeleton filtration of X^{an} is given by*

$$X^{\mathrm{an}}=X_n\supset X_{n-1}\supset\cdots\supset X_0$$

where the pairs (X_i,X_{i-1}) have only cohomology in degree i.

Remark 2.3.7. The Basic Lemma of Nori and Beilinson, see Thm. 2.5.7, shows that there is even an algebraic variant of this topological skeleton filtration.

Corollary 2.3.8 (Artin vanishing for relative cohomology)**.** *Let X be an affine variety of dimension n and $Z\subset X$ a closed subvariety. Then*

$$H^i(X^{\mathrm{an}},Z^{\mathrm{an}},\mathbb{Z})=0 \text{ for } i>n.$$

Proof. Consider the long exact sequence for relative cohomology and use Artin vanishing for X and Z from Thm.2.3.5. □

The following theorem is strongly related to the Artin vanishing theorem.

Theorem 2.3.9 (Lefschetz hyperplane theorem)**.** *Let $X\subset\mathbb{P}^N_{\mathbb{C}}$ be an integral projective variety of dimension n, and $H\subset\mathbb{P}^N_{\mathbb{C}}$ a hyperplane section such that $H\cap X$ contains the singularity set X_{sing} of X. Then the inclusion $H\cap X\subset X$ is $(n-1)$-connected. In particular, one has $H^q(X,\mathbb{Z})=H^q(X\cap H,\mathbb{Z})$ for $q\le n$.*

Proof. See for example [AF]. □

2.4 Künneth formula and Poincaré duality

Assume that we have given two topological spaces X and Y, and two closed subsets $j:A\hookrightarrow X$, and $j':C\hookrightarrow Y$. By the above, we have

$$H^*(X,A;\mathbb{Z})=H^*(X,j_!\mathbb{Z})$$

and

$$H^*(Y,C;\mathbb{Z})=H^*(Y,j'_!\mathbb{Z})\ .$$

The relative cohomology group

$$H^*(X\times Y,X\times C\cup A\times Y;\mathbb{Z})$$

can be computed as $H^*(X \times Y, \tilde{j}_!\mathbb{Z})$, where

$$\tilde{j} : X \times C \cup A \times Y \hookrightarrow X \times Y$$

is the inclusion map. One has $\tilde{j}_! = j_! \boxtimes j'_!$. Hence, we have a natural exterior product map

$$H^i(X, A; \mathbb{Z}) \otimes H^j(Y, C; \mathbb{Z}) \xrightarrow{\times} H^{i+j}(X \times Y, X \times C \cup A \times Y; \mathbb{Z}).$$

This is related to the so-called *Künneth formula*:

Theorem 2.4.1 (Künneth formula for pairs)**.** *Let $A \subset X$ and $C \subset Y$ be closed subsets. The exterior product map induces a natural isomorphism*

$$\bigoplus_{i+j=n} H^i(X, A; \mathbb{Q}) \otimes H^j(Y, C; \mathbb{Q}) \xrightarrow{\cong} H^n(X \times Y, X \times C \cup A \times Y; \mathbb{Q}).$$

The same result holds with $\mathbb{Z}$-coefficients, provided all cohomology groups of (X, A) and (Y, C) in all degrees are free.

Proof. Using the sheaves of singular cochains, see the proof of Theorem 2.2.5, one has fine resolutions $j_!\mathbb{Z} \to F^\bullet$ on X, and $j'_!\mathbb{Z} \to G^\bullet$ on Y. The tensor product $F^\bullet \boxtimes G^\bullet$ thus is a fine resolution of $\tilde{j}_!\mathbb{Z} = j_!\mathbb{Z} \boxtimes j'_!\mathbb{Z}$. Here one uses that the tensor product of fine sheaves is fine [Wa, pg. 193]. The cohomology of the tensor product complex $F^\bullet \otimes G^\bullet$ induces a short exact sequence

$$0 \to \bigoplus_{i+j=n} H^i(X, A; \mathbb{Z}) \otimes H^j(Y, C; \mathbb{Z}) \to H^n(X \times Y, X \times C \cup A \times Y; \mathbb{Z})$$

$$\to \bigoplus_{i+j=n+1} \mathrm{Tor}_1^{\mathbb{Z}}(H^i(X, A; \mathbb{Z}), H^j(Y, C; \mathbb{Z})) \to 0$$

by [God, thm. 5.5.1] or [We, thm. 3.6.3]. If all cohomology groups are free, the last term vanishes. □

Proposition 2.4.2. *The Künneth isomorphism of Theorem 2.4.1 is associative and graded commutative.*

Proof. This is a standard consequence of the definition of the Künneth isomorphism from complexes of groups. □

In later constructions, we will need a certain compatibility of the exterior product with coboundary maps. Assume that $X \supset A \supset B$ and $Y \supset C$ are closed subsets.

Proposition 2.4.3. *The diagram involving coboundary maps*

$$\begin{array}{ccc} H^i(A, B; \mathbb{Z}) \otimes H^j(Y, C; \mathbb{Z}) & \longrightarrow & H^{i+j}(A \times Y, A \times C \cup B \times Y; \mathbb{Z}) \\ \downarrow{\scriptstyle \delta\otimes\mathrm{id}} & & \downarrow{\scriptstyle \delta} \\ H^{i+1}(X, A; \mathbb{Z}) \otimes H^j(Y, C; \mathbb{Z}) & \longrightarrow & H^{i+j+1}(X \times Y, X \times C \cup A \times Y; \mathbb{Z}) \end{array}$$

commutes up to a sign $(-1)^j$. *The diagram*

$$\begin{array}{ccc} H^i(Y,C;\mathbb{Z})\otimes H^j(A,B;\mathbb{Z}) & \longrightarrow & H^{i+j}(Y\times A, Y\times B\cup C\times A;\mathbb{Z}) \\ {\scriptstyle \mathrm{id}\otimes\delta}\downarrow & & \downarrow{\scriptstyle \delta} \\ H^i(Y,C;\mathbb{Z})\otimes H^{j+1}(X,A;\mathbb{Z}) & \longrightarrow & H^{i+j+1}(Y\times X, Y\times A\cup C\times X;\mathbb{Z}) \end{array}$$

commutes (without a sign).

Proof. We indicate the argument, without going into full details. Let $F^\bullet$ be a complex computing $H^\bullet(Y,C;\mathbb{Z})$ Let $G_1^\bullet$ and $G_2^\bullet$ be complexes computing $H^\bullet(A,B;\mathbb{Z})$ and $H^\bullet(X,A;\mathbb{Z})$. Let $K_1^\bullet$ and $K_2^\bullet$ be the complexes computing cohomology of the corresponding product varieties. Cup product is induced from maps of complexes $F_i^\bullet\otimes G^\bullet\to K_i^\bullet$. In order to get compatibility with the boundary map, we have to consider the diagram of the form

$$\begin{array}{ccc} F_1\otimes G & \longrightarrow & K_1 \\ \downarrow & & \downarrow \\ (F_2[1])\otimes G & \longrightarrow & K_2[1] \end{array}$$

However, by Lemma 1.3.6, the complexes $(F_2[1])\otimes G$ and $(F_2\otimes G)[1]$ are not equal. We need to introduce the sign $(-1)^j$ in bidegree (i,j) to make the identification and get a commutative diagram.

The argument for the second type of boundary map is the same, but does not need the introduction of signs by Lemma 1.3.6. □

Assume now that $X=Y$ and $A=C$. Then, $j_!\mathbb{Z}$ has an algebra structure, and we obtain the *cup product* maps:

$$H^i(X,A;\mathbb{Z})\otimes H^j(X,A;\mathbb{Z})\longrightarrow H^{i+j}(X,A;\mathbb{Z})$$

via the multiplication maps

$$H^{i+j}(X\times X,\tilde{j}_!\mathbb{Z})\to H^{i+j}(X,j_!\mathbb{Z}),$$

induced by

$$\tilde{j}_! = j_!\boxtimes j_!\to j_!\ .$$

In the case where $A=\emptyset$, the cup product induces Poincaré duality:

Proposition 2.4.4 (Poincaré Duality)**.** *Let X be a compact, orientable topological manifold of dimension m. Then the cup product pairing*

$$H^i(X,\mathbb{Q})\times H^{m-i}(X,\mathbb{Q})\longrightarrow H^m(X,\mathbb{Q})\cong\mathbb{Q}$$

is non-degenerate in both factors. With $\mathbb{Z}$-coefficients, the same result holds for the two left groups modulo torsion.

Proof. We will give a proof of a slightly more general statement in the algebraic situation below. A proof of the stated theorem can be found in [GH, pg. 53]. There it is shown that $H^{2n}(X)$ is torsion-free of rank one, and the cup-product pairing is unimodular modulo torsion, using simplicial cohomology, and the relation between Poincaré duality and the dual cell decomposition. □

We will now prove a relative version in the algebraic case. It implies the version above in the case where $A = B = \emptyset$. By abuse of notation, we again do not distinguish between an algebraic variety over $\mathbb{C}$ and its underlying topological space.

Theorem 2.4.5 (Poincaré duality for algebraic pairs)**.** *Let X be a smooth and proper complex variety of dimension n over $\mathbb{C}$ and $A, B \subset X$ two normal crossing divisors, such that $A \cup B$ is also a normal crossing divisor. Then there is a non-degenerate duality pairing*

$$H^d(X\setminus A, B\setminus(A\cap B);\mathbb{Q}) \times H^{2n-d}(X\setminus B, A\setminus(A\cap B);\mathbb{Q}) \longrightarrow H^{2n}(X,\mathbb{Q}) \cong \mathbb{Q}(-n).$$

Again, with $\mathbb{Z}$-coefficients this is true modulo torsion by unimodularity of the cup-product pairing.

Proof. We give a sheaf theoretic proof using Verdier duality and some formulas and ideas of Beilinson (see [Be1]). Look at the commutative diagram:

$$\begin{array}{ccc} U = X\setminus(A\cup B) & \xrightarrow{\ell_U} & X\setminus A \\ {\scriptstyle\kappa_U}\downarrow & & \downarrow{\scriptstyle\kappa} \\ X\setminus B & \xrightarrow{\ell} & X. \end{array}$$

Assuming $A \cup B$ is a normal crossing divisor, we want to show first that the natural map

$$\ell_! R\kappa_{U*}\mathbb{Q}_U \longrightarrow R\kappa_* \ell_{U!}\mathbb{Q}_U,$$

extending $\mathrm{id} : \mathbb{Q}_U \to \mathbb{Q}_U$, is an isomorphism. This is a local computation. We look without loss of generality at a neighborhood of an intersection point $x \in A \cap B$, since the computation at other points is even easier. If we work in the analytic topology, we may choose a polydisk neighborhood D in X around x such that D decomposes as

$$D = D_A \times D_B$$

and such that

$$A \cap D = A_0 \times D_B, \quad B \cap D = D_A \times B_0$$

for some suitable topological spaces A_0, B_0. Using the same symbols for the maps as in the above diagram, the situation looks locally like

$$\begin{array}{ccc} (D_A \setminus A_0) \times (D_B \setminus B_0) & \xrightarrow{\ell_U} & (D_A \setminus A_0) \times D_B \\ \kappa_U \downarrow & & \downarrow \kappa \\ D_A \times (D_B \setminus B_0) & \xrightarrow{\ell} & D = D_A \times D_B. \end{array}$$

Using the Künneth formula, one concludes that both sides $\ell_! R\kappa_{U*}\mathbb{Q}_U$ and $R\kappa_* \ell_{U!}\mathbb{Q}_U$ are isomorphic to

$$R\kappa_{U*}\mathbb{Q}_{D_A \setminus A_0} \otimes \ell_! \mathbb{Q}_{D_B \setminus B_0}$$

near the point x, and the natural map provides an isomorphism.

Now, one has

$$H^d(X \setminus A, B \setminus (A \cap B)); \mathbb{Q}) = H^d(X, \ell_! \kappa_{U*}\mathbb{Q}_U),$$

(using that the maps involved are affine), and

$$H^{2n-d}(X \setminus B, A \setminus (A \cap B)); \mathbb{Q}) = H^{2n-d}(X, \kappa_! \ell_{U*}\mathbb{Q}_U).$$

We have to show that there is a perfect pairing

$$H^d(X \setminus A, B \setminus (A \cap B)); \mathbb{Q}) \times H^{2n-d}(X \setminus B, A \setminus (A \cap B)); \mathbb{Q}) \to \mathbb{Q}(-n).$$

However, by Verdier duality, we have a perfect pairing

$$\begin{aligned} H^{2n-d}(X \setminus B, A \setminus (A \cap B)); \mathbb{Q})^\vee &= H^{2n-d}(X, \kappa_! \ell_{U*}\mathbb{Q}_U)^\vee \\ &= H^{-d}(X, \kappa_! \ell_{U*}\mathbb{D}\mathbb{Q}_U)(-n) \\ &= H^{-d}(X, \mathbb{D}(\kappa_* \ell_{U!}\mathbb{Q}_U))(-n) \\ &= H^d(X, \kappa_* \ell_{U!}\mathbb{Q}_U)(-n) \\ &= H^d(X, \ell_! \kappa_{U*}\mathbb{Q}_U)(-n) \\ &= H^d(X \setminus A, B \setminus (A \cap B)); \mathbb{Q}). \end{aligned}$$

□

Remark 2.4.6. The normal crossing condition is necessary, as one can see in the example of $X = \mathbb{P}^2$, where A consists of two distinct lines meeting in a point, and B a line going through the same point.

2.5 Basic Lemma

In this section we prove the basic lemma of Nori [N, N1, N2], a topological result, which was also known to Beilinson [Be1] and Vilonen (unpublished). Let $k \subset \mathbb{C}$ be a subfield. The proof of Beilinson works more generally in positive characteristics as we will see below.

Convention 2.5.1. We fix an embedding $k \hookrightarrow \mathbb{C}$. All sheaves and all cohomology groups in the following section are to be understood in the analytic topology on $X(\mathbb{C})$.

Theorem 2.5.2 (Basic Lemma I)**.** *Let $k \subset \mathbb{C}$. Let X be an affine variety over k of dimension n and $W \subset X$ be a Zariski closed subset with $\dim(W) < n$. Then there exists a Zariski closed subset $Z \supset W$ defined over k with $\dim(Z) < n$ and*

$$H^q(X, Z; \mathbb{Z}) = 0, \text{ for } q \neq n$$

and, moreover, the cohomology group $H^n(X, Z; \mathbb{Z})$ is a free $\mathbb{Z}$-module.

We formulate the Lemma for coefficients in $\mathbb{Z}$, but by the universal coefficient theorem [We, thm. 3.6.4] it will hold with other coefficients as well.

Example 2.5.3. There is an example where there is an easy way to obtain Z. Assume that X is of the form $\bar{X} \setminus H$ for some smooth projective $\bar{X}$ and a hyperplane H. Let $W = \emptyset$. Then take another hyperplane section H' meeting $\bar{X}$ and H transversally. Then $Z := H' \cap X$ will have the property that $H^q(X, Z; \mathbb{Z}) = 0$ for $q \neq n$ by the Lefschetz hyperplane theorem, see Thm. 2.3.9. This argument will be generalized in two of the proofs below.

An inductive application of this Basic Lemma in the case $Z = \emptyset$ yields a filtration of X by closed subsets

$$X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset$$

with $\dim(X_i) = i$ such that the complex of free $\mathbb{Z}$-modules

$$\cdots \xrightarrow{\delta_{i-1}} H^i(X_i, X_{i-1}) \xrightarrow{\delta_i} H^{i+1}(X_{i+1}, X_i) \xrightarrow{\delta_{i+1}} \cdots,$$

where the maps $\delta_\bullet$ arise from the coboundary in the long exact sequence associated to the triples $X_{i+1} \supset X_i \supset X_{i-1}$, computes the cohomology of X.

Remark 2.5.4. This means that we can understand this filtration as algebraic analogue of the skeletal filtraton of simplicial complexes, see Corollary 2.3.6. Note that the filtration is not only algebraic, but even defined over the base field k.

The Basic Lemma is deduced from the following variant, which was also known to Beilinson [Be1]. To state it, we need the notion of a (weakly) constructible sheaf.

Definition 2.5.5. A sheaf of abelian groups on a variety X over k is *weakly constructible*, if there is a stratification of X into a disjoint union of finitely many Zariski locally closed subsets Y_i defined over k, and such that the restriction of F to Y_i is locally constant. It is called *constructible* if, in addition, the stalks of F are finitely generated abelian groups.

Remark 2.5.6. This combination of sheaves in the analytic topology and strata algebraic and defined over k, is not very much discussed in the literature. In fact, the formalism works in the same way as with algebraic strata over k. What we need are enough Whitney stratifications algebraic over k. That this is possible can be deduced from [Tei, Théorème 1.2 p. 455] (characterization of Whitney stratifications) and [Tei, Proposition 2.1] (Whitney stratifications are generic).

We will also need some basic facts about sheaf cohomology. If $j : U \hookrightarrow X$ is a Zariski open subset with closed complement $i : W \hookrightarrow X$ and F a sheaf of abelian groups on X, then there is an exact sequence of sheaves

$$0 \to j_! j^* F \to F \to i_* i^* F \to 0.$$

In addition, for F the constant sheaf $\mathbb{Z}$, one has $H^q(X, j_! j^* F) = H^q(X, W; \mathbb{Z})$ and $H^q(X, i_* i^* F) = H^q(W, \mathbb{Z})$, see section 2.1.

Theorem 2.5.7 (Basic Lemma II)**.** *Let X be an affine variety over k of dimension n and F be a weakly constructible sheaf on X. Then there exists a Zariski open subset $j : U \hookrightarrow X$ such the following three properties hold:*

1. $\dim(X \setminus U) < n$.
2. $H^q(X, F') = 0$ *for* $q \neq n$, *where* $F' := j_! j^* F \subset F$.
3. *If F is constructible, then $H^n(X, F')$ is finitely generated.*
4. *If the stalks of F are torsion free, then $H^n(X, F')$ is torsion free.*

Version II of the Lemma implies version I. Let $V = X \setminus W$ with open immersion $h : V \hookrightarrow X$, and the sheaf $F = h_! h^* \mathbb{Z}$ on X. Version II for F gives an open subset $\ell : U \hookrightarrow X$ such that the sheaf $F' = \ell_! \ell^* F$ has non-vanishing cohomology only in degree n. Let $W' = X \setminus U$. Since F was zero on W, we have that F' is zero on $Z = W \cup W'$ and it is the constant sheaf on $X \setminus Z$, i.e., $F' = j_! j^* F$ for $j : X \setminus Z \hookrightarrow X$. In particular, F' computes the relative cohomology $H^q(X, Z; \mathbb{Z})$ and it vanishes for $q \neq n$. Freeness follows from property (3). □

We will give two proofs of the Basic Lemma II below.

2.5.1 Direct proof of Basic Lemma I

We start by giving a direct proof of Basic Lemma I. It was given by Nori in the unpublished notes [N1]. Close inspection shows that it is actually a variant of Beilinson's argument in this very special case.

Lemma 2.5.8. *Let X be affine, $W \subset X$ closed. Then there exist*

1. $\tilde{X}$ *smooth projective;*

2. *$D_0, D_\infty \subset \tilde{X}$ closed such that $D_0 \cup D_\infty$ is a simple normal crossings divisor and $\tilde{X} \setminus D_0$ is affine;*
3. *$\pi : \tilde{X} \setminus D_\infty \to X$ proper surjective, an isomorphism outside of D_0 such that $Y := \pi(D_0 \setminus D_\infty \cap D_0)$ contains W and $\pi^{-1}(Y) = D_0 \setminus D_\infty \cap D_0$.*

Proof. We may assume without loss of generality that $X \setminus W$ is smooth. Let $\bar{X}$ be a projective closure of X and $\bar{W}$ the closure of W in $\bar{X}$. By resolution of singularities, there is $\tilde{X} \to X$ proper surjective and an isomorphism above $X \setminus W$ such that $\tilde{X}$ is smooth. Let $D_\infty \subset \tilde{X}$ be the complement of the preimage of X. Let $\tilde{W}$ be the closure of the preimage of X. By resolution of singularities, we can also assume that $\tilde{W} \cup D_\infty$ is a divisor with normal crossings.

Note that $\bar{X}$ and hence also $\tilde{X}$ are projective. We choose a generic hyperplane $\tilde{H}$ such that $\bar{W} \cup D_\infty \cup \tilde{H}$ is a divisor with normal crossings. This is possible as the ground field k is infinite and the condition is satisfied in a Zariski open subset of the space of hyperplane sections. We put $D_0 = \tilde{H} \cup \bar{W}$. As $\tilde{H}$ is a hyperplane section, it is an ample divisor. Therefore, $D_0 = \tilde{H} \cup \bar{W}$ is the support of the ample divisor $\tilde{H} + m\bar{W}$ for m sufficiently large [Ha2, Exer. II 7.5(b)]. Hence $\tilde{X} \setminus D_0$ is affine, as the complement of an ample divisor in a projective variety is affine. □

Proof of Basic Lemma I. We use the varieties constructed in the last lemma. We claim that Y has the right properties if the coefficients form an arbitrary field K. We have $Y \supset W$. From Artin vanishing, see Corollary 2.3.8, we immediately have vanishing of $H^i(X, Y; K)$ for $i > n$.

By excision (see Proposition 2.1.7)

$$H^i(X, Y; K) = H^i(\tilde{X} \setminus D_\infty, D_0 \setminus D_0 \cap D_\infty; K).$$

By Poincaré duality for pairs (see Theorem 2.4.5), it is dual to

$$H^{2n-i}(\tilde{X} \setminus D_0, D_\infty \setminus D_0 \cap D_\infty; K).$$

The variety $\tilde{X} \setminus D_0$ is affine. Hence by Artin vanishing, the cohomology group vanishes for all $i \neq n$ and any coefficient field K.

It remains to treat the case of integral coefficients. Let i be the smallest index such that $H^i(X, Y; \mathbb{Z})$ is non-zero. By Artin vanishing for $\mathbb{Z}$-coefficients 2.3.5, we have $i \leq n$.

If $i < n$, then the group $H^i(X, Y; \mathbb{Z})$ has to be torsion because the cohomology vanishes with $\mathbb{Q}$-coefficients. By the universal coefficient theorem [We, thm. 3.6.4]

$$H^{i-1}(X, Y; \mathbb{F}_p) = \mathrm{Tor}_1^{\mathbb{Z}}(H^i(X, Y; \mathbb{Z}), \mathbb{F}_p) ,$$

which implies that $H^{i-1}(X, Y; \mathbb{F}_p)$ is non-trivial for the occuring torsion primes. This is a contradiction to the vanishing for $K = \mathbb{F}_p$. Hence $i = n$. The same argument shows that $H^n(X, Y; \mathbb{Z})$ is torsion-free. □

2.5.2 Nori's proof of Basic Lemma II

We now present the proof of the stronger Basic Lemma II published by Nori in [N2].

We start with a couple of lemmas on weakly constructible sheaves.

Lemma 2.5.9. *Let $0 \to F_1 \to F_2 \to F_3 \to 0$ be a short exact sequence of sheaves on X with F_1, F_3 (weakly) constructible. Then F_2 is (weakly) constructible.*

Proof. By assumption, there are stratifications of X such that F_1 and F_3 become locally constant, respectively. We take a common refinement. We replace X by one of the strata and are now in the situation that F_1 and F_3 are locally constant. Then F_3 is also locally constant. Indeed, by passing to a suitable open cover (in the analytic topology), F_1 and F_3 become even constant. If $V \subset U$ is an inclusion of open connected subsets, then the restrictions $F_1(U) \to F_1(V)$ and $F_3(U) \to F_3(U)$ are isomorphisms. This implies the same statement for F_2, because $H^1(U, F_1) = H^1(V, F_1) = 0$, as constant sheaves do not have higher cohomology. □

Lemma 2.5.10. *The notion of (weak) constructibility is stable under $j_!$ for j an open immersion and π_* for π finite.*

Proof. The assertion of $j_!$ is obvious, same as for i_* for closed immersions.

Now assume $\pi : X \to Y$ is finite and in addition étale. Let F be (weakly) constructible on X. Let $X_0, \dots, X_n \subset X$ be the stratification such that $F|_{X_i}$ is locally constant. Let Y_i be the image of X_i. These are locally closed subvarieties of Y because π is closed and open. We refine them into a stratification of Y. As π is finite étale, it is locally in the analytic topology of the form $I \times B$ with I finite and $B \subset Y(\mathbb{C})$ an open in the analytic topology. Obviously $\pi_* F|_B$ is locally constant on the strata we have defined.

Now let π be finite. As we have already discussed closed immersion, it suffices to assume π surjective. There is an open dense subscheme $U \subset Y$ such π is étale above U. Let $U' = \pi^{-1}(U)$, $Z = Y \setminus U$ and $Z' = X \setminus U'$. We consider the exact sequence on X

$$0 \to j_{U'!} j_{U'}^* F \to F \to i_{Z'*} i_{Z'}^* F \to 0.$$

As π is finite, the functor π_* is exact and hence

$$0 \to \pi_* j_{U'!} j_{U'}^* F \to \pi_* F \to \pi_* i_{Z'*} i_{Z'}^* F \to 0.$$

By Lemma 2.5.9, it suffices to consider the outer terms. We have

$$\pi_* j_{U'!} j_{U'}^* F = j_{U*} \pi|_{U'*} j_{U'}^* F,$$

and this is (weakly) constructible by the étale case and the assertion on open immersions. We also have

$$\pi_* i_{Z'*} i_{Z'}^* F = i_{Z*} \pi|_{Z'*} i_{Z'}^* F,$$

and this is (weakly) constructible by noetherian induction and the case of closed immersions. □

Nori's proof of Basic Lemma II. The argument will show a more precise version of property (3): There exists a finite subset $E \subset U(\mathbb{C})$ such that $H^{\dim(X)}(X, F')$ is isomorphic to a direct sum $\oplus_x F_x$ of stalks of F at points of E.

Let $n := \dim(X)$. In the first step, we reduce to $X = \mathbb{A}^n$. We use Noether normalization to obtain a finite morphism $\pi : X \to \mathbb{A}^n$. By Lemma 2.5.10, the sheaf $\pi_* F$ is (weakly) constructible.

Let then $v : V \hookrightarrow \mathbb{A}^n$ be a Zariski open set with the property that $F' := v_! v^* \pi_* F$ satisfies the Basic Lemma II on $\mathbb{A}^n$. Let $U := \pi^{-1}(V) \overset{j}{\hookrightarrow} X$ be the preimage in X. One has an equality of sheaves:

$$\pi_* j_! j^* F = v_! v^* \pi_* F.$$

Therefore, $H^q(X, j_! j^* F) = H^q(\mathbb{A}^n, v_! v^* \pi_* F)$ and the latter vanishes for $q \leq n$. The formula for the n-cohomology on $\mathbb{A}^n$ implies the one on X.

So let us now assume that F is weakly constructible on $X = \mathbb{A}^n$. We argue by induction on n and all F. The case $n = 0$ is trivial.

By replacing F by $j_! j^* F$ for an appropriate open $j : U \to \mathbb{A}^n$, we may assume that F is locally constant on U and that $\mathbb{A}^n \setminus U = V(f)$. By Noether normalization or its proof, there is a surjective projection map $\pi : \mathbb{A}^n \to \mathbb{A}^{n-1}$ such that $\pi|_{V(f)} : V(f) \to \mathbb{A}^{n-1}$ is surjective and finite.

We will see in Lemma 2.5.11 that $R^q \pi_* F = 0$ for $q \neq 1$ and $R^1 \pi_* F$ is weakly constructible. The Leray spectral sequence now gives that

$$H^q(\mathbb{A}^n, F) = H^{q-1}(\mathbb{A}^{n-1}, R^1 \pi_* F).$$

In the induction procedure, we apply the Basic Lemma II to $R^1 \pi_* F$ on $\mathbb{A}^{n-1}$. By induction, there exists a Zariski open $h : V \hookrightarrow \mathbb{A}^{n-1}$ such that $h_! h^* R^1 \pi_* F$ has cohomology only in degree $n-1$. Let $U := \pi^{-1}(V)$ and $j : U \hookrightarrow \mathbb{A}^n$ be the inclusion. Then $j_! j^* F$ has cohomology only in degree n. The explicit description of cohomology in degree n follows from the description of the stalks of $R^1 \pi_* F$ in the proof of Lemma 2.5.11. □

Lemma 2.5.11. *Let π be as in the above proof. Then $R^q \pi_* F = 0$ for $q \neq 1$ and $R^1 \pi_* F$ is weakly constructible.*

Proof. This is a standard fact, but Nori gives a direct proof.

The stalk of $R^q \pi_* F$ at $y \in \mathbb{A}^{n-1}$ is given by $H^q(\{y\} \times \mathbb{A}^1, F_{\{y\} \times \mathbb{A}^1})$ by the variation of proper base change in Theorem 2.5.12 below.

Let, more generally, G be a sheaf on $\mathbb{A}^1$ which is locally constant outside a finite, non-empty set S. Let T be a tree in $\mathbb{A}^1(\mathbb{C})$ with vertex set S. Then the restriction map to the tree defines a retraction isomorphism $H^q(\mathbb{A}^1, G) \cong H^q(T, G_T)$

for all $q \geq 0$. Using Čech cohomology, we can compute that $H^q(T, G_T)$: For each vertex $v \in S$, let U_s be the star of half edges of length more than half the length of all outgoing edges at the vertex s. Then U_a and U_b only intersect if the vertices a and b have a common edge $e = e(a,b)$. The intersection $U_a \cap U_b$ is contractible and contains the center $t(e)$ of the edge e. There are no triple intersections. Hence $H^q(T, G_T) = 0$ for $q \geq 2$. Since G is zero on S, U_s is simply connected, and G is locally constant, $G(U_s) = 0$ for all s. Therefore also $H^0(T, G_T) = 0$ and $H^1(T, G_T)$ is isomorphic to $\bigoplus_e G_{t(e)}$.

This implies already that $R^q\pi_*F = 0$ for $q \neq 1$.

To show that $R^1\pi_*F$ is weakly constructible, means to show that it is locally constant on some stratification. We see that the stalks $(R^1\pi_*F)_y$ depend only on the set of points in $\{y\} \times \mathbb{A}^1 = \pi^{-1}(y)$ where $F_{\{y\}\times\mathbb{A}^1}$ vanishes. But the sets of points where the vanishing set has the same degree (cardinality) defines a suitable stratification. Note that the stratification only depends on the branching behaviour of $V(f) \to \mathbb{A}^{n-1}$, hence the stratification is algebraic and defined over k. □

Theorem 2.5.12 (Variation of Proper Base Change)**.** *Let $\pi : X \to Y$ be a continuous map between locally compact, locally contractible topological spaces which is a fiber bundle and let G be a sheaf on X. Assume $W \subset X$ is closed and such that G is locally constant on $X \setminus W$ and π restricted to W is proper. Then $(R^q\pi_*G)_y \cong H^q(\pi^{-1}(y), G_{\pi^{-1}(y)})$ for all q and all $y \in Y$.*

Proof. The statement is local on Y, so we may assume that $X = T \times Y$ is a product with π the projection. Since Y is locally compact and locally contractible, we may assume that Y is compact by passing to a compact neighbourhood of y. As $W \to Y$ is proper, this implies that W is compact. By enlarging W, we may assume that $W = K \times Y$ is a product of compact sets for some compact subset $K \subset X$. Since Y is locally contractible, we replace Y be a contractible neighbourhood. (We may loose compactness, but this does not matter anymore.) Let $i : K \times Y \to X$ be the inclusion and and $j : (T \setminus K) \times Y \to X$ the complement.

Look at the exact sequence

$$0 \to j_! G_{(T\setminus K)\times Y} \to G \to i_* G_{K\times Y} \to 0.$$

The result holds for $G_{K\times Y}$ by the usual proper base change.

Since Y is contractible, we may assume that $G_{(T\setminus K)\times Y}$ is the pull-back of constant sheaf on $T \setminus K$. Now the result for $j_! G_{(T\setminus K)\times Y}$ follows from the Künneth formula. □

2.5.3 Beilinson's proof of Basic Lemma II

We follow Beilinson [Be1] Proof 3.3.1. His proof is even more general, as he does not assume X to be affine. Note that Beilinson's proof is in the setting of étale sheaves, independent of the characteristic of the ground field. We have translated it to weakly constructible sheaves. The argument is intrinsically about perverse sheaves, even though we have downplayed their use as far as possible. For a complete development of the theory of perverse sheaves in the (weakly) constructible setting see Schürmann's monograph [Schü].

Let X be affine reduced of dimension n over a field $k \subset \mathbb{C}$. Let F be a (weakly) constructible sheaf on X. We choose a projective compactification $\kappa : X \hookrightarrow \bar{X}$ such that κ is an affine morphism. Let W be a divisor on X such that F is a locally constant sheaf on $h : X \setminus W \hookrightarrow X$ and $X \setminus W$ is smooth. Then define $M := h_! h^* F$.

Let $\bar{H} \subset \bar{X}$ be a generic hyperplane. (We will see in the proof of Lemma 2.5.13 below what the conditions on $\bar{H}$ are.) Let $H = X \cap \bar{H}$ be the hyperplane in X.

We denote by $V = \bar{X} \setminus \bar{H}$ the complement and by $\ell : V \hookrightarrow \bar{X}$ the open inclusion. Furthermore, let $\kappa_V : V \cap X \hookrightarrow V$ and $\ell_X : V \cap X \hookrightarrow X$ be the open inclusion maps, and $i : \bar{H} \hookrightarrow \bar{X}$ and $i_X : H \hookrightarrow X$ the closed immersions. We set $U := X \setminus (W \cup H)$ and consider the open inclusion $j : U \hookrightarrow X$ with complement $Z = W \cup (H \cap X)$. Let $M_{V \cap X}$ be the restriction of M to $V \cap X$. Summarizing, we have a commutative diagram

$$\begin{array}{ccccc}
 & & U & & \\
 & & \big\downarrow{\scriptstyle j} & & \\
V \cap X & \xrightarrow{\ell_X} & X & \xleftarrow{i_X} & H \\
{\scriptstyle \kappa_V}\big\downarrow & & \big\downarrow{\scriptstyle \kappa} & & \big\downarrow{\scriptstyle \tilde{\kappa}} \\
V & \xrightarrow{\ell} & \bar{X} & \xleftarrow{i} & \bar{H}.
\end{array}$$

Lemma 2.5.13. *For generic $\bar{H}$ in the above set-up, there is an isomorphism*

$$\ell_! \ell^* R\kappa_* M \xrightarrow{\cong} R\kappa_* \ell_{X*} M_{V \cap X}$$

extending naturally $\mathrm{id} : M_{V \cap X} \to M_{V \cap X}$.

Proof. We consider the map of distinguished triangles

$$\begin{array}{ccccc}
\ell_! \ell^* R\kappa_* M & \longrightarrow & R\kappa_* M & \longrightarrow & i_* i^* R\kappa_* M \\
\big\downarrow & & {\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow \\
R\kappa_* \ell_{X!} M_{V \cap X} & \longrightarrow & R\kappa_* M & \longrightarrow & i_* R\tilde{\kappa}_* i_X^* M
\end{array}$$

(the existence of the arrows follows from standard adjunctions together with proper base change in the simple form $\kappa^* \ell_! = \ell_{X!} \kappa_V^*$ and $\kappa^* i_* = i_{X*} \bar{\kappa}^*$, respectively).

Hence it is sufficient to prove that

$$i^* R\kappa_* M \xrightarrow{\cong} R\tilde{\kappa}_* i_X^* M. \qquad (2.1)$$

To prove this, we make a base change to the universal hyperplane section. In detail: Let $\mathbb{P}$ be the space of hyperplanes in $\bar{X}$. Let

$$\overline{\mathcal{H}}_{\mathbb{P}} \to \mathbb{P}$$

be the universal family. It comes with a natural map

$$i_{\mathbb{P}} : \overline{\mathcal{H}} \to \bar{X}.$$

Let $\mathcal{H}$ be the preimage of X. By [Gro2, pg. 9] and [Jo, Thm. 6.10] there is a dense Zariski open subset $T \subset \mathbb{P}$ such that the induced map

$$i_T : \overline{\mathcal{H}}_T \hookrightarrow \bar{X} \times T \longrightarrow \bar{X}$$

is smooth.

We apply smooth base change in the square

$$\begin{array}{ccc} \mathcal{H}_T & \xrightarrow{i_{X,T}} & X \\ {\scriptstyle \bar{\kappa}_T}\downarrow & & \downarrow{\scriptstyle \kappa} \\ \overline{\mathcal{H}}_T & \xrightarrow{i_T} & \bar{X} \end{array}$$

and obtain a quasi-isomorphism

$$i_T^* R\kappa_* M \xrightarrow{\cong} R\tilde{\kappa}_{T*} i_{X,T}^* M$$

of complexes of sheaves on $\overline{\mathcal{H}}_T$.

We specialize to some $t \in T(k)$ and get a hyperplane $t : \bar{H} \subset \overline{\mathcal{H}}_T$ to which we restrict. The left hand side turns into $i^* R\kappa_* M$.

The right hand side turns into

$$t^* R\bar{\kappa}_{T*} i_{X,T}^* M = R\bar{\kappa}_* t_X^* i_{X,T}^* M = R\bar{\kappa} i_X^* M$$

by applying the generic base change theorem 2.5.16 to $\bar{\kappa}_T$ over the base T and $\mathcal{G} = i_{X,T}^* M$. This requires to shrink T further.

Putting these equations together, we have verified equation 2.1. $\square$

Proof of Basic Lemma II. We keep the notation fixed in this section. By Artin vanishing for constructible sheaves (see Theorem 2.5.14), the group $H^i(X, j_! j^* F)$ vanishes for $i > n$. It remains to show that $H^i(X, j_! j^* F)$ vanishes for $i < n$. We obviously have $j_! j^* F = \ell_{X!} M_{V \cap X}$. Therefore,

$$\begin{aligned} H^i(X, j_! j^* F) &= H^i(X, \ell_{X!} M_{V \cap X}) \\ &= H^i(\bar{X}, R\kappa_* \ell_{X!} M_{V \cap X}) \\ &= H^i(\bar{X}, \ell_! \ell^* R\kappa_* M) \quad \text{by 2.5.13} \\ &= H^i_c(V, (R\kappa_* M)_V). \end{aligned}$$

The last group vanishes for $i < n$ by Artin's vanishing theorem 2.5.14 for compact supports once we have checked that $R\kappa_* M_V[n]$ is perverse. Recall that $M = h_! h^* F|_{X\setminus W}$ with $F|_{X\setminus W}$ locally constant sheaf on a smooth variety. Hence $F|_{X\setminus W}[n]$ is perverse. Both h and κ are affine, hence the same is true for $R\kappa_* h_! F|_{X\setminus W}$ by Theorem 2.5.14 3.

If, in addition, F is constructible, then by the same theorem, $R\kappa_* h_! F|_{X\setminus W}$ is perverse for the second t-structure mentionend in Remark 2.5.15. Hence our cohomology with compact support is also finitely generated.

If the stalks of F are torsion free, then $R\kappa_* h_! F|_{X\setminus W}$ is perverse for the third t-structure mentionend in Remark 2.5.15. Hence our cohomology with compact support is also torsion free. □

We now formulate the version of Artin vanishing used in the above proof. If X is a topological space, and $j : X \hookrightarrow \bar{X}$ an arbitrary compactification, then cohomology with supports with coefficients in a weakly constructible sheaf $\mathcal{G}$ is defined by

$$H^i_c(X, \mathcal{G}) := H^i(\bar{X}, j_! \mathcal{G}).$$

It follows from proper base change that this is independent of the choice of compactification.

Theorem 2.5.14 (Artin vanishing for constructible sheaves)**.** *Let X be affine of dimension n.*

1. *Let $\mathcal{G}$ be weakly constructible on X. Then $H^q(X, \mathcal{G}) = 0$ for $q > n$;*

2. *Let $\mathcal{F}_\bullet$ be a perverse sheaf on X for the middle perversity. Then $H^q_c(X, \mathcal{F}_\bullet) = 0$ for $q < 0$. More precisely, the complex $R_c(X, \mathcal{F}_\bullet)$ computing cohomology with compact support is in $D^{\geq 0}$.*

3. *Let $g : U \to X$ be an open immersion and $\mathcal{F}_\bullet$ a perverse sheaf on U. Then both $g_! \mathcal{F}_\bullet$ and $Rg_* \mathcal{F}_\bullet$ are perverse on X.*

Proof. The first two statements are [Schü, Corollary 6.0.4, p. 391]. Note that a weakly constructible sheaf lies in ${}^m D^{\leq n}(X)$ in the notation of loc.cit.

The last statement combines the vanishing results for affine morphisms [Schü, Theorem 6.0.4, p. 409] with the standard vanishing for all compactifiable morphisms [Schü, Corollary 6.0.5, p. 397] for a morphism of relative dimension 0. □

Remark 2.5.15. For the notion of a t-structure on a triangulated category and perverse sheaves, see the original reference [BBD]. Actually, as explained in [Schü, Example 6.0.2, p. 377], there are different possible choices for the triangulated category and inital t-structure $D^{\geq 0}$. In each case there is corresponding middle perverse t-structure by [Schü, Definition 6.0.3, p. 379]. The theorem applies in all of them.

1. The category of complexes of sheaves with weakly constructible cohomology. It is denoted $D(\mathbb{Z})$ in [Schü, Chapter 6]. The prototype of a perverse sheaf is a complex of the form $F[n]$ with F a local system on a smooth variety of dimension n.

2. The category of bounded complexes of sheaves with constructible cohomology. It is denoted $D(\mathbb{Z})_{\mathrm{perf}}$ in loc.cit. The prototype of a perverse sheaf is $F[n]$, with F a local system with finitely generated stalks.

3. In both cases, we can also use the t-structure based on ${}^+D^{\leq 0}$, the subcategory of complexes in positive degrees with H^0 torsion free. The prototype of a perverse sheaf in this case is $F[n]$, with F a local system with torsion free stalks.

Theorem 2.5.16 (Generic base change)**.** *Let S be of finite type over k, $f : X \to Y$ a morphism of S-varieties. Let $\mathcal{F}$ be a (weakly) constructible sheaf on X. Then there is a dense open subset $U \subset S$ such that:*

1. *over U, the sheaves $R^if_*\mathcal{F}$ are (weakly) constructible and almost all vanish;*

2. *the formation of $R^if_*\mathcal{F}$ is compatible with any base change $S' \to U \subset S$.*

This is the analogue of [SGA 4 1/2, Théorème 1.9 in sect. Thm. finitude], which is for constructible étale sheaves in the étale setting.

Proof. The case $S = Y$ was treated by Arapura, see [Ara, Theorem 3.1.10]. We explain the reduction to this case, using the same arguments as in the étale case.

All schemes can be assumed reduced.

Using Nagata, we can factor f as a composition of an open immersion and a proper map. The assertion holds for the latter by the proper base change theorem, hence it suffices to consider open immersions.

As the question is local on Y, we may assume that it is affine over S. We can then cover X by affines. Using the hypercohomology spectral sequence for the covering, we may reduce to the case X affine. In this case (X and Y affine, f an open immersion) we argue by induction on the dimension of the generic fibre of $X \to S$.

If $n = 0$, then, at least after shrinking S, we are in the situation where f is the inclusion of a connected component and the assertion is trivial.

We now assume the case $n-1$. We embed Y into $\mathbb{A}^m_S$ and consider the coordinate projections $p_i : Y \to \mathbb{A}^1_S$. We apply the inductive hypothesis to the map f over $\mathbb{A}^1_S$. Hence there is an open dense $U_i \subset \mathbb{A}^1_S$ such that the conclusion is valid over $p^{-1}U_i$. Hence the conclusion is valid over their union, i.e., outside a closed subvariety $Y_1 \subset Y$ finite over S. By shrinking S, we may assume that it is finite étale.

We fix the notation in the resulting diagram as follows:

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xleftarrow{i} & Y_1 \\ & {\scriptstyle a}\searrow & \downarrow{\scriptstyle b} & \swarrow{\scriptstyle b_1} & \\ & & S & & \end{array}$$

Let j be the open complement of i. We have checked that $j^*Rf_*\mathcal{G}$ is (weakly) constructible and compatible with any base change. We apply Rb_* to the triangle defined by the sequence

$$j_!j^*Rf_*\mathcal{G} \to Rf_*\mathcal{G} \to i_*i^*Rf_*\mathcal{G}$$

and obtain

$$Rb_*j_!j^*Rf_*\mathcal{G} \to Ra_*\mathcal{G} \to b_{1*}i^*Rf_*\mathcal{G}.$$

The first two terms are (after shrinking of S) (weakly) constructible by the previous considerations and the case $S = Y$. We also obtain that they are compatible with any base change. Hence the same is true for the third term. As b_1 is finite étale this also implies that $i^*Rf_*\mathcal{G}$ is (weakly) constructible and compatible with base change. (Indeed, this follows because a direct sum of sheaves is constant if and only if every summand is constant.) The same is true for $j_!j^*Rf_*\mathcal{G}$ by the previous considerations and base change for $j_!$. Hence the conclusion also holds for the middle term of the first triangle and we are done.

□

2.6 Triangulation of Algebraic Varieties

If X is a variety defined over $\mathbb{Q}$, we may ask whether any singular homology class $\gamma \in H_\bullet^{\mathrm{sing}}(X^{\mathrm{an}};\mathbb{Q})$ can be represented by an object described by polynomials. This is indeed the case: for a precise statement we need several definitions. The result will be formulated in Proposition 2.6.8.

This section follows closely the Diploma thesis of Benjamin Friedrich, see [Fr]. The results are due to him.

We work over $k = \widetilde{\mathbb{Q}}$, i.e., the integral closure of $\mathbb{Q}$ in $\mathbb{R}$. Note that $\widetilde{\mathbb{Q}}$ is a field.

In this section, we use X to denote a variety over $\widetilde{\mathbb{Q}}$, and X^{an} for the associated analytic space over $\mathbb{C}$ (cf. Subsection 1.2).

2.6.1 Semi-algebraic Sets

Definition 2.6.1 ([Hi2, Def. 1.1., p.166])**.** A subset of $\mathbb{R}^n$ is said to be $\widetilde{\mathbb{Q}}$-*semi-algebraic*, if it is of the form

$$\{\underline{x} \in \mathbb{R}^n | f(\underline{x}) \geq 0\}$$

for some polynomial $f \in \widetilde{\mathbb{Q}}[x_1, \ldots, x_n]$, or can be obtained from sets of this form in a finite number of steps, where each step consists of one of the following basic operations:

1. complementary set,
2. finite intersection,
3. finite union.

We need also a definition for maps:

Definition 2.6.2 ($\widetilde{\mathbb{Q}}$-semi-algebraic map [Hi2, p. 168])**.** A continuous map f between $\widetilde{\mathbb{Q}}$-semi-algebraic sets $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ is said to be $\widetilde{\mathbb{Q}}$-*semi-algebraic*, if its graph

$$\Gamma_f := \big\{ \big(a, f(a)\big) \,|\, a \in A \big\} \subseteq A \times B \subseteq \mathbb{R}^{n+m}$$

is $\widetilde{\mathbb{Q}}$-semi-algebraic.

Example 2.6.3. Any polynomial map

$$\begin{aligned} f : A &\longrightarrow B \\ (a_1, \ldots, a_n) &\mapsto (f_1(a_1, \ldots, a_n), \ldots, f_m(a_1, \ldots, a_n)) \end{aligned}$$

between $\widetilde{\mathbb{Q}}$-semi-algebraic sets $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ with $f_i \in \widetilde{\mathbb{Q}}[x_1, \ldots, x_n]$ for $i = 1, \ldots, m$ is $\widetilde{\mathbb{Q}}$-semi-algebraic, since it is continuous and its graph $\Gamma_f \subseteq \mathbb{R}^{n+m}$ is cut out from $A \times B$ by the polynomials

$$y_i - f_i(x_1, \ldots, x_n) \in \widetilde{\mathbb{Q}}[x_1, \ldots, x_n, y_1, \ldots, y_m] \quad \text{for} \quad i = 1, \ldots, m. \tag{2.2}$$

We can even allow f to be a rational map with rational component functions

$$f_i \in \widetilde{\mathbb{Q}}(x_1, \ldots, x_n), \quad i = 1, \ldots, m$$

as long as none of the denominators of the f_i vanish at a point of A. The argument remains the same except that the expression (2.2) has to be multiplied by the denominator of f_i.

Fact 2.6.4 ([Hi2, Prop. II, p. 167], [Sb, Thm. 3, p. 370])**.**
By a result of Seidenberg-Tarski, the image (respectively preimage) of a $\widetilde{\mathbb{Q}}$-semi-algebraic set under a $\widetilde{\mathbb{Q}}$-semi-algebraic map is again $\widetilde{\mathbb{Q}}$-semi-algebraic.

As the name suggests, any algebraic set should be in particular $\widetilde{\mathbb{Q}}$-semi-algebraic.

Lemma 2.6.5. *Let X be a quasi-projective algebraic variety defined over $\widetilde{\mathbb{Q}}$. Then we can regard the complex analytic space X^{an} associated to the base change $X_{\mathbb{C}} = X \times_{\widetilde{\mathbb{Q}}} \mathbb{C}$ as a bounded $\widetilde{\mathbb{Q}}$-semi-algebraic subset*

$$X^{\text{an}} \subseteq \mathbb{R}^N \tag{2.3}$$

for some N. Moreover, if $f : X \to Y$ is a morphism of varieties defined over $\widetilde{\mathbb{Q}}$, we can consider $f^{\text{an}} : X^{\text{an}} \to Y^{\text{an}}$ as a $\widetilde{\mathbb{Q}}$-semi-algebraic map.

Remark 2.6.6. We will mostly need the case when X is even *affine*. Then $X \subset \mathbb{C}^n$ is defined by polynomial equations with coefficients in $\widetilde{\mathbb{Q}}$. We identify $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and rewrite the equations for the real and imaginary part. Hence X is obviously $\widetilde{\mathbb{Q}}$-semialgebraic. In the lemma, we will show in addition that X can be embedded as a *bounded* $\widetilde{\mathbb{Q}}$-semialgebraic set.

Proof of Lemma 2.6.5. First step $X = \mathbb{P}^n_{\widetilde{\mathbb{Q}}}$: Consider

- $\mathbb{P}^n_{\mathbb{C}} = (\mathbb{P}^n_{\widetilde{\mathbb{Q}}} \times_{\widetilde{\mathbb{Q}}} \mathbb{C})^{\mathrm{an}}$ with homogenous coordinates $x_0, \ldots, x_n$, which we split as $x_m = a_m + ib_m$ with $a_m, b_m \in \mathbb{R}$ in real and imaginary part, and
- $\mathbb{R}^N$, $N = 2(n+1)^2$, with coordinates $\{y_{kl}, z_{kl}\}_{k,l=0,\ldots,n}$.

We define an explicit map

$$\psi : \underset{[x_0:\ldots:x_n]}{\mathbb{P}^n_{\mathbb{C}}} \longrightarrow \underset{(y_{00},z_{00},\ldots,y_{nn},z_{nn})}{\mathbb{R}^N}$$

$$[x_0 : \ldots : x_n] \mapsto \Bigg(\ldots, \underbrace{\frac{\operatorname{Re} x_k\overline{x}_l}{\sum_{m=0}^n |x_m|^2}}_{y_{kl}}, \underbrace{\frac{\operatorname{Im} x_k\overline{x}_l}{\sum_{m=0}^n |x_m|^2}}_{z_{kl}}, \ldots\Bigg)$$

$$[a_0 + ib_0 : \ldots : a_n + ib_n] \mapsto \Bigg(\ldots, \underbrace{\frac{a_k a_l + b_k b_l}{\sum_{m=0}^n a_m^2 + b_m^2}}_{y_{kl}}, \underbrace{\frac{b_k a_l - a_k b_l}{\sum_{m=0}^n a_m^2 + b_m^2}}_{z_{kl}}, \ldots\Bigg).$$

We can understand this map as a section of a natural fibre bundle on $\mathbb{P}^n_{\mathbb{C}}$. Its total space is given by the set E of hermetian $(n+1) \times (n+1)$-matrices or rank 1. The map

$$\phi : E \to \mathbb{P}^n_{\mathbb{C}}$$

maps a linear map M to its image in $\mathbb{C}^{n+1}$. We get a section of ϕ by mapping a 1-dimensional subspace L of $\mathbb{C}^{n+1}$ to the matrix of the orthogonal projection from $\mathbb{C}^{n+1}$ to L with respect to the standard hermetian product on $\mathbb{C}^{n+1}$. We can describe this section in coordinates. Let $(x_0, \ldots, x_n) \in \mathbb{C}^{n+1}$ be a vector of length 1. Then an elementary computation shows that $M = (x_i\bar{x}_j)_{i,j}$ is the hermetian projector to the line $L = \mathbb{C}(x_0, \ldots, x_n)$. Writing the real and imaginary part of the matrix M separately gives us precisely the formula for ψ. In particular, ψ is injective.

Therefore, we can consider $\mathbb{P}^n_{\mathbb{C}}$ via ψ as a subset of $\mathbb{R}^N$. It is bounded since it is contained in the unit sphere $S^{N-1} \subset \mathbb{R}^N$. We claim that $\psi(\mathbb{P}^n_{\mathbb{C}})$ is also $\widetilde{\mathbb{Q}}$-semi-algebraic. The composition of the projection

$$\begin{aligned} \pi : \mathbb{R}^{2(n+1)} \setminus \{(0, \ldots, 0)\} &\longrightarrow \mathbb{P}^n_{\mathbb{C}} \\ (a_0, b_0, \ldots, a_n, b_n) &\mapsto [a_0 + ib_0 : \ldots : a_n + ib_n] \end{aligned}$$

with the map ψ is a polynomial map, hence $\widetilde{\mathbb{Q}}$-semi-algebraic by Example 2.6.3. Thus

$$\operatorname{Im}\psi\circ\pi = \operatorname{Im}\psi \subseteq \mathbb{R}^N$$

is $\widetilde{\mathbb{Q}}$-semi-algebraic by Fact 2.6.4.

Second step (zero set of a polynomial): We use the notation

$$\begin{aligned} V(g) &:= \{\underline{x}\in\mathbb{P}^n_{\mathbb{C}} \mid g(\underline{x})=0\} \quad \text{for} \quad g\in\mathbb{C}[x_0,\ldots,x_n] \text{ homogenous,} \quad \text{and} \\ W(h) &:= \{\underline{t}\in\mathbb{R}^N \mid h(\underline{t})=0\} \quad \text{for} \quad h\in\mathbb{C}[y_{00},\ldots,z_{nn}]. \end{aligned}$$

Let $X^{\mathrm{an}} = V(g)$ for some homogenous $g\in\widetilde{\mathbb{Q}}[x_0,\ldots,x_n]$. Then $\psi(X^{\mathrm{an}})\subseteq\mathbb{R}^N$ is a $\widetilde{\mathbb{Q}}$-semi-algebraic subset, as a little calculation shows. Setting for $k=0,\ldots,n$

$$\begin{aligned} g_k &:= \text{“}g(\underline{x}\,\overline{x}_k)\text{”} \\ &= g(x_0\overline{x}_k,\ldots,x_n\overline{x}_k) \\ &= g\big((a_0a_k+b_0b_k)+i(b_0a_k-a_0b_k),\ldots,(a_na_k+b_nb_k)+i(b_na_k-a_nb_k)\big), \end{aligned}$$

where $x_j = a_j + ib_j$ for $j=0,\ldots,n$, and

$$h_k := g(y_{0k}+iz_{0k},\ldots,y_{nk}+iz_{nk}),$$

we obtain

$$\begin{aligned} \psi(X^{\mathrm{an}}) &= \psi(V(g)) \\ &= \bigcap_{k=0}^{n} \psi(V(g_k)) \\ &= \bigcap_{k=0}^{n} \psi(\mathbb{P}^n_{\mathbb{C}})\cap W(h_k) \\ &= \bigcap_{k=0}^{n} \psi(\mathbb{P}^n_{\mathbb{C}})\cap W(\operatorname{Re} h_k)\cap W(\operatorname{Im} h_k). \end{aligned}$$

Final step: We can choose an embedding

$$X\subseteq\mathbb{P}^n_{\widetilde{\mathbb{Q}}},$$

thus getting

$$X^{\mathrm{an}}\subseteq\mathbb{P}^n_{\mathbb{C}}.$$

Since X is a locally closed subvariety of $\mathbb{P}^n_{\widetilde{\mathbb{Q}}}$, the space X^{an} can be expressed in terms of subvarieties of the form $V(g)$ with $g\in\widetilde{\mathbb{Q}}[x_0,\ldots,x_n]$, using only the following basic operations

1. complementary set,

2. finite intersection,
3. finite union.

Now $\widetilde{\mathbb{Q}}$-semi-algebraic sets are stable under these operations as well and the first assertion is proved.

Second assertion: The first part of the lemma provides us with $\widetilde{\mathbb{Q}}$-semi-algebraic inclusions

$$\psi : X^{\mathrm{an}} \subseteq \underset{\underline{x}=[x_0:\ldots:x_n]}{\mathbb{P}^n_{\mathbb{C}}} \subseteq \underset{(y_{00},z_{00},\ldots,y_{nn},z_{nn})}{\mathbb{R}^N},$$
$$\phi : Y^{\mathrm{an}} \subseteq \underset{\underline{u}=[u_0:\ldots:u_m]}{\mathbb{P}^m_{\mathbb{C}}} \subseteq \underset{(v_{00},w_{00},\ldots,v_{mm},w_{mm})}{\mathbb{R}^M},$$

and a choice of coordinates as indicated. We use the notation

$$V(g) := \{(\underline{x},\underline{u}) \in \mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^m_{\mathbb{C}} \mid g(\underline{x},\underline{u}) = 0\},$$
$$\text{for } g \in \mathbb{C}[x_0,\ldots,x_n,u_0,\ldots,u_m] \text{ homogenous in both } \underline{x} \text{ and } \underline{u}, \quad \text{and}$$
$$W(h) := \{\underline{t} \in \mathbb{R}^{N+M} \mid h(\underline{t}) = 0\}, \quad \text{for } h \in \mathbb{C}[y_{00},\ldots,z_{nn},v_{00},\ldots,w_{mm}].$$

Let $\{U_i\}$ be a finite open affine covering of X such that $f(U_i)$ satisfies

- $f(U_i)$ does not meet the hyperplane $\{u_j = 0\} \subset \mathbb{P}^m_{\widetilde{\mathbb{Q}}}$ for some j, and
- $f(U_i)$ is contained in an open affine subset V_i of Y.

This is always possible, since we can start with the open covering $Y \cap \{u_j \neq 0\}$ of Y, take a subordinated open affine covering $\{V_{i'}\}$, and then choose a finite open affine covering $\{U_i\}$ subordinated to $\{f^{-1}(V_{i'})\}$. Now each of the maps

$$f_i := f^{\mathrm{an}}{}_{|U_i} : U_i^{\mathrm{an}} \longrightarrow Y^{\mathrm{an}}$$

has image contained in V_i^{an} and does not meet the hyperplane $\{\underline{u} \in \mathbb{P}^m_{\mathbb{C}} \mid u_j = 0\}$ for an appropriate j

$$f_i : U_i^{\mathrm{an}} \longrightarrow V_i^{\mathrm{an}}.$$

Being associated to an algebraic map between affine varieties, this map is rational

$$f_i : \underline{x} \mapsto \left[\frac{g_0'(\underline{x})}{g_0''(\underline{x})} : \ldots : \frac{g_{j-1}'(\underline{x})}{g_{j-1}''(\underline{x})} : \underset{j}{1} : \frac{g_{j+1}'(\underline{x})}{g_{j+1}''(\underline{x})} : \ldots : \frac{g_m'(\underline{x})}{g_m''(\underline{x})}\right],$$

with $g_k', g_k'' \in \widetilde{\mathbb{Q}}[x_0,\ldots,x_n]$, $k = 0,\ldots,\widehat{j},\ldots,m$. Since the graph $\Gamma_{f^{\mathrm{an}}}$ of f^{an} is the finite union of the graphs Γ_{f_i} of the f_i, it is sufficient to prove that $(\psi \times \phi)(\Gamma_{f_i})$ is a $\widetilde{\mathbb{Q}}$-semi-algebraic subset of $\mathbb{R}^{N+M}$. Now

$$\Gamma_{f_i} = (U_i^{\mathrm{an}} \times V_i^{\mathrm{an}}) \cap \bigcap_{\substack{k=0\\k\neq j}}^{n} V\left(\frac{y_k}{y_j} - \frac{g_k'(\underline{x})}{g_k''(\underline{x})}\right) = (U_i^{\mathrm{an}} \times V_i^{\mathrm{an}}) \cap \bigcap_{\substack{k=0\\k\neq j}}^{n} V(y_k g_k''(\underline{x}) - y_j g_k'(\underline{x})),$$

so all we have to deal with is

$$V(y_k g''_k(\underline{x}) - y_j g'_k(\underline{x})).$$

Again a little calculation is necessary. Setting

$$\begin{aligned}
g_{pq} &:= \text{“}u_k\overline{u}_q g''_k(\underline{x}\,\overline{x}_p) - u_j\overline{u}_q g'_k(\underline{x}\,\overline{x}_p)\text{”} \\
&= u_k\overline{u}_q g''_k(x_0\overline{x}_p,\dots,x_n\overline{x}_p) - u_j\overline{u}_q g'_k(x_0\overline{x}_p,\dots,x_n\overline{x}_p) \\
&= \big((c_kc_q+d_kd_q)+i(d_kc_q-c_kd_q)\big) \\
&\qquad g''_k\big((a_0a_p+b_0b_p)+i(b_0a_p-a_0b_p),\dots,(a_na_p+b_nb_p)+i(b_na_p-a_nb_p)\big) \\
&\quad - \big((c_jc_q+d_jd_q)+i(d_jc_q-c_jd_q)\big) \\
&\qquad g'_k\big((a_0a_p+b_0b_p)+i(b_0a_p-a_0b_p),\dots,(a_na_p+b_nb_p)+i(b_na_p-a_nb_p)\big),
\end{aligned}$$

where $x_l = a_l + ib_l$ for $l = 0,\dots,n$, $u_l = c_l + id_l$ for $l = 0,\dots,m$, and

$$h_{pq} := (v_{kq}+iw_{kq})g''_k(y_{0p}+iz_{0p},\dots,y_{np}+iz_{np})-(v_{jq}+iw_{jq})g'_k(y_{0p}+iz_{0p},\dots,y_{np}+iz_{np}),$$

we obtain

$$\begin{aligned}
(\psi\times\phi)\Big(V\big(y_kg''_k(\underline{x})-y_jg'_k(\underline{x})\big)\Big) &= \\
&= \bigcap_{p=0}^{n}\bigcap_{q=0}^{m}(\psi\times\phi)(V(g_{pq})) \\
&= \bigcap_{p=0}^{n}\bigcap_{q=0}^{m}(\psi\times\phi)(U_i^{\mathrm{an}}\times V_j^{\mathrm{an}})\cap W(h_{pq}) \\
&= \bigcap_{p=0}^{n}\bigcap_{q=0}^{m}(\psi\times\phi)(U_i^{\mathrm{an}}\times V_j^{\mathrm{an}})\cap W(\operatorname{Re}h_{pq})\cap W(\operatorname{Im}h_{pq}).
\end{aligned}$$

□

2.6.2 Semi-algebraic singular chains

We need further prerequisites in order to state the promised Proposition 2.6.8.

Definition 2.6.7 ([Hi2, p. 168])**.** By an *open simplex* $\triangle^\circ$ we mean the interior of a simplex (= the convex hull of $r+1$ points in $\mathbb{R}^n$ which span an r-dimensional subspace). For convenience, a point is considered as an open simplex as well.

The notation $\triangle_d^{\mathrm{std}}$ will be reserved for the *closed standard simplex* spanned by the standard basis

$$\{e_i = (0,\dots,0,\underset{i}{1},0,\dots,0)\,|\,i=1,\dots,d+1\}$$

of $\mathbb{R}^{d+1}$.

Consider the following data $(*)$:

- X a variety defined over $\widetilde{\mathbb{Q}}$,
- D a divisor in X with normal crossings,
- and finally $\gamma \in H_p^{\text{sing}}(X^{\text{an}}, D^{\text{an}}; \mathbb{Q}),\ p \in \mathbb{N}_0$.

As before, we have denoted by X^{an} (resp. D^{an}) the complex analytic space associated to the base change $X_{\mathbb{C}} = X \times_{\widetilde{\mathbb{Q}}} \mathbb{C}$ (resp. $D_{\mathbb{C}} = D \times_{\widetilde{\mathbb{Q}}} \mathbb{C}$).

By Lemma 2.6.5, we may consider both X^{an} and D^{an} as bounded $\widetilde{\mathbb{Q}}$-semi-algebraic subsets of $\mathbb{R}^N$.

We are now able to formulate our proposition.

Proposition 2.6.8. *With data $(*)$ as above, we can find a representative of γ that is a rational linear combination of singular simplices each of which is $\widetilde{\mathbb{Q}}$-semi-algebraic.*

The proof of this proposition relies on the following proposition due to Lojasiewicz which has been written down by Hironaka.

Proposition 2.6.9 ([Hi2, p. 170])**.** *For $\{X_i\}$ a finite system of bounded $\widetilde{\mathbb{Q}}$-semi-algebraic sets in $\mathbb{R}^n$, there exists a simplicial decomposition*

$$\mathbb{R}^n = \coprod_j \overset{\circ}{\triangle}_j$$

by open simplices $\overset{\circ}{\triangle}_j$ and a $\widetilde{\mathbb{Q}}$-semi-algebraic automorphism

$$\kappa : \mathbb{R}^n \to \mathbb{R}^n$$

such that each X_i is a finite union of some of the $\kappa(\overset{\circ}{\triangle}_j)$.

Note 2.6.10. Although Hironaka considers $\mathbb{R}$-semi-algebraic sets, we can safely replace $\mathbb{R}$ by $\widetilde{\mathbb{Q}}$ in this result (including the fact he cites from [Sb]). The only problem that could possibly arise concerns a "good direction lemma":

Lemma 2.6.11 (Good direction lemma for $\mathbb{R}$, [Hi2, p. 172], [KB, Thm. 5.I, p. 242])**.** *Let Z be a $\mathbb{R}$-semi-algebraic subset of $\mathbb{R}^n$, which is nowhere dense. A direction $v \in \mathbb{P}_{\mathbb{R}}^{n-1}(\mathbb{R})$ is called* good*, if any line l in $\mathbb{R}^n$ parallel to v meets Z in a discrete (maybe empty) set of points; otherwise v is called* bad*. Then the set $B(Z)$ of bad directions is a Baire category set in $\mathbb{P}_{\mathbb{R}}^{n-1}(\mathbb{R})$.*

This gives immediately good directions $v \in \mathbb{P}_{\mathbb{R}}^{n-1}(\mathbb{R}) \setminus B(Z)$, but not necessarily $v \in \mathbb{P}_{\widetilde{\mathbb{Q}}}^{n-1}(\widetilde{\mathbb{Q}}) \setminus B(Z)$. However, in Remark 2.1 of [Hi2], which follows directly after the lemma, the following statement is made: If Z is compact, then $B(Z)$ is closed in $\mathbb{P}_{\mathbb{R}}^{n-1}(\mathbb{R})$. In particular $\mathbb{P}_{\widetilde{\mathbb{Q}}}^{n-1}(\widetilde{\mathbb{Q}}) \setminus B(Z)$ will be non-empty. Since we only consider *bounded* $\widetilde{\mathbb{Q}}$-semi-algebraic sets Z', we may take $Z := \overline{Z'}$ (which is compact by Heine-Borel), and thus find a good direction $v \in \mathbb{P}^{n-1}_{\widetilde{\mathbb{Q}}}(\widetilde{\mathbb{Q}}) \setminus B(Z')$ using $B(Z') \subseteq B(Z)$. Hence:

Lemma 2.6.12 (Good direction lemma for $\widetilde{\mathbb{Q}}$)**.** *Let Z' be a bounded $\widetilde{\mathbb{Q}}$-semi-algebraic subset of $\mathbb{R}^n$, which is nowhere dense. Then the set $\mathbb{P}^{n-1}_{\mathbb{R}}(\mathbb{R}) \setminus B(Z)$ of good directions is non-empty.*

Proof of Proposition 2.6.8. Applying Proposition 2.6.9 to the two-element system of $\widetilde{\mathbb{Q}}$-semi-algebraic sets $X^{\mathrm{an}}, D^{\mathrm{an}} \subseteq \mathbb{R}^N$, we obtain a $\widetilde{\mathbb{Q}}$-semi-algebraic decomposition

$$\mathbb{R}^N = \coprod_j \triangle^\circ_j$$

of $\mathbb{R}^N$ by open simplices $\triangle^\circ_j$ and a $\widetilde{\mathbb{Q}}$-semi-algebraic automorphism

$$\kappa : \mathbb{R}^N \to \mathbb{R}^N.$$

We write $\triangle_j$ for the closure of $\triangle^\circ_j$. The sets

$$K := \{\triangle^\circ_j \,|\, \kappa(\triangle^\circ_j) \subseteq X^{\mathrm{an}}\} \quad \text{and} \quad L := \{\triangle^\circ_j \,|\, \kappa(\triangle^\circ_j) \subseteq D^{\mathrm{an}}\}$$

can be thought of as finite simplicial complexes, but built out of open simplices instead of closed ones. We define their *geometric realizations*

$$|K| := \bigcup_{\triangle^\circ_j \in K} \triangle^\circ_j \quad \text{and} \quad |L| := \bigcup_{\triangle^\circ_j \in L} \triangle^\circ_j.$$

Then Proposition 2.6.9 states that κ maps the pair of topological spaces $(|K|, |L|)$ homeomorphically to $(X^{\mathrm{an}}, D^{\mathrm{an}})$.

Easy case: If X is complete, so is $X_{\mathbb{C}}$ (by [Ha2, Cor. II.4.8(c), p. 102]), hence X^{an} and D^{an} will be compact [Ha2, B.1, p. 439]. In this situation,

$$\overline{K} := \{\triangle_j \,|\, \kappa(\triangle_j) \subseteq X^{\mathrm{an}}\} \quad \text{and} \quad \overline{L} := \{\triangle_j \,|\, \kappa(\triangle_j) \subseteq D^{\mathrm{an}}\}$$

are (ordinary) simplicial complexes, whose geometric realizations coincide with those of K and L, respectively. In particular

$$\begin{aligned} H^{\mathrm{simpl}}_\bullet(\overline{K}, \overline{L}; \mathbb{Q}) &\cong H^{\mathrm{sing}}_\bullet(|\overline{K}|, |\overline{L}|; \mathbb{Q}) \\ &\cong H^{\mathrm{sing}}_\bullet(|K|, |L|; \mathbb{Q}) \\ &\cong H^{\mathrm{sing}}_\bullet(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q}). \end{aligned} \tag{2.4}$$

Here $H^{\mathrm{simpl}}_\bullet(\overline{K}, \overline{L}; \mathbb{Q})$ denotes simplicial homology of course.

We write $\gamma_{\mathrm{simpl}} \in H^{\mathrm{simpl}}_p(\overline{K}, \overline{L}; \mathbb{Q})$ and $\gamma_{\mathrm{sing}} \in H^{\mathrm{sing}}_p(|\overline{K}|, |\overline{L}|; \mathbb{Q})$ for the image of γ under this isomorphism. Any representative Γ_{simpl} of γ_{simpl} is a rational linear combination

$$\Gamma_{\mathrm{simpl}} = \textstyle\sum_j a_j \,\triangle_j, \quad a_j \in \mathbb{Q}$$

of oriented closed simplices $\triangle_j \in \overline{K}$. We can choose orientation-preserving affine-linear maps of the standard simplex $\triangle_p^{\mathrm{std}}$ to $\triangle_j$

$$\sigma_j : \triangle_p^{\mathrm{std}} \longrightarrow \triangle_j \quad \text{for} \quad \triangle_j \in \Gamma_{\mathrm{simpl}}.$$

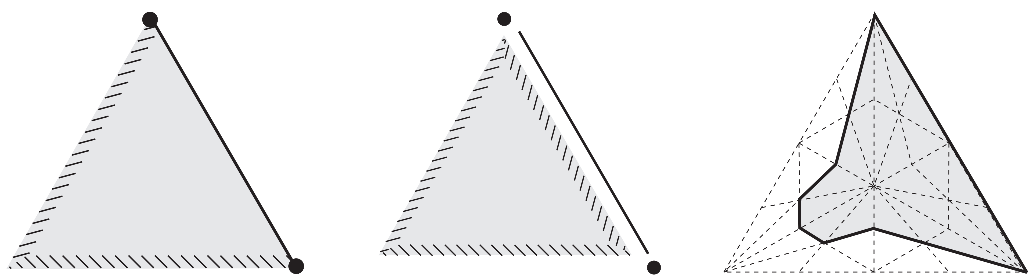
These maps yield a representative

$$\Gamma_{\mathrm{sing}} := \textstyle\sum_j a_j \, \sigma_j$$

of γ_{sing}. Composing with κ yields $\Gamma := \kappa_* \Gamma_{\mathrm{sing}} \in \gamma$, where Γ has the desired properties.

In the *general case*, we perform a barycentric subdivision $\mathcal{B}$ on K twice (once is not enough) and define $|K|$ and $|L|$ not as the "closure" of K and L, but as follows (see Figure 2.1)

$$\begin{aligned} \overline{K} &:= \{\triangle \mid \triangle^\circ \in \mathcal{B}^2(K) \text{ and } \triangle \subseteq |K|\}, \\ \overline{L} &:= \{\triangle \mid \triangle^\circ \in \mathcal{B}^2(K) \text{ and } \triangle \subseteq |L|\}. \end{aligned} \tag{2.5}$$



$\kappa^{-1}(X^{\mathrm{an}}) \cap \triangle_j$ — Intersection of $\kappa^{-1}(X^{\mathrm{an}})$ with a closed 2-simplex $\triangle_j$, where we assume that part of the boundary $\partial\triangle_j$ does not belong to $\kappa^{-1}(X^{\mathrm{an}})$

$K \cap \triangle_j$ — Open simplices of K contained in $\triangle_j$

$\left|\overline{K}\right| \cap \triangle_j$ — Intersection of $\left|\overline{K}\right|$ with $\triangle_j$ (the dashed lines show the barycentric subdivision)

Figure 2.1: Definition of $\overline{K}$

The point is that the pair of topological spaces $(\left|\overline{K}\right|, \left|\overline{L}\right|)$ is a strong deformation retract of $(|K|, |L|)$. Assuming this, we see that in the general case with $\overline{K}$, $\overline{L}$ defined as in (2.5), the isomorphism (2.4) still holds and we can proceed as in the easy case to prove the proposition.

We define the retraction map

$$\rho : (|K| \times [0,1], |L| \times [0,1]) \to (\left|\overline{K}\right|, \left|\overline{L}\right|)$$

as follows: Let $\triangle_j^\circ \in K$ be an open simplex which is not contained in the boundary of any other simplex of K and set

$$\mathit{inner} := \triangle_j \cap \overline{K}, \qquad \mathit{outer} := \triangle_j \setminus \overline{K}.$$

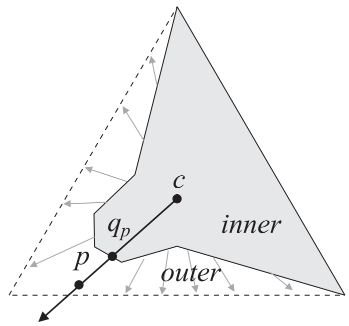


Figure 2.2: Definition of q_p

Note that *inner* is closed. For any point $p \in outer$ the ray $\overrightarrow{cp}$ from the center c of $\triangle_j^\circ$ through p "leaves" the set *inner* at a point q_p, i.e. $\overrightarrow{cp} \cap inner$ equals the line segment $c\,q_p$; see Figure 2.2. The map

$$\rho_j : \triangle_j \times [0,1] \to \triangle_j$$

$$(p,t) \mapsto \begin{cases} p & \text{if } p \in inner, \\ q_p + t \cdot (p - q_p) & \text{if} \quad p \in outer \end{cases}$$

retracts $\triangle_j$ onto *inner*.

Now these maps ρ_j glue together to give the desired homotopy ρ. □

We want to state one of the intermediate results of this proof explicitly:

Corollary 2.6.13. *Let X and D be as above. Then the pair of topological spaces $(X^{\mathrm{an}}, D^{\mathrm{an}})$ is homotopy equivalent to a pair of (realizations of) simplicial complexes $(|X^{\mathrm{simpl}}|, |D^{\mathrm{simpl}}|)$.*

2.7 Singular cohomology via the h'-topology

In order to give a simple description of the period isomorphism for singular varieties, we are going to need a more sophisticated description of singular cohomology.

We work in the category of complex analytic spaces An.

Definition 2.7.1. Let X be a complex analytic space. The h'-*topology* on the category $(\mathrm{An}/X)_{\mathrm{h}'}$ of complex analytic spaces over X is the smallest Grothendieck topology such that the following are covering maps:

1. proper surjective morphisms;
2. open covers.

If $\mathcal{F}$ is a presheaf of An/X we denote $\mathcal{F}_{\mathrm{h}'}$ its sheafification in the h'-topology.

Remark 2.7.2. This definition is inspired by Voevodsky's *h*-topology on the category of schemes, see Section 3.2. We are not sure if it is the correct analogue in the analytic setting. However, it is good enough for our purposes.

Lemma 2.7.3. *For $Y \in \mathrm{An}$ let $\mathbb{C}_Y$ be the (ordinary) sheaf associated to the presheaf $\mathbb{C}$. Then*

$$Y \mapsto \mathbb{C}_Y(Y)$$

is an h'-sheaf on An.

Proof. We have to check the sheaf condition for the generators of the topology. By assumption it is satisfied for open covers. Let $\tilde{Y} \to Y$ be proper surjective. Without loss of generality Y is connected. Let $\tilde{Y}_i$ for $i \in I$ be the collection of connected components of $\tilde{Y}$. Then

$$\tilde{Y} \times_Y \tilde{Y} = \bigcup_{i,j \in I} \tilde{Y}_i \times_Y \tilde{Y}_j$$

We have to compute the kernel of

$$\prod_{i \in I} \mathbb{C} \to \prod_{i,j} \mathbb{C}(\tilde{Y}_i \times_Y \tilde{Y}_j)$$

via the difference of the two natural restriction maps. Comparing a_i and a_j in $\mathbb{C}(\tilde{Y}_i \times_Y \tilde{Y}_j)$ we see that they agree. Hence the kernel is just one copy of $\mathbb{C} = \mathbb{C}_Y(Y)$. □

Proposition 2.7.4. *Let X be an analytic space and $i : Z \subset X$ a closed subspace. Then there is a morphism of sites $\rho : (\mathrm{An}/X)_{\mathrm{h}'} \to X$. It induces an isomorphism*

$$H^i_{\mathrm{sing}}(X, Z; \mathbb{C}) \to H^i_{\mathrm{h}'}((\mathrm{An}/X)_{\mathrm{h}'}, \mathrm{Ker}(\mathbb{C}_{\mathrm{h}'} \to i_*\mathbb{C}_{\mathrm{h}'}))$$

compatible with long exact sequences and products.

Remark 2.7.5. This statement and the following proof can be extended to more general sheaves $\mathcal{F}$.

The argument uses the notion of a hypercover, see Definition 1.5.8.

Proof. We first treat the absolute case with $Z = \emptyset$. We use the theory of cohomological descent as developed in [SGA4Vbis]. Singular cohomology satisfies cohomological descent for open covers and also for proper surjective maps (see Theorem 2.7.6). (The main ingredient for the second case is the proper base change theorem.) Hence it satisfies cohomological descent for h'-covers. This implies that singular cohomology can be computed as a direct limit

$$\lim_{\mathfrak{X}_\bullet} \mathbb{C}(\mathfrak{X}_\bullet),$$

where $\mathfrak{X}_\bullet$ runs through all h′-hypercovers. On the other hand, the same limit computes h′-cohomology, see Proposition 1.6.9 For the general case, recall that we have a short exact sequence

$$0 \to j_!\mathbb{C} \to \mathbb{C} \to i_*\mathbb{C} \to 0$$

of sheaves on X. Its pull-back to An/X maps naturally to the short exact sequence

$$0 \to \mathrm{Ker}(\mathbb{C}_{\mathrm{h}'} \to i_*\mathbb{C}_{\mathrm{h}'})) \to \mathbb{C}_{\mathrm{h}'} \to i_*\mathbb{C}_{\mathrm{h}'} \to 0\ .$$

This reduces the comparison in the relative case to the absolute case once we have shown that $Ri_*\mathbb{C}_{\mathrm{h}'} = i_*\mathbb{C}_{\mathrm{h}'}$. The sheaf $R^n i_*\mathbb{C}_{\mathrm{h}'}$ is given by the h′-sheafification of the presheaf

$$X' \mapsto H^n_{\mathrm{h}'}(Z \times_X X', \mathbb{C}_{\mathrm{h}'}) = H^n_{\mathrm{sing}}(Z \times_X X', \mathbb{C})$$

for $X' \to X$ in An/X. By resolution of singularities for analytic spaces we may assume that X' is smooth and $Z' = X' \times_X Z$ a divisor with normal crossings. By passing to an open cover, we may assume that Z' an open ball in a union of coordinate hyperplanes, in particular contractible. Hence its singular cohomology is trivial. This implies that $R^n i_*\mathbb{C}_{\mathrm{h}'} = 0$ for $n \geq 1$. □

Theorem 2.7.6 (Descent for proper hypercoverings)**.** *Let $D \subset X$ be a closed subvariety and $D_\bullet \to D$ a proper hypercover(see Definition 1.5.8), such that there is a commutative diagram*

$$\begin{array}{ccc} D_\bullet & \longrightarrow & X_\bullet \\ \downarrow & & \downarrow \\ D & \longrightarrow & X \end{array}$$

Then one has cohomological descent for singular cohomology:

$$H^*(X, D; \mathbb{Z}) = H^*\left(\mathrm{Cone}(\mathrm{Tot}(X_\bullet) \to \mathrm{Tot}(D_\bullet))[-1]; \mathbb{Z}\right).$$

Here, $\mathrm{Tot}(-)$ *denotes the total complex in* $\mathbb{Z}[\mathrm{Var}]$ *associated to the corresponding simplicial variety, see Definition 1.5.11.*

Proof. The relative case follows from the absolute case. The essential ingredient is proper base change, which allows to reduce to the case where X is a point. The statement then becomes a completely combinatorial assertion on contractibility of simplicial sets. The results are summed up in [D5] (5.3.5). For a complete reference see [SGA4Vbis], in particular Corollaire 4.1.6. □

Chapter 3

Algebraic de Rham cohomology

Let k be a field of characteristic zero. We are going to define relative algebraic de Rham cohomology for general varieties over k, not necessarily smooth.

3.1 The smooth case

In this section, all varieties are smooth over k. In this case, de Rham cohomology is defined as hypercohomology of the complex of sheaves of differentials.

3.1.1 Definition

Definition 3.1.1. Let X be a smooth variety over k. Let Ω^1_X be the sheaf of k-differentials on X. For $p \geq 0$ let

$$\Omega^p_X = \Lambda^p \Omega^1_X$$

be the exterior power in the category of $\mathcal{O}_X$-modules. The universal k-derivation $d : \mathcal{O}_X \to \Omega^1_X$ induces

$$d^p : \Omega^p_X \to \Omega^{p+1}_X \ .$$

We call $(\Omega^\bullet_X, d)$ the *algebraic de Rham complex* of X.

If X is smooth of dimension n, the sheaf Ω^1_X is locally free of rank n. This allows to define exterior powers. Note that Ω^i_X vanishes for $i > n$. The differential is uniquely characterized by the properties:

1. $d^0 = d$ on $\mathcal{O}_X$;

2. $d^{p+1}d^p = 0$ for all $p \geq 0$;
3. $d(\omega \wedge \omega') = d\omega \wedge \omega' + (-1)^p \omega \wedge d\omega'$ for all local sections ω of Ω_X^p and ω' of $\Omega_X^{p'}$.

Indeed, if $t_1, \ldots, t_n$ is a system of local parameters at $x \in X$, then local sections of Ω_X^p near x can be expressed as

$$\omega = \sum_{1 \leq i_1 < \cdots < i_p \leq n} f_{i_1 \ldots i_p} dt_{i_1} \wedge \cdots \wedge dt_{i_p}$$

and we have

$$d^p \omega = \sum_{1 \leq i_1 < \cdots < i_p \leq n} df_{i_1 \ldots i_p} \wedge dt_{i_1} \wedge \cdots \wedge dt_{i_p} \ .$$

Definition 3.1.2. Let X be smooth variety over a field k of characteristic 0. We define *algebraic de Rham cohomology* of X as the hypercohomology

$$H^i_{\mathrm{dR}}(X) = H^i(X, \Omega_X^\bullet) \ .$$

For background material on hypercohomology see Section 1.4.

If X is smooth and affine, this simplifies to

$$H^i_{\mathrm{dR}}(X) = H^i(\Omega_X^\bullet(X)) \ .$$

Example 3.1.3. 1. Consider the affine line $X = \mathbb{A}^1_k = \operatorname{Spec} k[t]$. Then

$$\Omega^\bullet_{\mathbb{A}^1}(\mathbb{A}^1) = \left[k[t] \xrightarrow{d} k[t]dt \right] \ .$$

We have

$$\operatorname{Ker}(d) = \{P \in k[t] | P' = 0\} = k \ , \qquad \operatorname{Im}(d) = k[t]dt \ ,$$

because we have assumed characteristic zero. Hence

$$H^i_{\mathrm{dR}}(\mathbb{A}^1) = \begin{cases} k & i = 0, \\ 0 & i > 0. \end{cases}$$

2. Consider the multiplicative group $X = \mathbb{G}_m = \operatorname{Spec} k[t, t^{-1}]$. Then

$$\Omega^\bullet_{\mathbb{G}_m}(\mathbb{G}_m) = \left[k[t, t^{-1}] \xrightarrow{d} k[t, t^{-1}]dt \right] \ .$$

We have

$$\operatorname{Ker}(d) = \{P \in k[t] | P' = 0\} = k \ ,$$

$$\operatorname{Im}(d) = \{\sum_{i=n}^{N} a_i t^i dt | a_{-1} = 0\} \ ,$$

again because of characteristic zero. Hence

$$H^i_{\mathrm{dR}}(\mathbb{G}_m) = \begin{cases} k & i = 0, 1, \\ 0 & i > 1. \end{cases}$$

The isomorphism for $i = 1$ is induced by the residue for meromorphic differential forms.

3. Let X be a connected smooth projective curve of genus g. We use the stupid filtration on the de Rham complex

$$0 \to \Omega^1_X[-1] \to \Omega^\bullet_X \to \mathcal{O}_X[0] \to 0 \ .$$

The cohomological dimension of any variety X is the index i above which the cohomology $H^i(X, \mathcal{F})$ of any coherent sheaf $\mathcal{F}$ vanishes, see [Ha2, Chap. III, Section 4]. The cohomological dimension of a smooth, projective curve is 1, hence the long exact sequence reads

$$\begin{aligned} 0 \to H^{-1}(X, \Omega^1_X) \to H^0_{\mathrm{dR}}(X) \to H^0(X, \mathcal{O}_X) & \\ \xrightarrow{\partial} H^0(X, \Omega^1_X) \to H^1_{\mathrm{dR}}(X) \to H^1(X, \mathcal{O}_X) & \\ \xrightarrow{\partial} H^1(X, \Omega^1_X) \to H^2_{\mathrm{dR}}(X) \to 0 & \end{aligned}$$

This is a special case of the Hodge spectral sequence. It is known to degenerate (e.g. [D4]). Hence the boundary maps ∂ vanish. By Serre duality, this yields

$$H^i_{\mathrm{dR}}(X) = \begin{cases} H^0(X, \mathcal{O}_X) = k & i = 0, \\ H^1(X, \Omega^1_X) = H^0(X, \mathcal{O}_X)^\vee = k & i = 2, \\ 0 & i > 2. \end{cases}$$

The most interesting group for $i = 1$ sits in an exact sequence

$$0 \to H^0(X, \Omega^1_X)^\vee \to H^1_{\mathrm{dR}}(X) \to H^0(X, \Omega^1_X) \to 0$$

and hence

$$\dim H^1_{\mathrm{dR}}(X) = 2g \ .$$

Remark 3.1.4. In these cases, the explicit computation shows that algebraic de Rham cohomology computes the standard Betti numbers of these varieties. We are going to show in chapter 5 that this is always true. In particular, it is always finite dimensional. A second algebraic proof of this fact will also be given in Corollary 3.1.17.

Lemma 3.1.5. *Let X be a smooth variety of dimension d. Then $H^i_{\mathrm{dR}}(X)$ vanishes for $i > 2d$. If in addition X is affine, it vanishes for $i > d$.*

Proof. We use the stupid filtration on the de Rham complex. This induces a system of long exact sequences relating the groups $H^i(X,\Omega^p_X)$ to algebraic de Rham cohomology.

Any variety of dimension d has cohomological dimension $\leq d$ for coherent sheaves [Ha2, ibid.]. All Ω^p_X are coherent, hence $H^i(X,\Omega^p_X)$ vanishes for $i>d$. The complex $\Omega^\bullet_X$ is concentrated in degrees at most d. This adds up to cohomological dimension $2d$ for algebraic de Rham cohomology. Affine varieties have cohomological dimension 0, hence $H^i(X,\Omega^p_X)$ vanishes already for $i>0$. □

3.1.2 Functoriality

Let $f:X\to Y$ be morphism of smooth varieties over k. We want to explain the functoriality

$$f^*:H^i_{\mathrm{dR}}(Y)\to H^i_{\mathrm{dR}}(X)\ .$$

We use the Godement resolution (see Definition 1.4.8) and put

$$R\Gamma_{\mathrm{dR}}(X)=\Gamma(X,Gd(\Omega^\bullet_X))\ .$$

The natural map $f^{-1}\mathcal{O}_X\to\mathcal{O}_X$ induces a unique multiplicative map

$$f^{-1}\Omega^\bullet_X\to\Omega^\bullet_Y\ .$$

By functoriality of the Godement resolution, we have

$$f^{-1}Gd_X(\Omega^\bullet_X)\to Gd_Y(f^{-1}\Omega^\bullet_X)\to Gd_Y(\Omega^\bullet_Y)\ .$$

Taking global sections, this yields

$$R\Gamma_{\mathrm{dR}}(Y)\to R\Gamma_{\mathrm{dR}}(X)\ .$$

We have shown:

Lemma 3.1.6. *De Rham cohomology $H^i_{\mathrm{dR}}(\cdot)$ is a contravariant functor on the category of smooth varieties over k with values in k-vector spaces. It is induced by a functor*

$$R\Gamma_{\mathrm{dR}}:\mathrm{Sm}\to C^+(k\mathrm{-Mod})\ .$$

Note that $\mathbb{Q}\subset k$, so the functor can be extended $\mathbb{Q}$-linearly to $\mathbb{Q}[\mathrm{Sm}]$. This allows to extend the definition of algebraic de Rham cohomology to complexes of smooth varieties in the next step. Explicitly: let $X^\bullet$ be an object of $C^-(\mathbb{Q}[\mathrm{Sm}])$. Then there is a double complex $K^{\bullet,\bullet}$ with

$$K^{n,m}=\Gamma(X^{-n},Gd^m(\Omega^\bullet))\ .$$

Definition 3.1.7. Let $X^\bullet$ be a object of $C^-(\mathbb{Z}[\mathrm{Sm}])$. We denote the total complex by

$$R\Gamma_{\mathrm{dR}}(X^\bullet)=\mathrm{Tot}(K^{\bullet,\bullet})$$

and set

$$H^i_{\mathrm{dR}}(X^\bullet)=H^i(R\Gamma_{\mathrm{dR}}(X^\bullet)\ .$$

We call this the *algebraic de Rham cohomology* of $X^\bullet$.

3.1.3 Cup product

Let X be a smooth variety over k. Wedge product of differential forms turns $\Omega^\bullet_X$ into a differential graded algebra:

$$\mathrm{Tot}(\Omega^\bullet_X \otimes_k \Omega^\bullet_X) \to \Omega^\bullet_X \ .$$

The compatibility with differentials was built into the definition of d in Definition 3.1.1.

Lemma 3.1.8. *$H^\bullet_{\mathrm{dR}}(X)$ carries a natural multiplication*

$$\cup : H^i_{\mathrm{dR}}(X) \otimes_k H^j_{\mathrm{dR}}(X) \to H^{i+j}_{\mathrm{dR}}(X)$$

induced from wedge product of differential forms.

Proof. We need to define

$$R\Gamma_{\mathrm{dR}}(X) \otimes_k R\Gamma_{\mathrm{dR}}(X) \to R\Gamma_{\mathrm{dR}}(X)$$

as a morphism in the derived category. We have quasi-isomorphisms

$$\Omega^\bullet_X \otimes \Omega^\bullet_X \to Gd(\Omega^\bullet_X) \otimes Gd(\Omega^\bullet_X)$$

and hence a quasi-isomorphism of flasque resolutions of $\Omega^\bullet_X \otimes \Omega^\bullet_X$

$$s : Gd(\Omega^\bullet_X \otimes \Omega^\bullet_X) \to Gd\left(Gd(\Omega^\bullet_X) \otimes Gd(\Omega^\bullet_X)\right)$$

In the derived category, this allows the composition

$$\begin{aligned} R\Gamma_{\mathrm{dR}}(X) \otimes_k R\Gamma_{\mathrm{dR}}(X) &= \Gamma(X, Gd(\Omega^\bullet_X)) \otimes_k \Gamma(X, Gd(\Omega^\bullet_X)) \\ &\to \Gamma(X, Gd(\Omega^\bullet) \otimes Gd(\Omega^\bullet_X)) \\ &\to \Gamma\left(X, Gd\left(Gd(\Omega^\bullet_X) \otimes Gd(\Omega^\bullet_X)\right)\right) \\ &\leftarrow s\Gamma(X, Gd(\Omega^\bullet_X \otimes \Omega^\bullet_X)) \\ &\to \Gamma(X, Gd(\Omega^\bullet_X)) = R\Gamma_{\mathrm{dR}}(X) \ . \end{aligned}$$

□

The same method also allows the construction of an exterior product.

Proposition 3.1.9 (Künneth formula)**.** *Let X, Y be smooth varieties. There is a natural multiplication induced from wedge product of differential forms*

$$H^i_{\mathrm{dR}}(X) \otimes_k H^j_{\mathrm{dR}}(Y) \to H^{i+j}_{\mathrm{dR}}(X \times Y) \ .$$

It induces an isomorphism

$$H^n_{\mathrm{dR}}(X \times Y) \cong \bigoplus_{i+j=n} H^i_{\mathrm{dR}}(X) \otimes_k H^j_{\mathrm{dR}}(Y) \ .$$

Proof. Let $p : X \times Y \to X$ and $q : X \times Y \to Y$ be the projection maps. The exterior multiplication is given by

$$H^i_{\mathrm{dR}}(X) \otimes H^j_{\mathrm{dR}}(Y) \xrightarrow{p^* \otimes q^*} H^i_{\mathrm{dR}}(X \times Y) \otimes H^j_{\mathrm{dR}}(X \times Y) \xrightarrow{\cup} H^{i+j}_{\mathrm{dR}}(X \times Y) \ .$$

The Künneth formula is most easily proved by comparison with singular cohomology. We postpone the proof to Lemma 5.3.2 in Chap. 5. □

Corollary 3.1.10 (Homotopy invariance)**.** *Let X be a smooth variety. Then the natural map*

$$H^n_{\mathrm{dR}}(X) \to H^n_{\mathrm{dR}}(X \times \mathbb{A}^1)$$

is an isomorphism.

Proof. We combine the Künneth formula with the compuation in the case of $\mathbb{A}^1$ in Example 3.1.3. □

3.1.4 Change of base field

Let K/k be an extension of fields of characteristic zero. We have the corresponding base change functor

$$X \mapsto X_K$$

from (smooth) varieties over k to (smooth) varieties over K. Let

$$\pi : X_K \to X$$

be the natural map of schemes. By standard calculus of differential forms,

$$\Omega^\bullet_{X_K/K} \cong \pi^*\Omega^\bullet_{X/k} = \pi^{-1}\Omega^\bullet_{X/k} \otimes_k K \ .$$

Lemma 3.1.11. *Let K/k be an extension of fields of characteristic zero. Let X be a smooth variety over k. Then there are natural isomorphisms*

$$H^i_{\mathrm{dR}}(X) \otimes_k K \to H^i_{\mathrm{dR}}(X_K) \ .$$

They are induced by a natural quasi-isomorphism

$$R\Gamma_{\mathrm{dR}}(X) \otimes_k K \to R\Gamma_{\mathrm{dR}}(X_K) \ .$$

Proof. By functoriality of the Godement resolution (see Lemma 1.4.10) and k-linarity, we get natural quasi-isomorphisms

$$\pi^{-1}Gd_X(\Omega^\bullet_{X/k}) \otimes_k K \to Gd_{X_K}(\pi^{-1}\Omega^\bullet_{X/k}) \to Gd_{X_K}(\Omega^\bullet_{X_K/K}) \ .$$

As K is flat over k, taking global sections induces a sequence of quasi-isomorphisms

$$\begin{aligned} R\Gamma_{\mathrm{dR}}(X)\otimes_k K &= \Gamma(X, Gd_X(\Omega^\bullet_{X/k}))\otimes_k K \\ &\cong \Gamma(X_K, \pi^{-1}Gd_X(\Omega^\bullet_{X/k})\otimes_k K \\ &\cong \Gamma(X_K, \pi^{-1}Gd_X(\Omega^\bullet_{X/k})\otimes_k K) \\ &\to \Gamma(X_K, Gd_{X_K}(\Omega^\bullet_{X_K/K}) \\ &= R\Gamma_{\mathrm{dR}}(X_K)\ . \end{aligned}$$

□

Remark 3.1.12. This immediately extends to algebraic de Rham cohomology of complexes of smooth varieties.

Conversely, we can also restrict scalars.

Lemma 3.1.13. *Let K/k be a finite field extension. Let Y be a smooth variety over K. Then there are a natural isomorphism*

$$H^i_{\mathrm{dR}}(Y/k)\to H^i_{\mathrm{dR}}(Y/K).$$

They are induced by a natural isomorphism

$$R\Gamma_{\mathrm{dR}}(Y/k)\to R\Gamma_{\mathrm{dR}}(Y/K).$$

Proof. We use the sequence of sheaves on Y ([Ha2] Proposition 8.11)

$$\pi^*\Omega^1_{K/k}\to\Omega^1_{Y/k}\to\Omega^1_{Y/K}\to 0$$

where $\pi : Y\to \mathrm{Spec}K$ is the structural map. As we are in characteristic 0, we have $\Omega^1_{K/k}=0$. This implies that we actually have identical de Rham complexes

$$\Omega^\bullet_{Y/K}=\Omega^\bullet_{Y/k}$$

and identical Godement resolutions. □

3.1.5 Étale topology

In this section, we give an alternative interpretation of algebraic de Rham cohomology using the étale topology. The results are not used in our discussions of periods.

Let X_{et} be the small étale site on X, see section 1.6. The complex of differential forms $\Omega^\bullet_X$ can be viewed as a complex of sheaves on X_{et} (see [Mi], Chap. II, Example 1.2 and Proposition 1.3). We write $\Omega^\bullet_{X_{\mathrm{et}}}$ for distinction.

Lemma 3.1.14. *There is a natural isomorphism*

$$H^i_{\mathrm{dR}}(X)\to H^i(X_{\mathrm{et}},\Omega^\bullet_{X_{\mathrm{et}}})\ .$$

Proof. The map of sites $s : X_{\text{et}} \to X$ induces a map on cohomology

$$H^i(X, \Omega^\bullet_X) \to H^i(X_{\text{et}}, \Omega^\bullet_{X_{\text{et}}}) \ .$$

We filter $\Omega^\bullet_X$ by the stupid filtration $F^p\Omega^\bullet_X$

$$0 \to F^{p+1}\Omega^\bullet_X \to F^p\Omega^\bullet_X \to \Omega^p_X[-p] \to 0$$

and compare the induced long exact sequences in cohomology on X and X_{et}. As the Ω^p_X are coherent, the comparison maps

$$H^i(X, \Omega^p_X) \to H^i(X_{\text{et}}, \Omega^p_{X_{\text{et}}})$$

are isomorphisms by [Mi] Chap. III, Proposition 3.7. By descending induction on p, this implies that we have isomorphisms for all $F^p\Omega^\bullet_X$, in particular for $\Omega^\bullet_X$ itself. □

3.1.6 Differentials with log poles

We give an alternative description of algebraic de Rham cohomology using differentials with log poles as introduced by Deligne, see [D4], Chap. 3. We are not going to use this point of view in our study of periods.

Let X be a smooth variety and $j : X \to \bar{X}$ an open immersion into a smooth projective variety such that $D = \bar{X} \smallsetminus X$ is a simple divisor with normal crossings (see Definition 1.1.2).

Definition 3.1.15. Let

$$\Omega^1_{\bar{X}}\langle D\rangle \subset j_*\Omega^1_X$$

be the locally free $\mathcal{O}_{\bar{X}}$-module with the following basis: if $U \subset X$ is an affine open subvariety étale over $\mathbb{A}^n$ via coordinates $t_1, \dots, t_n$ and $D|_U$ given by the equation $t_1 \dots t_r = 0$, then $\Omega^1_{\bar{X}}\langle D\rangle|_U$ has $\mathcal{O}_{\bar{X}}$-basis

$$\frac{dt_1}{t_1}, \dots, \frac{dt_r}{t_r}, dt_{r+1}, \dots, dt_n \ .$$

For $p > 1$ let

$$\Omega^p_{\bar{X}}\langle D\rangle = \Lambda^p\Omega^1_{\bar{X}}\langle D\rangle \ .$$

We call the $\Omega^\bullet_{\bar{X}}\langle D\rangle$ the *complex of differentials with log poles along* D.

Note that the differential of $j_*\Omega^\bullet_X$ respects $\Omega^\bullet_{\bar{X}}\langle D\rangle$, so that this is indeed a complex.

Proposition 3.1.16. *The inclusion induces a natural isomorphism*

$$H^i(\bar{X}, \Omega^\bullet_{\bar{X}}\langle D\rangle) \to H^i(X, \Omega^\bullet_X) \ .$$

Proof. This is the algebraic version of [D4], Prop. 3.1.8. We indicate the argument. Note that $j : X \to \bar{X}$ is affine, hence j_* is exact and we have

$$H^i(X, \Omega^\bullet_X) \cong H^i(\bar{X}, j_*\Omega^\bullet_X) \ .$$

It remains to show that

$$\iota : \Omega^\bullet_{\bar{X}}\langle D\rangle \to j_*\Omega^\bullet_X$$

is a quasi-isomorphism, or, equivalently, that $\mathrm{Coker}(\iota)$ is exact. By Lemma 3.1.14 we can work in the étale topology. It suffices to check exactness in stalks in geometric points of $\bar{X}$ over closed points. As $\bar{X}$ is smooth and D a divisor with normal crossings, it suffices to consider the case $D = V(t_1 \dots t_r) \subset \mathbb{A}^n$ and the stalk in 0. As in the proof of the Poincaré lemma, it suffices to consider the case $n = 1$. If $r = 0$, then there is nothing to show.

In remains to consider the following situation: let $k = \bar{k}$, $\mathcal{O}$ be the henselization of $k[t]$ with respect to the ideal (t). We have to check that the complex

$$\mathcal{O}[t^{-1}]/\mathcal{O} \to \mathcal{O}[t^{-1}]/t^{-1}\mathcal{O}dt$$

is acyclic. The term in degree 0 has the $\mathcal{O}$-basis t^{-i} for $i > 0$. The term in degree 1 has the $\mathcal{O}$-basis $t^{-i}dt$ for $i > 1$. In this basis, the differential has the form

$$f\frac{dt}{t^i} \mapsto \begin{cases} f'\frac{dt}{t^i} - if\frac{dt}{t^{i+1}} & i > 1, \\ -f\frac{dt}{t^2} & i = 1. \end{cases}$$

It is injective because $\mathrm{char}(k) = 0$. By induction on i we also check that it is surjective. □

Corollary 3.1.17. *Let X be a smooth variety over k. Then the algebraic de Rham cohomology groups $H^i_{\mathrm{dR}}(X)$ are finite dimensional k-vector spaces.*

Proof. By resolution of singularities, we can embed X into a projective $\bar{X}$ such that D is a simple divisor with normal crossings. By the proposition

$$H^i_{\mathrm{dR}}(X) = H^i(\bar{X}, \Omega^\bullet_{\bar{X}}\langle D\rangle) \ .$$

Note that all $\Omega^\bullet_{\bar{X}}\langle D\rangle$ are coherent sheaves on a projective variety, hence the cohomology groups $H^p(\bar{X}, \Omega^q_{\bar{X}}\langle D\rangle)$ are finite dimensional over k. We use the stupid filtration on $\Omega^\bullet_{\bar{X}}\langle D\rangle$ and the induced long exact cohomology sequence. By induction, all $H^q(\bar{X}, F^p\Omega^\bullet_{\bar{X}}\langle D\rangle)$ are finite dimensional. □

Remark 3.1.18. The complex of differentials with log poles is studied intensively in the theory of mixed Hodge structures. Indeed, Deligne uses it in [D4] in order to define the Hodge and the weight filtration on cohomology of a smooth variety X. We are not going to use Hodge structures in the sequel though.

3.2 The general case: via the h-topology

We now want to extend the definition to the case of singular varieties and even to relative cohomology. The most simple minded idea – use Definition 3.1.2 – does not give the desired dimensions.

Example 3.2.1. Consider $X = \mathrm{Spec} A$ with $A = k[X,Y]/XY$, the union of two affine lines. This variety is homotopy equivalent to a point, so we expect its cohomology to be trivial. We compute the cohomology of the de Rham complex

$$A \to \langle dX, dY \rangle_A / \langle XdY + YdX \rangle_A \ .$$

Elements of A can be represented uniquely by polynomials of the form

$$P = \sum_{i=0}^{n} a_i X^i + \sum_{j=1}^{m} b_j Y^j$$

with

$$dP = \sum_{i=1}^{n} i a_i X^{i-1} dX + \sum_{j=1}^{m} b_j j Y^{j-1} dY \ .$$

P is in the kernel of d if it is constant. On the other hand d is not surjective because it misses differentials of the form $Y^i dX$.

There are different ways of adapting the definition in order to get a well-behaved theory.

The h-topology introduced by Voevodsky makes the handling of singular varieties straightforward. In this topology, any variety is locally smooth by resolution of singularities. The h-sheafification of the presheaf of Kähler differentials was studied in detail by Huber and Jörder in [HJ]. The weaker notion of eh-differential was already introduced by Geisser in [Ge].

We review a definition given by Voevodsky in [Voe].

Definition 3.2.2 ([Voe] Section 3.1)**.** A morphism of schemes $p : X \to Y$ is called *topological epimorphism* if Y has the quotient topology of X. It is a *universal topological epimorphism* if any base change of p is a topological epimorphism.

The h-topology on the category $(\mathrm{Sch}/X)_\mathrm{h}$ of separated schemes of finite type over X is the Grothendieck topology with coverings finite families $\{p_i : U_i \to Y\}$ such that $\bigcup_i U_i \to Y$ is a universal topological epimorphism.

By [Voe] the following are h-covers:

1. finite flat covers (in particular étale covers);
2. proper surjective morphisms;

3. quotients by finite groups actions.

The assignment $X \mapsto \Omega^p_{X/k}(X)$ is a presheaf on Sch. We denote by Ω^p_{h} (resp. $\Omega^p_{\mathrm{h}/X}$, if X needs to be specified) its sheafification in the h-topology, and by $\Omega^p_{\mathrm{h}}(X)$ its value as abelian group.

Definition 3.2.3. Let X be a separated k-scheme of finite type over k. We define

$$H^i_{\mathrm{dR}}(X_{\mathrm{h}}) = H^i((\mathrm{Sch}/X)_{\mathrm{h}}, \Omega^\bullet_{\mathrm{h}}) \ .$$

Proposition 3.2.4 ([HJ] Theorem 3.6, Proposition 7.4)**.** *Let X be smooth over k. Then*

$$\Omega^p_{\mathrm{h}}(X) = \Omega^p_{X/k}(X)$$

and

$$H^i_{\mathrm{dR}}(X_{\mathrm{h}}) = H^i_{\mathrm{dR}}(X) \ .$$

Proof. The statement on $\Omega^p_{\mathrm{h}}(X)$ is [HJ], Theorem 3.6. The statement on the de Rham cohomology is loc.cit., Proposition 7.4. together with the comparison of loc. cit., Lemma 7.22. □

Remark 3.2.5. The main ingredients of the proof are a normal form for h-covers established by Voevodsky in [Voe] Theorem 3.1.9, an explicit computation for the blow-up of a smooth variety in a smooth center and étale descent for the coherent sheaves $\Omega^p_{Y/k}$.

A particular useful h-cover are *abstract blow-ups*, covers of the form $(f : X' \to X, i : Z \to X)$ where Z is a closed immersion and f is proper and an isomorphism above $X - Z$.

Then, the above implies that there is a long exact blow-up sequence

$$\dots \to H^i_{\mathrm{dR}}(X) \to H^i_{\mathrm{dR}}(X') \oplus H^i_{\mathrm{dR}}(Z) \to H^i_{\mathrm{dR}}(f^{-1}(Z)) \to \dots$$

induced by the blow-up triangle

$$[f^{-1}(Z)] \to [X'] \oplus [Z] \to [X]$$

in SmCor.

Definition 3.2.6. Let $X \in \mathrm{Sch}$ and $i : Z \to X$ a closed subscheme. Put

$$\Omega^p_{\mathrm{h}/(X,Z)} = \mathrm{Ker}(\Omega^p_{\mathrm{h}/X} \to i_*\Omega^p_{\mathrm{h}/Z})$$

in the category of abelian sheaves on $(\mathrm{Sch}/X)_{\mathrm{h}}$.

We define *relative algebraic de Rham cohomology* as

$$H^p_{\mathrm{dR}}(X, Z) = H^p_{\mathrm{h}}(X, \Omega^\bullet_{\mathrm{h}/(X,Z)}) \ .$$

Lemma 3.2.7 ([HJ] Lemma 7.26)**.** *Let $i : Z \to X$ be a closed immersion.*

1. *Then*
$$Ri_*\Omega^p_{\mathrm{h}/Z} = i_*\Omega^p_{\mathrm{h}/Z}$$
and hence
$$H^q_{\mathrm{h}}(X, i_*\Omega^p_{\mathrm{h}/Z}) = H^q_{\mathrm{h}}(Z, \Omega^p_{\mathrm{h}}) \ .$$
2. *The natural map of sheaves of abelian groups on* $(\mathrm{Sch}/X)_{\mathrm{h}}$
$$\Omega^p_{\mathrm{h}/X} \to i_*\Omega^p_{\mathrm{h}/Z}$$
is surjective.

Remark 3.2.8. The main ingredient of the proof is resolution of singularities and the computation of $\Omega^p_{\mathrm{h}}(Z)$ for Z a divisor with normal crossings: it is given as Kähler differentials modulo torsion, see [HJ] Proposition 4.9.

Proposition 3.2.9 ((Long exact sequence) [HJ] Proposition 2.7)**.** *Let $Z \subset Y \subset X$ be closed immersions. Then there is a natural long exact sequence*

$$\cdots \to H^q_{\mathrm{dR}}(X, Y) \to H^q_{\mathrm{dR}}(X, Z) \to H^q_{\mathrm{dR}}(Y, Z) \to H^{q+1}_{\mathrm{dR}}(X, Y) \to \cdots$$

Remark 3.2.10. The sequence is the long exact cohomology sequence attached to the exact sequence of h-sheaves on X

$$0 \to \Omega^p_{\mathrm{h}/(X,Y)} \to \Omega^p_{\mathrm{h}/(X,Z)} \to i_{Y*}\Omega^p_{\mathrm{h}/(Y,Z)} \to 0$$

where $i_Y : Y \to X$ is the closed immersion.

Proposition 3.2.11 ((Excision) [HJ] Proposition 7.28)**.** *Let $\pi : \tilde{X} \to X$ be a proper surjective morphism, which is an isomorphism outside of $Z \subset X$. Let $\tilde{Z} = \pi^{-1}(Z)$. Then*
$$H^q_{\mathrm{dR}}(\tilde{X}, \tilde{Z}) \cong H^q_{\mathrm{dR}}(X, Z) \ .$$

Remark 3.2.12. This is an immediate consequence of the blow-up triangle.

Proposition 3.2.13 ((Künneth formula) [HJ] Proposition 7.29)**.** *Let $Z \subset X$ and $Z' \subset X'$ be closed immersions. Then there is a natural isomorphism*

$$H^n_{\mathrm{dR}}(X \times X', X \times Z' \cup Z \times X') = \bigoplus_{a+b=n} H^a_{\mathrm{dR}}(X, Z) \otimes_k H^b_{\mathrm{dR}}(X', Z') \ .$$

Proof. We explain the construction of the map. We work in the category of h-sheaves of k-vector spaces on $X \times X'$. Note that h-cohomology of an h-sheaf of k-vector spaces computed in the category of sheaves of abelian groups agrees with its h-cohomology computed in the category of sheaves of k-vector spaces because an injective sheaf of k-vector spaces is also injective as sheaf of abelian groups.

We abbreviate $T = X \times Z' \cup Z \times X'$. By h-sheafification of the product of Kähler differentials we have a natural multiplication

$$\mathrm{pr}_X^*\Omega^a_{\mathrm{h}/X} \otimes_k \mathrm{pr}_{X'}^*\Omega^b_{\mathrm{h}/X'} \to \Omega^{a+b}_{\mathrm{h}/X\times X'} \ .$$

It induces, with $i_Z : Z \to X$, $i_{Z'} : Z' \to X'$, and $i : T \to X \times X'$

$$\mathrm{pr}_X^*\mathrm{Ker}(\Omega^a_{\mathrm{h}/X} \to i_{Z*}\Omega^a_{\mathrm{h}/Z})\otimes_k \mathrm{pr}_{X'}^*\mathrm{Ker}(\Omega^b_{\mathrm{h}/X'} \to i_{Z'*}\Omega^b_{\mathrm{h}/Z'}) \to \mathrm{Ker}(\Omega^{a+b}_{\mathrm{h}/X\times X'} \to i_*\Omega^{a+b}_{\mathrm{h}/T}) \ .$$

The resulting morphism

$$\mathrm{pr}_X^\bullet\Omega^*_{\mathrm{h}/(X,Z)} \otimes_k \mathrm{pr}_{X'}^\bullet\Omega^*_{\mathrm{h}/(X',Z')} \to \Omega^\bullet_{\mathrm{h}/(X\times X',T)} \ .$$

induces a natural Künneth morphism

$$\bigoplus_{a+b=n} H^a_{\mathrm{dR}}(X,Z) \otimes_k H^b_{\mathrm{dR}}(X',Z') \to H^n_{\mathrm{dR}}(X \times X', T) \ .$$

We refer to the proof of [HJ] Proposition 7.29 for the argument that this is an isomorphism. □

Lemma 3.2.14. *Let K/k be an extension of fields of characteristic zero. Let X be a variety over k and $Z \subset X$ a subvariety. Then there are natural isomorphisms*

$$H^i_{\mathrm{dR}}(X,Z) \otimes_k K \to H^i_{\mathrm{dR}}(X_K, Z_K) \ .$$

They are induced by a natural quasi-isomorphism

$$R\Gamma_{\mathrm{dR}}(X) \otimes_k K \to R\Gamma_{\mathrm{dR}}(X_K) \ .$$

Proof. Via the long exact cohomology sequence for pairs, and the long exact sequence for a blow-up, it suffices to consider the case when X is a single smooth variety, where it follows from Lemma 3.1.11. □

Lemma 3.2.15. *Let K/k be a finite extension of fields of characteristic 0. Let Y be variety over K and $W \subset Y$ a subvariety. We denote Y_k and W_k the same varieties when considered over k.*

Then there are natural isomorphisms

$$H^i_{\mathrm{dR}}(Y,W) \to H^i_{\mathrm{dR}}(Y_k, W_k) \ .$$

They are induced by a natural quasi-isomorphism

$$R\Gamma_{\mathrm{dR}}(Y) \to R\Gamma_{\mathrm{dR}}(Y_K) \ .$$

Proof. Note that if a variety is smooth over K, then it is also smooth when viewed over k.

The morphism on cohomology is induced by a morphism of sites from the category of k-varieties over Y to the category of K-varieties over k, both equipped with the h-topology. The pull-back of the de Rham complex over Y maps to the de Rham complex over Y_k. Via the long exact sequence for pairs and the blow-up sequence, it suffices to show the isomorphism for a single smooth Y. This was settled in Lemma 3.1.13. □

3.3 The general case: alternative approaches

We are now going to present a number of earlier definitions in the literature. They all give the same results in the cases where they are defined.

3.3.1 Deligne's method

We present the approach of Deligne in [D5]. A singular variety is replaced by a suitable simplicial variety whose terms are smooth.

3.3.2 Hypercovers

See Section 1.5 for basics on simplicial objects. In particular, we have the notion of an S-hypercover for a class of covering maps of varieties.

We will need two cases:

1. S is the class of open covers, i.e., $X = \coprod_{i=1}^n U_i$ with $U_i \subset Y$ open and such that $\bigcup_{i=1}^n U_i = Y$.
2. S the class of proper surjective maps.

Lemma 3.3.1. *Let $X \to Y$ be in S. We put*

$$X_\bullet = \mathrm{cosq}_0^Y X\ .$$

In explicit terms,

$$X_p = X \times_Y \cdots \times_Y X \quad (p+1 \text{ factors})$$

where we number the factors from 0 to p. The face map ∂_i is the projection forgetting the factor number i. The degeneration s_i is induced by the diagonal from the factor i into the factors i and $i+1$.

Then $X_\bullet \to Y$ is an S-hypercover.

Proof. By [SGA4.2] Exposé V, Proposition 7.1.2, the morphism

$$\mathrm{cosq}_0 \to \mathrm{cosq}_{n-1}\mathrm{sq}_{n-1}\mathrm{cosq}_0$$

is an isomorphism of functors for $n \geq 1$. (This follows directly from the adjunction properties of the coskeleton functor.) Hence the condition on X_n is satisfied trivially for $n \geq 1$. In degree 0 we consider

$$X_0 = X \to (\mathrm{cosq}_{-1}^Y \mathrm{sq}_{-1} \mathrm{cosq}_0^Y)_0 = Y\ .$$

By assumption, it is in S. □

It is worth spelling this out in complete detail.

Example 3.3.2. Let $X = \coprod_{i=1}^n U_i$ with $U_i \subset Y$ open. For $i_0, \dots, i_p \in \{1, \dots, n\}$ we abbreviate

$$U_{i_0,\dots,i_p} = U_{i_0} \cap \dots \cap U_{i_p} \ .$$

Then the open hypercover $X_\bullet$ is nothing but

$$X_p = \coprod_{i_0,\dots,i_p=0^n} U_{i_0,\dots,i_p}$$

with face and degeneracy maps given by the natural inclusions. Let $\mathcal{F}$ be a sheaf of abelian groups on X. Then the complex associated to the cosimplicial abelian group $\mathcal{F}(X_\bullet)$ is given by

$$\bigoplus_{i=1}^n \mathcal{F}(U_i) \to \bigoplus_{i_0,i_1=1}^n \mathcal{F}(U_{i_0,i_1}) \to \bigoplus_{i_0,i_1,i_2=1}^n \mathcal{F}(U_{i_0,i_1,i_2}) \to \dots$$

with differential

$$\delta^p(\alpha)_{i_0,\dots,i_p} = \sum_{j=0}^{p+1} \alpha_{i_0,\dots,\hat{i}_j,\dots,i_{p+1}}|_{U_{i_0,\dots,i_{p+1}}} \ ,$$

i.e., the differential of the Čech complex. Indeed, the natural projection

$$\mathcal{F}(X_\bullet) \to C^\bullet(\mathfrak{U}, \mathcal{F})$$

to the Čech complex (see Definition 1.4.12) is a quasi-isomorphism.

Definition 3.3.3. We say that $X_\bullet \to Y_\bullet$ is a smooth proper hypercover if it is a proper hypercover with all X_n smooth.

Example 3.3.4. Let $Y = Y_1 \cup \dots Y_n$ with $Y_i \subset Y$ closed. For $i_0, \dots, i_p = 1, \dots, n$ put

$$Y_{i_0,\dots,i_p} = Y_{i_0} \cap \dots Y_{i_p} \ .$$

Assume that all Y_i and all $Y_{i_0,\dots,i_p}$ are smooth.

Then $X = \coprod_{i=1}^n Y_i \to Y$ is proper and surjective. The proper hypercover $X_\bullet$ is nothing but

$$X_n = \coprod_{i_0,\dots,i_n=0^n} Y_{i_0} \cap \dots Y_{i_n}$$

with face and degeneracy maps given by the natural inclusions. Hence $X_\bullet \to Y$ is a smooth proper hypercover. As in the open case, the projection to Čech complex of the closed cover $\mathfrak{Y} = \{V_i\}_{i=1}^n$ is a quasi-isomorphism.

Proposition 3.3.5. *Let $Y_\bullet$ be a simplicial variety. Then the system of all proper hypercovers of $Y_\bullet$ is filtered up to simplicial homotopy. It is functorial in $Y_\bullet$. The subsystem of smooth proper hypercovers is cofinal.*

Proof. The first statement is [SGA4.2], Exposé V, Théorème 7.3.2. For the second assertion, it suffices to construct a smooth proper hypercover for any $Y_\bullet$. Recall that by Hironaka's resolution of singularities [Hi1], or by de Jong's theorem on alterations [dJ], we have for any variety Y a proper surjective map $X \to Y$ with X smooth. By the technique of [SGA4.2], Exposé Vbis, Proposition 5.1.3 (see also [D5] 6.2.5), this allows to construct $X_\bullet$. □

3.3.3 Definition of de Rham cohomology in the general case

Let again k be a field of characteristic 0.

Definition 3.3.6. Let X be a variety over k and $X_\bullet \to X$ a smooth proper hypercover. Let $C(X_\bullet) \in \mathbb{Z}\mathrm{Sm}$ be the associated complex We define *algebraic de Rham cohomology* of X by

$$H^i_{\mathrm{dR}}(X) = H^i\left(R\Gamma_{\mathrm{dR}}(X_\bullet)\right)$$

with $R\Gamma_{\mathrm{dR}}$ as in Definition 3.1.7. Let $D \subset X$ be a closed subvariety and $D_\bullet \to D$ a smooth proper hypercover such that there is a commutative diagram

$$\begin{array}{ccc} D_\bullet & \longrightarrow & X_\bullet \\ \downarrow & & \downarrow \\ D & \longrightarrow & X \end{array}$$

We define *relative algebraic de Rham cohomology* of the pair (X, D) by

$$H^i_{\mathrm{dR}}(X, D) = H^i\left(\mathrm{Cone}(R\Gamma(X_\bullet) \to R\Gamma(D_\bullet))[-1]\right) .$$

Proposition 3.3.7. *Algebraic de Rham cohomology is a well-defined functor, independent of the choice of hypercoverings of X and D.*

Remark 3.3.8. $R\Gamma_{\mathrm{dR}}$ defines a functor

$$\mathrm{Var} \to K^+(k\mathrm{-Vect})$$

but *not* to $C^+(k\mathrm{-Vect})$. Hence it does *not* extend directly to $C^b(\mathbb{Q}[\mathrm{Var}])$. We avoid addressing this point by the use of the h-topology instead.

Proof. This is a special case of descent for h-covers and hence a consequence of Proposition 3.2.4.

Alternatively, we can deduce if from the case of singular cohomology. Recall that algebraic de Rham cohomology is well-behaved with respect to extensions of the ground field. Without loss of generality, we may assume that k is finitely generated over $\mathbb{Q}$ and hence embeds into $\mathbb{C}$. Then we apply the period isomorphism of Definition 5.3.1. It remains to check the analogue for singular cohomology. This is Theorem 2.7.6. □

Example 3.3.9. Let X be a smooth affine variety and D a simple divisor with normal crossings. Let $D_1, \dots, D_n$ be the irreducible components. Let $X_\bullet$ be the constant simplicial variety X and $D_\bullet$ as in Example 3.3.4. Then algebraic de Rham cohomology D is computed by the total complex of the double complex ($D_{i_0,\dots,i_p}$ being the $(p+1)$-fold intersection of components)

$$K^{p,q} = \bigoplus_{i_0<\dots<i_p} \Omega^q_{D_{i_0,\dots,i_p}}\left(D_{i_0,\dots,i_p}\right)$$

with differential $d^{p,q} = \sum_{j=0}^{p}(-1)^j \partial_j^*$ the Čech differential and $\delta^{p,q}$ differentiation of differential forms.

Relative algebraic de Rham cohomology of (X, D) is computed by the total complex of the double complex

$$L^{p,q} = \begin{cases} K^{p-1,q} & p > 0, \\ \Omega^q_X(X) & p = 0. \end{cases}$$

Remark 3.3.10. Establishing the expected properties of relative algebraic de Rham cohomology is lengthy. Particularly complicated is the handling of the multiplicative structure which uses the the functor between complexes in $\mathbb{Z}[\mathrm{Sm}]$ and simplicial objects in $\mathbb{Z}[\mathrm{Sm}]$ and the product for simplicial objects. We do not go into the details but rely on the comparison with h-cohomology instead.

3.3.4 Hartshorne's method

We want to review Hartshorne's definition from [Ha1]. As before let k be a field of characteristic 0.

Definition 3.3.11. Let X be a smooth variety over k, $i : Y \subset X$ a closed subvariety. We define *algebraic de Rham cohomology* of Y as

$$H^i_{H-\mathrm{dR}}(Y) = H^i(\hat{X}, \hat{\Omega}^\bullet_X),$$

where $\hat{X}$ is the formal completion of X along Y and $\hat{\Omega}^\bullet_X$ the formal completion of the complex of algebraic differential forms on X.

Proposition 3.3.12 ([Ha1] Theorem (1.4))**.** *Let Y be as in Definition 3.3.11. Then $H^i_{H-\mathrm{dR}}(Y)$ is independent of the choice of X. In particular, if Y is smooth, the definition agrees with the one in Definition 3.1.2.*

Theorem 3.3.13. *The three definition of algebraic de Rham cohomology (Definition 3.3.6 via hypercovers, Definition 3.3.11 via embedding into smooth varieties, Definition 3.2.3 using the h-topology) agree.*

Proof. The comparison of $H^i_{H-\mathrm{dR}}(X)$ and $H^i_{\mathrm{dR}}(X_{\mathrm{eh}})$ is [Ge], Theorem 4.10. It agrees with $H^i_{\mathrm{dR}}(X_{\mathrm{h}})$ by [HJ], Proposition 6.1. By [HJ], Proposition 7.4 it agrees also with the definition via hypercovers. □

3.3.5 Using geometric motives

In Chapter 10 we are going to introduce the triangulated category of effective geometric motives $DM^{\mathrm{eff}}_{\mathrm{gm}}$ over k with coefficients in $\mathbb{Q}$. We only review the most important properties here and refer to Chapter 10 for more details. For technical reasons, it is easier to work with the affine version.

The objects in $DM^{\mathrm{eff}}_{\mathrm{gm}}$ are the same as the objects in $C^b(\mathrm{SmCor})$ where SmCor is the category of correspondences, see Section 1.1 and we denote SmCorAff the full subcategory with objects smooth affine varieties.

Lecomte and Wach in [LW] explain how to define an operation of correspondences on $\Omega^\bullet_X(X)$. We give a quick survey of their method.

For any normal variety Z let $\Omega^{p,**}_Z$ be the $\mathcal{O}_Z$-double dual of the sheaf of p-differentials. This is nothing but the sheaf of *reflexive differentials* on Z.

If $Z' \to Z$ is a finite morphism between normal varietes which is generically Galois with covering group G, then by [Kn]

$$\Omega^{p,**}_Z(Z) \cong \Omega^{p,**}_{Z'}(Z')^G \ .$$

Let X and Y be smooth affine varieties. Assume for simplicity that X and Y are connected. Let $\Gamma \in \mathrm{Cor}(X,Y)$ be a prime correspondence, i.e., $\Gamma \subset X \times Y$ an integral closed subvariety which is finite and dominant over X. Choose a finite $\tilde{\Gamma} \to \Gamma$ such that $\tilde{\Gamma}$ is normal and the covering $\tilde{\Gamma} \to X$ generically Galois with covering group G. In this case, $X = \tilde{\Gamma}/G$.

Definition 3.3.14. For a correspondence $\Gamma \in \mathrm{Cor}(X,Y)$ as above, we define

$$\Gamma^* : \Omega^\bullet_Y(Y) \to \Omega^\bullet_X(X)$$

as the composition

$$\Omega^\bullet_Y(Y) \to \Omega^\bullet_{\tilde{\Gamma}}(\tilde{\Gamma}) \to \Omega^{\bullet,**}_{\tilde{\Gamma}}(\tilde{\Gamma}) \xrightarrow{\frac{1}{|G|}\sum_{g\in G} g^*} \Omega^{\bullet,**}_{\tilde{\Gamma}}(\tilde{\Gamma})^G = \Omega^\bullet_X(X) \ .$$

This is well-defined and compatible with composition of correspondences. We can now define de Rham cohomology for complexes of correspondences.

Definition 3.3.15. Let $X^\bullet \in C^b(\mathrm{SmCorAff})$. We define

$$R\Gamma_{\mathrm{dR}}(X_\bullet) = \mathrm{Tot} R\Gamma_{\mathrm{dR}}(X_n)_{n\in\mathbb{Z}} \ .$$

and

$$H^i_{\mathrm{dR}}(X_\bullet) = H^i R\Gamma_{\mathrm{dR}}(X_\bullet) \ .$$

Note that there is a simple functor SmAff $\to$ SmCor. It assigns an object to itself and a morphism to its graph. This induces

$$i : C^b(\mathbb{Q}[\mathrm{SmAff}]) \to DM^{\mathrm{eff}}_{\mathrm{gm}} \ .$$

By construction,
$$f^* = \Gamma_f^* : \Omega_Y^\bullet(Y) \to \Omega_X^\bullet(X)$$
for any morphism $f : X \to Y$ between smooth affine varieties. Hence,
$$R\Gamma_{\mathrm{dR}}(X_\bullet) = R\Gamma_{\mathrm{dR}}(i(X_\bullet)),$$
where the left hand side was defined in Definition 3.1.7.

Proposition 3.3.16 (Voevodsky)**.** *The functor i extends naturally to a functor*
$$i : C^b(\mathbb{Q}[\mathrm{Var}]) \to DM_{\mathrm{gm}}^{\mathrm{eff}} .$$

Proof. The category of geometric motives constructed from affine varieties only agrees with the original $DM_{\mathrm{gm}}^{\mathrm{eff}}$. For details, see [Ha].

The extension to all varieties is a highly non-trivial result of Voevodsky. By [VSF], Chapter V, Corollary 4.1.4, there is functor
$$\mathrm{Var} \to DM_{\mathrm{gm}} .$$
Indeed, the functor
$$X \mapsto C_*L(X)$$
of loc. cit., Section 4.1, which assigns to every variety a homotopy invariant complex of Nisnevich sheaves, extends to $C^b(\mathbb{Z}[\mathrm{Var}])$ by taking total complexes. We consider it in the derived category of Nisnevich sheaves. Then the functor factors via the homotopy category $K^b(\mathbb{Z}[\mathrm{Var}])$.

By induction on the length of the complex, it follows from the result quoted above that $C_*L(\cdot)$ takes values in the full subcategory of geometric motives. □

Definition 3.3.17. Let $D \subset X$ be a closed immersion of varieties. We define
$$H^i_{\mathrm{dR}}(X, D) = H^i R\Gamma_{\mathrm{dR}}(i([D \to X]) ,$$
where $[D \to X] \in C^b(\mathbb{Z}[\mathrm{Var}])$ is concentrated in degrees -1 and 0.

Proposition 3.3.18. *This definition agrees with the one given in Definition 3.3.6.*

Proof. The easiest way to formulate the proof is to invoke another variant of the category of geometric motives. It does not need transfers, but imposes h-descent instead. Scholbach [Sch1, Definition 3.10] defines the category $DM_{\mathrm{gm},h}^{\mathrm{eff}}$ as the localization of $K^-(\mathbb{Q}[\mathrm{Var}])$ with respect to the triangulated subcategory generated by complexes of the form $X \times \mathbb{A}^1 \to X$ and h-hypercovers $X_\bullet \to X$ and closed under certain infinite sums. By definition of $DM_{\mathrm{gm},h}^{\mathrm{eff}}$, any hypercovering $X_\bullet \to X$ induces an isomorphism of the associated complexes in $DM_{\mathrm{gm},h}^{\mathrm{eff}}$. By resolution of singularities, any object of $DM_{\mathrm{gm},h}^{\mathrm{eff}}$ is isomorphic to an object where all components are smooth. Hence we can replace $K^-(\mathbb{Q}[\mathrm{Var}])$ by

$K^-(\mathbb{Q}[\mathrm{Sm}])$ in the definition without any change. We have seen how algebraic de Rham cohomology is defined on $K^-(\mathbb{Q}[\mathrm{Sm}])$. By homotopy invariance (Corollary 3.1.10) and h-descent of the de Rham complex (Proposition 3.3.7), the definition of algebraic de Rham cohomology factors via $DM^{\mathrm{eff}}_{\mathrm{gm},h}$.

This gives a definition of algebraic de Rham cohomology for $K^-(\mathbb{Q}[\mathrm{Var}])$ which by construction agrees with the one in Definition 3.3.6. On the other hand, the main result of [Sch1] is that $DM^{\mathrm{eff}}_{\mathrm{gm}}$ can be viewed as full subcategory of $DM^{\mathrm{eff}}_{\mathrm{gm},h}$. This inclusion maps the motive of a (possibly singular) variety to the motive of a variety. As the two definitions of algebraic de Rham cohomology of motives agree on motives of smooth varieties, they agree on all motives. □

3.3.6 The case of divisors with normal crossings

We are going to need the following technical result in order to give a simplified description of periods.

Proposition 3.3.19. *Let X be a smooth affine variety of dimension d and $D \subset X$ a simple divisor with normal crossings. Then every class in $H^d_{\mathrm{dR}}(X, D)$ is represented by some $\omega \in \Omega^d_X(X)$.*

The proof will be given at the end of this section.

Let $D = D_1 \cup \cdots \cup D_n$ be the decomposition into irreducible components. For $I \subset \{1, \dots, n\}$, let again

$$D_I = \bigcap_{i \in I} D_i \ .$$

Recall from Example 3.3.9 that the de Rham cohomology of (X, D) is computed by the total complex of

$$\Omega^\bullet_X(X) \to \bigoplus_{i=1}^n \Omega^\bullet_{D_i}(D_i) \to \bigoplus_{i<j} \Omega^\bullet_{D_{i,j}}(D_{i,j}) \to \cdots \to \Omega^\bullet_{D_{1,2,\dots,n}}(D_{1,2,\dots,n}) \ .$$

Note that D_I has dimension $d - |I|$, hence the double complex is concentrated in degrees $p, q \geq 0$, $p + q \leq d$. By definition, the classes in the top cohomology group $H^d_{\mathrm{dR}}(X, D)$ are presented by a tuple

$$(\omega_0, \omega_1, \dots, \omega_n) \quad \omega_0 \in \Omega^d_X(X), \omega_i \in \bigoplus_{|I|=i} \Omega^{d-i}_{D_I}(D_I) \ , i > 0 \ .$$

All such tuples are cocycles for dimension reasons. We have to show that, modulo coboundaries, we can assume $\omega_i = 0$ for all $i > 0$.

Lemma 3.3.20. *The maps*

$$\Omega^{d-1}_X(X) \to \bigoplus_{i=1}^n \Omega^{d-1}_{D_i}(D_i)$$

$$\bigoplus_{|I|=s} \Omega^{d-s-1}_{D_I}(D_I) \to \bigoplus_{|J|=s+1} \Omega^{d-s-1}_{D_J}(D_J)$$

are surjective.

Proof. X and all D_i are assumed affine, hence the global section functor is exact. It suffices to check the assertion for the corresponding sheaves on X and hence locally for the étale topology. By replacing X by an étale neighbourhood of a point, we can assume that there is a global system of regular paramters $t_1, \dots, t_d$ on X such that $D_i = \{t_i = 0\}$ for $i = 1, \dots, n$. First consider the case $s = 0$. The elements of $\Omega^{d-1}_{D_i}(D_i)$ are locally of the form $f_i dt_1 \wedge \dots \wedge d\dot{t}_i \wedge \dots \wedge t_d$ (omitting the factor at i). Again by replacing X by an open subvariety, we can assume they are globally of this shape. The forms can all be lifted to X.

$$\omega = \sum_{i=1}^{n} f_i dt_1 \wedge \dots \wedge d\dot{t}_i \wedge \dots \wedge t_d$$

is the preimage we were looking for.

For $s \geq 1$ we argue by induction on d and n. If $n = 1$, there is nothing to show. This settles the case $d = 1$. If $n > 0$, consider the decomposition

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\bigoplus_{|I|=s, I\subset\{1,\dots,n-1\}} \Omega^{d-s-1}_{D_I}(D_I) & \longrightarrow & \bigoplus_{|J|=s+1, J\subset\{1,\dots,n-1\}} \Omega^{d-s-1}_{D_J}(D_J) \\
\downarrow & & \downarrow \\
\bigoplus_{|I|=s, I\subset\{1,\dots,n\}} \Omega^{d-s-1}_{D_I}(D_I) & \longrightarrow & \bigoplus_{|J|=s+1, J\subset\{1,\dots,n\}} \Omega^{d-s-1}_{D_J}(D_J) \\
\downarrow & & \downarrow \\
\bigoplus_{|I|=s, I\subset\{1,\dots,n\}, n\in I} \Omega^{d-s-1}_{D_I}(D_I) & \longrightarrow & \bigoplus_{|J|=s+1, J\subset\{1,\dots,n\}, n\in J} \Omega^{d-s-1}_{D_J}(D_J) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

The arrow on the top is surjective by induction on n. The arrow on the bottom reproduces the assertion for X replaced by D_n and D replaced by $D_n \cap (D_1 \cup \dots \cup D_{n-1})$. By induction, it is surjective. Hence, the arrow in the middle is surjective. □

Proof of Proposition 3.3.19. Consider a cocycle $\omega = (\omega_0, \omega_1, \dots, \omega_n)$ as explained above. We argue by descending induction on the degree i. Consider $\omega_n \in \bigoplus_{|I|=n} \Omega^{d-n}_{D_I}(D_I)$. By the lemma, there is

$$\omega'_{n-1} \in \bigoplus_{|I|=n-1} \Omega^{d-n}_{D_I}(D_I)$$

such that $\partial\omega'_{n-1} = \omega_n$. We replace ω by $\omega \pm d\omega'_{n-1}$ (depending on the signs in the double complex). By construction, its component in degree n vanishes.

Hence, without loss of generality, we have $\omega_n = 0$. Next, consider ω_{n-1} etc. □

Chapter 4

Holomorphic de Rham cohomology

We are going to define a natural comparison isomorphism between de Rham cohomology and singular cohomology of varieties over the complex numbers. The link is provided by holomorphic de Rham cohomology which we study in this chapter.

4.1 Holomorphic de Rham cohomology

Everything we did in the algebraic setting also works for complex manifolds, indeed this is the older notion.

We write $\mathcal{O}_X^{\mathrm{hol}}$ for the sheaf of holomorphic functions on a complex manifold X.

4.1.1 Definition

Definition 4.1.1. Let X be a complex manifold. Let Ω^1_X be the sheaf of *holomorphic differentials* on X. For $p \geq 0$ let

$$\Omega^p_X = \Lambda^p \Omega^1_X$$

be the exterior power in the category of $\mathcal{O}_X^{\mathrm{hol}}$-modules and $(\Omega^\bullet_X, d)$ the *holomorphic de Rham complex.*

The differential is defined as in the algebraic case, see Definition 3.1.1.

Definition 4.1.2. Let X be a complex manifold. We define *holomorphic de Rham cohomology* of X as hypercohomology

$$H^i_{\mathrm{dR^{an}}}(X) = H^i(X, \Omega^\bullet_X) \ .$$

As in the algebraic case, de Rham cohomology is a contravariant functor. The exterior products induces a cup-product.

Proposition 4.1.3 (Poincaré lemma)**.** *Let X be a complex manifold. The natural map of sheaves $\mathbb{C} \to \mathcal{O}_X^{\mathrm{hol}}$ induces an isomorphism*

$$H^i_{\mathrm{sing}}(X, \mathbb{C}) \to H^i_{\mathrm{dR}^{\mathrm{an}}}(X) \ .$$

Proof. By Theorem 2.2.5, we can compute singular cohomology as sheaf cohomology on X. It remains to show that the complex

$$0 \to \mathbb{C} \to \mathcal{O}_X^{\mathrm{hol}} \to \Omega^1_X \to \Omega^2_X \to \dots$$

is exact. Let Δ be the unit ball in $\mathbb{C}$. The question is local, hence we may assume that $X = \Delta^d$. There is a natural isomorphism

$$\Omega^\bullet_{\Delta^d} \cong (\Omega^\bullet_\Delta)^{\otimes d}$$

Hence it suffices to treat the case $X = \Delta$. In this case we consider

$$0 \to \mathbb{C} \to \mathcal{O}^{\mathrm{hol}}(\Delta) \to \mathcal{O}^{\mathrm{hol}}(\Delta)dt \to 0 \ .$$

The elements of $\mathcal{O}^{\mathrm{hol}}(\Delta)$ are of the form $\sum_{i\geq 0} a_i t^i$ with radius of convergence 1. The differential has the form

$$\sum_{i\geq 0} a_i t^i \mapsto \sum_{i\geq 0} i a_i t^{i-1} dt \ .$$

The kernel is given by the constants. It is surjective because the antiderivative has the same radius of convergence as the original power series. □

Proposition 4.1.4 (Künneth formula)**.** *Let X, Y be complex manifolds. There is a natural multiplication induced from wedge product of differential forms*

$$H^i_{\mathrm{dR}}(X) \otimes_k H^j_{\mathrm{dR}}(Y) \to H^{i+j}_{\mathrm{dR}}(X \times Y) \ .$$

It induces an isomorphism

$$H^n_{\mathrm{dR}}(X \times Y) \cong \bigoplus_{i+j=n} H^i_{\mathrm{dR}}(X) \otimes_k H^j_{\mathrm{dR}}(Y) \ .$$

Proof. The construction of the morphism is the same as in the algebraic case, see Proposition 3.1.9. The quasi-isomorphism $\mathbb{C} \to \Omega^\bullet$ is compatible with the exterior products. Hence the isomorphism reduces to the Künneth isomorphism for singular cohomology, see Proposition 2.4.1. □

4.1.2 Holomorphic differentials with log poles

Let $j : X \to \bar{X}$ be a an open immersion of complex manifolds. Assume that $D = \bar{X} \smallsetminus X$ is a divisor with normal crossings, i.e., locally on $\bar{X}$ there is a coordinate system $(t_1, \dots, t_n)$ such that D is given as the set of zeroes of $t_1 t_2 \dots t_r$ with $0 \le r \le n$.

Definition 4.1.5. Let

$$\Omega^1_{\bar{X}}\langle D\rangle \subset j_*\Omega^1_X$$

be the locally free $\mathcal{O}_{\bar{X}}$-module with the following basis: if $U \subset X$ is an open with coordinates $t_1, \dots, t_n$ and $D|_U$ given by the equation $t_1 \dots t_r = 0$, then $\Omega^1_{\bar{X}}\langle D\rangle|_U$ has $\mathcal{O}^{\mathrm{hol}}_{\bar{X}}$-basis

$$\frac{dt_1}{t_1}, \dots, \frac{dt_r}{t_r}, dt_{r+1}, \dots, dt_n \ .$$

For $p > 1$ let

$$\Omega^p_{\bar{X}}\langle D\rangle = \Lambda^p \Omega^1_{\bar{X}}\langle D\rangle \ .$$

We call the $\Omega^\bullet_{\bar{X}}\langle D\rangle$ the *complex of differentials with log poles along D*.

Note that the differential of $j_*\Omega^\bullet_X$ respects $\Omega^\bullet_{\bar{X}}\langle D\rangle$, so that this is indeed a complex.

Proposition 4.1.6. *The inclusion induces a natural isomorphism*

$$H^i(\bar{X}, \Omega^\bullet_{\bar{X}}\langle D\rangle) \to H^i(X, \Omega^\bullet_X) \ .$$

This is [D4] Proposition 3.1.8. The algebraic analogue was treated in Proposition 3.1.16.

Proof. Note that $j : X \to \bar{X}$ is Stein, hence j_* is exact and we have

$$H^i(X, \Omega^\bullet_X) \cong H^i(\bar{X}, j_*\Omega^\bullet_X) \ .$$

It remains to show that

$$\iota : \Omega^\bullet_{\bar{X}}\langle D\rangle \to j_*\Omega^\bullet_X$$

is a quasi-isomorphism, or, equivalently, that $\mathrm{Coker}(\iota)$ is exact. The statement is local, hence we may assume that $\bar{X}$ is a coordinate ball and $D = V(t_1 \dots t_r)$. We consider the stalk in 0. The complexes are tensor products of the complexes in the 1-dimensional situation. Hence it suffices to consider the case $n = 1$. If $r = 0$, then there is nothing to show.

In remains to consider the following situation: let $\mathcal{O}^{\mathrm{hol}}$ be ring of germs of holomorphic functions at $0 \in \mathbb{C}$ and $\mathcal{K}^{\mathrm{hol}}$ the ring of germs of holomorphic functions with an isolated singularity at 0. The ring $\mathcal{O}^{\mathrm{hol}}$ is given by power series with a positive radius of convergence. The field $\mathcal{K}^{\mathrm{hol}}$ is given by Laurent series converging on some punctured neighborhood $\{t \mid 0 < t < \epsilon\}$. We have to check that the complex

$$\mathcal{K}^{\mathrm{hol}}/\mathcal{O}^{\mathrm{hol}} \to (\mathcal{K}^{\mathrm{hol}}/t^{-1}\mathcal{O}^{\mathrm{hol}})dt$$

is acyclic.

We pass to the principal parts. The differential has the form

$$\sum_{i>0} a_i t^{-i} \mapsto \sum_{i>0} (-i) a_i t^{-i-1}$$

It is obviously injective. For surjectivity, note that the antiderivative

$$\int : \sum_{i>1} b_i t^{-i} \mapsto \sum_{i>1} \frac{b_i}{-i+1} t^{-i+1}$$

maps convergent Laurent series to convergent Laurent series. □

4.1.3 GAGA

We work over the field of complex numbers.

An affine variety $X \subset \mathbb{A}^n_{\mathbb{C}}$ is also a closed set in the analytic topology on $\mathbb{C}^n$. If X is smooth, the associated analytic space X^{an} in the sense of Section 1.2.1 is a complex submanifold. As in loc. cit., we denote by

$$\alpha : (X^{\mathrm{an}}, \mathcal{O}^{\mathrm{hol}}_{X^{\mathrm{an}}}) \to (X, \mathcal{O}_X)$$

the map of locally ringed spaces. Note that any algebraic differential form is holomorphic, hence there is a natural morphism of complexes

$$\alpha^{-1}\Omega^\bullet_X \to \Omega^\bullet_{X^{\mathrm{an}}} \ .$$

It induces

$$\alpha^* : H^i_{\mathrm{dR}}(X) \to H^i_{\mathrm{dR}^{\mathrm{an}}}(X^{\mathrm{an}}) \ .$$

Proposition 4.1.7 (GAGA for de Rham cohomology)**.** *Let X be a smooth variety over $\mathbb{C}$. Then the natural map*

$$\alpha^* : H^i_{\mathrm{dR}}(X) \to H^i_{\mathrm{dR}^{\mathrm{an}}}(X^{\mathrm{an}})$$

is an isomorphism.

If X is smooth and projective, this is a standard consequence of Serre's comparison result for cohomology of coherent sheaves (GAGA). We need to extend this to the open case.

Proof. Let $j : X \to \bar{X}$ be a compactification such that $D = \bar{X} \smallsetminus X$ is a simple divisor with normal crossings. The change of topology map α also induces

$$\alpha^{-1} j_* \Omega^\bullet_X \to j^{\mathrm{an}}_* \Omega^\bullet_{X^{\mathrm{an}}}$$

which respects differential with log-poles

$$\alpha^{-1} \Omega_{\bar{X}^\bullet} \langle D \rangle \to j^{\mathrm{an}}_* \Omega^\bullet_{\bar{X}^{\mathrm{an}}} \langle D^{\mathrm{an}} \rangle \ .$$

Hence we get a commutative diagram

$$\begin{array}{ccc} H^i_{\mathrm{dR}}(X) & \longrightarrow & H^i_{\mathrm{dR}^{\mathrm{an}}}(X^{\mathrm{an}}) \\ \uparrow & & \uparrow \\ H^i(\bar{X}, \Omega^\bullet_{\bar{X}} \langle D \rangle) & \longrightarrow & H^i(\bar{X}^{\mathrm{an}}, \Omega^\bullet_{\bar{X}^{\mathrm{an}}} \langle D^{\mathrm{an}} \rangle) \end{array}$$

By Proposition 3.1.16 in the algebraic, and Proposition 4.1.6 in the holomorphic case, the vertical maps are isomorphism. By considering the Hodge to de Rham spectral sequence (attached to the stupid filtration on $\Omega^\bullet_{\bar{X}} \langle D \rangle$), it suffices to show that

$$H^p(\bar{X}, \Omega^q_{\bar{X}} \langle D \rangle) \to H^p(\bar{X}^{\mathrm{an}}, \Omega^q_{\bar{X}^{\mathrm{an}}} \langle D^{\mathrm{an}} \rangle)$$

is an isomorphism for all p, q. Note that $\bar{X}$ is smooth, projective and $\Omega^q_{\bar{X}} \langle D \rangle$ is coherent. Its analytification $\alpha^{-1} \Omega^q_{\bar{X}} \langle D \rangle \otimes_{\alpha^{-1} \mathcal{O}_{\bar{X}}} \mathcal{O}^{\mathrm{hol}}_{\bar{X}^{\mathrm{an}}}$ is nothing but $\Omega^q_{\bar{X}^{\mathrm{an}}} \langle D^{\mathrm{an}} \rangle$. By GAGA [Se1], we have an isomorphism in cohomology. □

4.2 De Rham cohomology via the h′-topology

We address the singular case via the h′-topology on (An/X) introduced in Definition 2.7.1.

4.2.1 h′-differentials

Definition 4.2.1. Let $\Omega^p_{\mathrm{h}'}$ be the h′-sheafification of the presheaf

$$Y \mapsto \Omega^p_Y(Y)$$

on the category of complex analytic spaces An.

Theorem 4.2.2 (Jörder [Joe])**.** *Let X be a complex manifold. Then*

$$\Omega^p_X(X) = \Omega^p_{\mathrm{h}'}(X) \ .$$

Proof. Jörder defines in [Joe, Definition 1.4.1] what he calls h-differentials Ω^p_{h} as the presheaf pull-back of Ω^p from the category of manifolds to the category of complex analytic spaces. (There is no mention of a topology in loc.cit.) In

[Joe, Proposition 1.4.2 (4)] he establishes that $\Omega^p_{\mathrm{h}}(X) = \Omega^p_X(X)$ in the smooth case. It remains to show that $\Omega^p_{\mathrm{h}} = \Omega^p_{\mathrm{h}'}$. By resolution of singularities, every X is smooth locally for the h′-topology. Hence it suffices to show that Ω^p_{h} is an h′-sheaf. By [Joe, Lemma 1.4.5], the sheaf condition is satisfied for proper covers. The sheaf condition for open covers is satisfied because already Ω^p_X is a sheaf in the ordinary topology. □

Lemma 4.2.3 (Poincaré lemma)**.** *Let X be a complex analytic space. Then the complex*

$$\mathbb{C}_{\mathrm{h}'} \to \Omega^\bullet_{\mathrm{h}'}$$

of h′*-sheaves on* $(\mathrm{An}/X)_{\mathrm{h}'}$ *is exact.*

Proof. We may check this locally in the h′-topology. By resolution of singuarities it suffices to consider sections over some Y which is smooth and even an open ball in $\mathbb{C}^n$. By Theorem 4.2.2 the complex reads

$$\mathbb{C} \to \Omega^\bullet_Y(Y) \ .$$

By the ordinary holomorphic Poincaré Lemma 4.1.3, it is exact. □

Remark 4.2.4. The main topic of Jörder's thesis [Joe] is to treat the question of a Poincaré Lemma for h′-forms with respect to the usual topology. This is more subtle and fails in general.

4.2.2 De Rham cohomology

We now turn to de Rham cohomology.

Definition 4.2.5. Let X be a complex analytic space.

1. We define h′*-de Rham cohomology* as hypercohomology

$$H^i_{\mathrm{dR}^{\mathrm{an}}}(X_{\mathrm{h}'}) = H^i_{\mathrm{h}'}((\mathrm{Sch}/X)_{\mathrm{h}'}, \Omega^\bullet_{\mathrm{h}'}) \ .$$

2. Let $i : Z \to X$ a closed subspace. Put

$$\Omega^p_{\mathrm{h}/(X,Z)} = \mathrm{Ker}(\Omega^p_{\mathrm{h}/X} \to i_*\Omega^p_{\mathrm{h}/Z})$$

in the category of abelian sheaves on $(\mathrm{An}/X)_{\mathrm{h}'}$.

We define *relative* h′*-de Rham cohomology* as

$$H^p_{\mathrm{dR}^{\mathrm{an}}}(X_{\mathrm{h}'}, Z_{\mathrm{h}'}) = H^p_{\mathrm{h}'}((\mathrm{An}/X)_{\mathrm{h}'}, \Omega^*_{\mathrm{h}/(X,Z)}) \ .$$

Lemma 4.2.6. *The properties (long exact sequence, excision, Künneth formula) of relative algebraic H-de Rham cohomology (see Section 3.2) are also satisfied in relative* h′*-de Rham cohomology.*

Proof. The proofs are the same as Section 3.2, respectively in [HJ, Section 7.3]. The proof relies on the computation of $\Omega^p_{\mathrm{h}'}(D)$ when D is a normal crossings space. Indeed, the same argument as in the proof of [HJ, Proposition 4.9] shows that

$$\Omega^p_{\mathrm{h}'}(D) = \Omega^p_D(D)/\mathrm{torsion} \ .$$

□

As in the previous case, exterior multiplication of differential forms induces a product structure on h′-de Rham cohomology.

Corollary 4.2.7. *For all $X \in \mathrm{An}$ and closed immersions $i : Z \to X$ the inclusion of the Poincaré lemma induces a natural isomorphism*

$$H^i_{\mathrm{sing}}(X, Z, \mathbb{C}) \to H^i_{\mathrm{dR^{an}}}(X_{\mathrm{h}'}, Z_{\mathrm{h}'}) \ ,$$

compatible with long exact sequences and multiplication. Moreover, the natural map

$$H^i_{\mathrm{dR^{an}}}(X_{\mathrm{h}'}) \to H^i_{\mathrm{dR^{an}}}(X)$$

is an isomorphism if X is smooth.

Proof. By the Poincaré Lemma 4.2.3, we have a natural isomorphism

$$H^i_{\mathrm{h}'}(X_{\mathrm{h}'}, Z_{\mathrm{h}'}, \mathbb{C}_{\mathrm{h}'}) \to H^i_{\mathrm{dR^{an}}}(X_{\mathrm{h}'}, Z_{\mathrm{h}'}) \ .$$

We combine it with the comparison isomorphism with singular cohomology of Proposition 2.7.4.

The second statement holds because both compute singular cohomology by Prop. 2.7.4 and Prop. 4.1.3. □

4.2.3 GAGA

We work over the base field $\mathbb{C}$. As before we consider the analytification functor

$$X \mapsto X^{\mathrm{an}}$$

which takes a separated scheme of finite type over $\mathbb{C}$ to a complex analytic space. We recall the map of locally ringed spaces

$$\alpha : X^{\mathrm{an}} \to X \ .$$

We want to view it as a morphism of topoi

$$\alpha : (\mathrm{An}/X^{\mathrm{an}})_{\mathrm{h}'} \to (\mathrm{Sch}/X)_{\mathrm{h}} \ .$$

Definition 4.2.8. Let $X \in \mathrm{Sch}/\mathbb{C}$. We define the h′-topology on the category $(\mathrm{Sch}/X)_{\mathrm{h}'}$ to be the smallest Grothendieck topology such that the following are covering maps:

1. proper surjective morphisms;
2. open covers.

If $\mathcal{F}$ is a presheaf of An/X, we denote by $\mathcal{F}_{\mathrm{h}'}$ its sheafification in the h′-topology.

Lemma 4.2.9. *1. The morphism of sites* $(\mathrm{Sch}/X)_{\mathrm{h}} \to (\mathrm{Sch}/X)_{\mathrm{h}'}$ *induces an isomorphism on the categories of sheaves.*

2. The analytification functor induces a morphism of sites

$$(\mathrm{An}/X^{\mathrm{an}})_{\mathrm{h}'} \to (\mathrm{Sch}/X)_{\mathrm{h}'} .$$

Proof. By [Voe] Theorem 3.1.9 any h-cover can be refined by a cover in normal form which is a composition of open immersions followed by proper maps. This shows the first assertion. The second is clear by construction. □

By h′-sheafifiying, the natural morphism of complexes

$$\alpha^{-1}\Omega^\bullet_X \to \Omega^\bullet_{X^{\mathrm{an}}}$$

of Section 4.1.3, we also obtain

$$\alpha^{-1}\Omega^\bullet_{\mathrm{h}} \to \Omega^\bullet_{\mathrm{h}'}$$

on $(\mathrm{An}/X^{\mathrm{an}})_{\mathrm{h}'}$. It induces

$$\alpha^* : H^i_{\mathrm{dR}}(X_{\mathrm{h}}) \to H^i_{\mathrm{dR}^{\mathrm{an}}}(X^{\mathrm{an}}_{\mathrm{h}'}) .$$

Proposition 4.2.10 (GAGA for h′-de Rham cohomology)**.** *Let X be a variety over $\mathbb{C}$ and Z a closed subvariety. Then the natural map*

$$\alpha^* : H^i_{\mathrm{dR}}(X_{\mathrm{h}}, Z_{\mathrm{h}}) \to H^i_{\mathrm{dR}^{\mathrm{an}}}(X^{\mathrm{an}}_{\mathrm{h}'}, Z^{\mathrm{an}}_{\mathrm{h}'})$$

is an isomorphism. It is compatible with long exact sequences and products.

Proof. By naturality, the comparison morphism is compatible with long exact sequences. Hence it suffices to consider the absolute case.

Let $X_\bullet \to X$ be a smooth proper hypercover. This is a cover in h′-topology, hence we may replace X by $X_\bullet$ on both sides. As all components of $X_\bullet$ are smooth, we may replace h-cohomology by Zariski-cohomology in the algebraic setting (see Proposition 3.2.4). On the analytic side, we may replace h′-cohomology by ordinary sheaf cohomology (see Corollary 2.7.4). The statement then follows from the comparison in the smooth case, see Proposition 4.1.7. □

Chapter 5

The period isomorphism

The aim of this section is to define well-behaved isomorphisms between singular and de Rham cohomology of algebraic varieties.

5.1 The category $(k, \mathbb{Q})-\mathrm{Vect}$

We introduce a simple linear algebra category which will later allow to formalize the notion of periods. Throughout, let $k \subset \mathbb{C}$ be a subfield.

Definition 5.1.1. Let $(k, \mathbb{Q})-\mathrm{Vect}$ be the category of triples $(V_k, V_\mathbb{Q}, \phi_\mathbb{C})$ where V_k is a finite dimensional k-vector space, $V_\mathbb{Q}$ a finite dimensional $\mathbb{Q}$-vector space and

$$\phi_\mathbb{C} : V_k \otimes_k \mathbb{C} \to V_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{C}$$

a $\mathbb{C}$-linear isomorphism. Morphisms in $(k, \mathbb{Q})-\mathrm{Vect}$ are linear maps on V_k and $V_\mathbb{Q}$ compatible with comparison isomorphisms.

Note that $(k, \mathbb{Q})-\mathrm{Vect}$ is a $\mathbb{Q}$-linear additive tensor category with the obvious notion of tensor product. It is rigid, i.e., all objects have strong duals. It is even Tannakian with projection to the $\mathbb{Q}$-component as fibre functor.

For later use, we make the duality explicit:

Remark 5.1.2. Let $V = (V_k, V_\mathbb{Q}, \phi_\mathbb{C}) \in (k, \mathbb{Q})-\mathrm{Vect}$. The the dual $V^\vee$ is given by

$$V^\vee = (V_k^*, V_\mathbb{Q}^*, (\phi^*)^{-1}$$

where $\cdot^*$ denotes the vector space dual over k and $\mathbb{Q}$ or $\mathbb{C}$. Note that the inverse is needed in order to make the map go in the right direction.

Remark 5.1.3. The above is a simplification of the category of mixed Hodge structures introduced by Deligne, see [D4]. It does not take the weight and Hodge filtration into account. In other words: there is a faithful forgetful functor from mixed Hodge structures over k to $(k, \mathbb{Q})-\mathrm{Vect}$.

Example 5.1.4. The invertible objects are those where $\dim_k V_k = \dim_{\mathbb{Q}} V_{\mathbb{Q}} = 1$. Up to isomorphism they are of the form

$$L(\alpha) = (k, \mathbb{Q}, \alpha) \text{ with } \alpha \in \mathbb{C}^* \ .$$

5.2 A triangulated category

We introduce a triangulated category with a t-structure whose heart is $(k, \mathbb{Q})-\mathrm{Vect}$.

Definition 5.2.1. A *cohomological* $(k, \mathbb{Q})-\mathrm{Vect}$-*complex* consists of the following data:

- a bounded below complex $K_k^\bullet$ of k-vector spaces with finite dimensional cohomology;
- a bounded below complex $K_{\mathbb{Q}}^\bullet$ of $\mathbb{Q}$-vector spaces with finite dimensional cohomology;
- a bounded below complex $K_{\mathbb{C}}^\bullet$ of $\mathbb{C}$-vector spaces with finite dimensional cohomology;
- a quasi-isomorphism $\phi_{k,\mathbb{C}} : K_k^\bullet \otimes_k \mathbb{C} \to K_{\mathbb{C}}^\bullet$;
- a quasi-isomorphism $\phi_{\mathbb{Q},\mathbb{C}} : K_{\mathbb{Q}}^\bullet \otimes_{\mathbb{Q}} \mathbb{C} \to K_{\mathbb{C}}^\bullet$.

Morphisms of cohomological $(k, \mathbb{Q})-\mathrm{Vect}$-complexes are given by a pair of morphisms of complexes on the k-, $\mathbb{Q}$- and $\mathbb{C}$-component such that the obvious diagram commutes. We denote the category of cohomological $(k, \mathbb{Q})-\mathrm{Vect}$-complexes by $C^+_{(k,\mathbb{Q})}$.

Let K and L be objects of $C^+_{(k,\mathbb{Q})}$. A *homotopy* between K and L is a homotopy in the k-, $\mathbb{Q}$- and $\mathbb{C}$-component compatible under the comparison maps. Two morphisms in $C^+_{(k,\mathbb{Q})}$ are *homotopic* if they differ by a homotopy. We denote by $K^+_{(k,\mathbb{Q})}$ the *homotopy category of cohomological* $(k, \mathbb{Q})-\mathrm{Vect}$-*complexes.*

A morphism in $K^+_{(k,\mathbb{Q})}$ is called *quasi-isomorphism* if its k-, $\mathbb{Q}$-, and $\mathbb{C}$-components are quasi-isomorphisms. We denote by $D^+_{(k,\mathbb{Q})}$ the localization of $K^+_{(k,\mathbb{Q})}$ with respect to quasi-isomorphisms. It is called the *derived category of cohomological* $(k, \mathbb{Q})-\mathrm{Vect}$-*complexes.*

Remark 5.2.2. This is a simplification of the category of mixed Hodge complexes introduced by Beilinson [Be2]. A systematic study of this type of category can be found in [Hu1, §4]. In the language of loc.cit., it is the rigid glued category of the category of k-vector spaces and the category of $\mathbb{Q}$-vector spaces via the category of $\mathbb{C}$-vector spaces and the extension of scalars functors. Note that they are exact, hence the construction simplifies.

Lemma 5.2.3. *$D^+_{(k,\mathbb{Q})}$ is a triangulated category. It has a natural t-structure with*

$$H^i : D^+_{(k,\mathbb{Q})} \to (k,\mathbb{Q})-\mathrm{Vect}$$

defined componentwise. The heart of the t-structure is $(k,\mathbb{Q})-\mathrm{Vect}$.

Proof. This is straightforward. For more details see [Hu1, §4]. □

Remark 5.2.4. In [Hu1, 4.2, 4.3], explicit formulas are given for the morphisms in $D^+_{(k,\mathbb{Q})}$. The category has cohomological dimension 1. For $K, L \in (k,\mathbb{Q})-\mathrm{Vect}$, the group $\mathrm{Hom}_{D^+_{(k,\mathbb{Q})}}(K, L[1])$ is equal to the group of Yoneda extensions. As in [Be2], this implies that $D^+_{(k,\mathbb{Q})}$ is equivalent to the bounded derived category $D^+((k,\mathbb{Q})-\mathrm{Vect})$. We do not spell out the details because we are not going to need these properties.

There is an obvious definition of a tensor product on $C^+_{(k,\mathbb{Q})}$. Let $K^\bullet, L^\bullet \in C^+_{(k,\mathbb{Q})}$. We define $K^\bullet \otimes L^\bullet$ with $k, \mathbb{Q}, \mathbb{C}$-component given by the tensor product of complexes of vector spaces over k, $\mathbb{Q}$, and $\mathbb{C}$, respectively (see Example 1.3.4). Tensor product of two quasi-isomorphisms defines the comparison isomorphism on the tensor product.

It is associative and commutative. Note that the

Lemma 5.2.5. *$C^+_{(k,\mathbb{Q})}$, $K^+_{(k,\mathbb{Q})}$ and $D^+_{(k,\mathbb{Q})}$ are associative and commutative tensor categories with the above tensor product. The cohomology functor H^* commutes with $\otimes$. For $K^\bullet, L^\bullet$ in $D^+_{(k,\mathbb{Q})}$, we have a natural isomorphism*

$$H^*(K^\bullet) \otimes H^*(L^\bullet) \to H^*(K^\bullet \otimes L^\bullet).$$

It is compatible with the associativity constraint. It is compatible with the commutativity constraint up to the sign $(-1)^{pq}$ on $H^p(K^\bullet) \otimes H^q(L^\bullet)$.

Proof. The case of $D^+_{(k,\mathbb{Q})}$ follows immediately from the case of complexes of vector spaces, where it is well-known. The signs come from the signs in the total complex of a bicomplex, in this case, tensor product of complexes, see Section 1.3.3. □

Remark 5.2.6. This is again simpler than the case treated in [Hu1, Chapter 13], because we do not need to control filtrations and because our tensor products are exact.

5.3 The period isomorphism in the smooth case

Let k be a subfield of $\mathbb{C}$. We consider smooth varieties over k and the complex manifold X^{an} associated to $X \times_k \mathbb{C}$.

Definition 5.3.1. Let X be a smooth variety over k. We define the *period isomorphism*

$$\mathrm{per}: H^\bullet_{\mathrm{dR}}(X)\otimes_k\mathbb{C}\to H^\bullet_{\mathrm{sing}}(X,\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C}$$

to be the isomorphism given by the composition of the isomorphisms

1. $H^\bullet_{\mathrm{dR}}(X)\otimes_k\mathbb{C}\to H^\bullet_{\mathrm{dR}}(X\times_k\mathbb{C})$ of Lemma 3.1.11,
2. $H^\bullet_{\mathrm{dR}}(X\times_k\mathbb{C})\to H^\bullet_{\mathrm{dR}^{\mathrm{an}}}(X^{\mathrm{an}})$ of Proposition 4.1.7,
3. the inverse of $H^\bullet_{\mathrm{dR}^{\mathrm{an}}}(X^{\mathrm{an}})\to H^\bullet_{\mathrm{sing}}(X^{\mathrm{an}},\mathbb{C})$ of Proposition 4.1.3,
4. the inverse of the change of coefficients isomorphism $H^\bullet_{\mathrm{sing}}(X^{\mathrm{an}},\mathbb{C})\to H^\bullet_{\mathrm{sing}}(X^{\mathrm{an}},\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C}$.

We define the *period pairing*

$$\mathrm{per}: H^\bullet_{\mathrm{dR}}(X)\times H_\bullet^{\mathrm{sing}}(X^{\mathrm{an}},\mathbb{Q})\to\mathbb{C}$$

to be the map

$$(\omega,\gamma)\mapsto\gamma(\mathrm{per}(\omega))$$

where we view classes in singular homology as linear forms on singular cohomology.

Recall the category $(k,\mathbb{Q})-\mathrm{Vect}$ introduced in Section 5.1.

Lemma 5.3.2. *The assignment*

$$X\mapsto(H^\bullet_{\mathrm{dR}}(X),H^\bullet_{\mathrm{sing}}(X),\mathrm{per})$$

defines a functor

$$\mathrm{H}:\mathrm{Sm}\to(k,\mathbb{Q})-\mathrm{Vect}\ .$$

For all $X,Y\in\mathrm{Sm}$, the Künneth isomorphism induces an natural isomorphism

$$\mathrm{H}(X)\otimes\mathrm{H}(Y)\to\mathrm{H}(X\times Y)\ .$$

The image of H *is closed under direct sums and tensor product.*

Proof. Functoriality holds by construction. The Künneth morphism is induced from the Künneth isomorphism in singular cohomology (Proposition 2.4.1) and algebraic de Rham cohomology (see Proposition 3.1.9). All identifications in Definition 5.3.1 are compatible with the product structure. Hence we have defined a Künneth morphism in H. It is an isomorphism because it is an isomorphism in singular cohomology.

The direct sum realized by the disjoint union. The tensor product is realized by the product. □

In Chapter 9, we are going to study systematically the periods of the objects in H(Sm).

The period isomorphism has an explicit description in terms of integration.

Theorem 5.3.3. *Let X be a smooth affine variety over k and $\omega \in \Omega^i(X)$ a closed differential form with de Rham class $[\omega]$. Let $c \in H_d^{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{Q})$ be a singular homology class. Let $\sum a_j\gamma_j$ with $a_j \in \mathbb{Q}$ and $\gamma_j : \Delta_i \to X^{\mathrm{an}}$ differentiable singular cycles as in Definition 2.2.2. Then*

$$\mathrm{per}([\omega], c) = \sum a_j \int_{\Delta_i} \gamma^*(\omega) \ .$$

Remark 5.3.4. We could use the above formula as a definition of the period pairing, at least in the affine case. By Stokes' theorem, the value only depends on the class of ω.

Proof. Let $A^i(X^{\mathrm{an}})$ be group of $\mathbb{C}$-valued C^∞-differential forms and $\mathcal{A}^i_{X^{\mathrm{an}}}$ the associated sheaf. By the Poincaré lemma and its C^∞-analogue the morphisms

$$\mathbb{C} \to \Omega^\bullet_{X^{\mathrm{an}}} \to \mathcal{A}^\bullet_{X^{\mathrm{an}}}$$

are quasi-isomorphism. It induces a quasi-isomorphism

$$\Omega^\bullet_{X^{\mathrm{an}}}(X^{\mathrm{an}}) \to A^\bullet(X^{\mathrm{an}})$$

because both compute singular cohomology in the affine case. Hence it suffices to view ω as a C^∞-differential form. By the Theorem of de Rham, see [Wa], Sections 5.34-5.36, the period isomorphism is realized by integration over simplices. □

Example 5.3.5. For $X = \mathbb{P}^n_k$, we have

$$\mathrm{H}^{2j}(\mathbb{P}^n_k) = L((2\pi i)^j)$$

with $L(\alpha)$ the invertible object of Example 5.1.4.

5.4 The general case (via the h′-topology)

We generalize the period isomorphism to relative cohomology of arbitrary varieties.

Let k be a subfield of $\mathbb{C}$. We consider varieties over k and the complex analytic space X^{an} associated to $X \times_k \mathbb{C}$.

Definition 5.4.1. Let X be a variety over k, and $Z \subset X$ a closed subvariety. We define the *period isomorphism*

$$\mathrm{per} : H^\bullet_{\mathrm{dR}}(X, Z) \otimes_k \mathbb{C} \to H^\bullet_{\mathrm{sing}}(X, Z, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$$

to be the isomorphism given by the composition of the isomorphisms

1. $H^\bullet_{\mathrm{dR}}(X,Z)\otimes_k\mathbb{C}\to H^\bullet_{\mathrm{dR}}(X\times_k\mathbb{C},Z\times_k\mathbb{C})$ of Lemma 3.2.14,
2. $H^\bullet_{\mathrm{dR}}(X\times_k\mathbb{C},Z\times_k\mathbb{C})\to H^\bullet_{\mathrm{dR}^{\mathrm{an}}}(X^{\mathrm{an}}_{\mathrm{h}'},Z^{\mathrm{an}}_{\mathrm{h}'})$ of Proposition 4.2.10,
3. the inverse of $H^\bullet_{\mathrm{dR}^{\mathrm{an}}}(X^{\mathrm{an}}_{\mathrm{h}'},Z^{\mathrm{an}}_{\mathrm{h}'})\to H^\bullet_{\mathrm{sing}}(X^{\mathrm{an}},Z^{\mathrm{an}},\mathbb{C})$ of Corollary 4.2.7,
4. the inverse of the change of coefficients isomorphism $H^\bullet_{\mathrm{sing}}(X^{\mathrm{an}},Z^{\mathrm{an}},\mathbb{C})\to H^\bullet_{\mathrm{sing}}(X^{\mathrm{an}},Z^{\mathrm{an}},\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C}$.

We define the *period pairing*

$$\mathrm{per}:H^\bullet_{\mathrm{dR}}(X,Z)\times H_\bullet^{\mathrm{sing}}(X^{\mathrm{an}},Z^{\mathrm{an}},\mathbb{Q})\to\mathbb{C}$$

to be the map

$$(\omega,\gamma)\mapsto\gamma(\mathrm{per}(\omega)),$$

where we view classes in singular homology as linear forms on singular cohomology.

Lemma 5.4.2. *The assignment*

$$(X,Z)\mapsto(H^\bullet_{\mathrm{dR}}(X,Z),H^\bullet_{\mathrm{sing}}(X,Z),\mathrm{per})$$

defines a functor denoted H *on the category of pairs* $X\supset Z$ *with values in* $(k,\mathbb{Q})-\mathrm{Vect}$*. For all* $Z\subset Z$*,* $Z'\subset X'$*, the Künneth isomorphism induces a natural isomorphism*

$$\mathrm{H}(X,Z)\otimes\mathrm{H}(X',Z')\to\mathrm{H}(X\times X',X\times Z'\cup Z\times X')\ .$$

The image of H *is closed under direct sums and tensor product.*

If $Z\subset Y\subset X$ *is a triple, the there is a induced long exact sequence in* $(k,\mathbb{Q})-\mathrm{Vect}$*.*

$$\dots\to H^i(X,Y)\to H^i(X,Z)\to H^i(Y,Z)\xrightarrow{\partial}H^{i+1}(X,Y)\to\dots\ .$$

Proof. Functoriality and compatibility with long exact sequences holds by construction. The Künneth morphism is induced from the Künneth isomorphism in singular cohomology (Proposition 2.4.1) and algebraic de Rham cohomology (see Proposition 3.1.9). All identifications in Definition 5.3.1 are compatible with the product structure. Hence we have defined a Künneth morphism in H. It is an isomorphism because it is an isomorphism in singular cohomology.

The direct sum realized by the disjoint union. The tensor product is realized by the product. □

5.5 The general case (Deligne's method)

We generalize the period isomorphism to relative cohomology of arbitrary varieties.

Let k be a subfield of $\mathbb{C}$.

Recall from Section 3.1.2 the functor

$$R\Gamma_{\mathrm{dR}} : \mathbb{Z}[\mathrm{Sm}] \to C^+(k\text{−Mod})$$

which maps a smooth variety to a natural complex computing its de Rham cohomology. In the same way, we define using the Godement resolution (see Definition 1.4.8)

$$R\Gamma_{\mathrm{sing}}(X) = \Gamma(X^{\mathrm{an}}, Gd(\mathbb{Q})) \in C^+(\mathbb{Q}\text{−Mod})$$

a complex computing singular cohomology of X^{an}. Moreover, let

$$R\Gamma_{\mathrm{dR}^{\mathrm{an}}}(X) = \Gamma(X^{\mathrm{an}}, Gd(\Omega^\bullet_{X^{\mathrm{an}}}) \in C^+(\mathbb{C}\text{−Mod})$$

be a complex computing holomorphic de Rham cohomology of X^{an}.

Lemma 5.5.1. *Let X be a smooth variety over k.*

1. *As before let $\alpha : X^{\mathrm{an}} \to X \times_k \mathbb{C}$ be the morphism of locally ringed spaces and $\beta : X \times_k \mathbb{C} \to X$ the natural map. The inclusion $\alpha^{-1}\beta^{-1}\Omega^\bullet_X \to \Omega^\bullet_{X^{\mathrm{an}}}$ induces a natural quasi-isomorphism of complexes*

$$\phi_{\mathrm{dR},\mathrm{dR}^{\mathrm{an}}} : R\Gamma_{\mathrm{dR}}(X) \otimes_k \mathbb{C} \to R\Gamma_{\mathrm{dR}^{\mathrm{an}}}(X) \ .$$

2. *The inclusion $\mathbb{Q} \to \Omega^\bullet_{X^{\mathrm{an}}}$ induces a natural quasi-isomorphism of complexes*

$$\phi_{\mathrm{sing},\mathrm{dR}^{\mathrm{an}}} : R\Gamma_{\mathrm{sing}}(X) \otimes_{\mathbb{Q}} \mathbb{C} \to R\Gamma_{\mathrm{dR}^{\mathrm{an}}}(X) \ .$$

3. *We have*

$$\mathrm{per} = H^\bullet(\phi_{\mathrm{sing},\mathrm{dR}^{\mathrm{an}}})^{-1} \circ H^\bullet(\phi_{\mathrm{sing},\mathrm{dR}^{\mathrm{an}}}) : H^\bullet_{\mathrm{dR}}(X) \otimes_k \mathbb{C}) \to H^\bullet_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{Q}) \ .$$

Proof. The first assertion follows from applying Lemma 1.4.10 to β and α. As before, we identify sheaves on $X \times_k \mathbb{C}$ with sheaves on the set of closed points of $X \times_k \mathbb{C}$. This yields a quasi-isomorphism

$$\alpha^{-1}\beta^{-1}Gd_X(\Omega^\bullet_X) \to Gd_{X^{\mathrm{an}}}(\alpha^{-1}\beta^{-1}\Omega^\bullet_X) \ .$$

We compose with

$$Gd_{X^{\mathrm{an}}}(\alpha^{-1}\beta^{-1}\Omega^\bullet_X) \to Gd_{X^{\mathrm{an}}}(\Omega^\bullet_{X^{\mathrm{an}}}) \ .$$

Taking global sections yields by definition a natural $\mathbb{Q}$-linear map of complexes

$$R\Gamma_{\mathrm{dR}}(X) \to R\Gamma_{\mathrm{dR}^{\mathrm{an}}}(X) \ .$$

By extension of scalars we get $\phi_{\mathrm{dR},\mathrm{dR}^{\mathrm{an}}}$. It is a quasi-isomorphism because on cohomology it defines the maps from Lemma 3.1.11 and Proposition 4.1.7.

The second assertion follows from ordinary functoriality of the Godement resolution. The last holds by construction. □

In other words:

Corollary 5.5.2. *The assignment*

$$X \mapsto (R\Gamma_{\mathrm{dR}}(X), R\Gamma_{\mathrm{sing}}(X), R\Gamma_{\mathrm{dR}^{\mathrm{an}}}(X), \phi_{\mathrm{dR},\mathrm{dR}^{\mathrm{an}}}, \phi_{\mathrm{sing},\mathrm{dR}^{\mathrm{an}}})$$

defines a functor

$$R\Gamma : \mathrm{Sm} \to C^+_{(k,\mathbb{Q})}$$

where $C^+_{(k,\mathbb{Q})}$ *is the category of cohomological* $(k, \mathbb{Q})-\mathrm{Vect}$*-complexes introduced in Definition 5.2.1.*

Moreover,

$$H^\bullet(R\Gamma(X)) = \mathrm{H}(X) \ ,$$

where the functor H *is defined as above.*

Proof. Clear from the lemma. □

By naturality, these definitions extend to objects in $\mathbb{Z}[\mathrm{Sm}]$.

Definition 5.5.3. Let

$$R\Gamma : K^-(\mathbb{Z}\mathrm{Sm}) \to D^+_{(k,\mathbb{Q})}$$

be defined componentwise as the total complex complex of the complex in $C^+_{(k,\mathbb{Q})}$. For $X_\bullet \in C^-(\mathbb{Z}\mathrm{Sm})$ and $i \in \mathbb{Z}$ we put

$$\mathrm{H}^i(X_\bullet) = H^i R\Gamma(X_\bullet) \ .$$

Definition 5.5.4. Let k be a subfield of $\mathbb{C}$ and X a variety over k with a closed subvariety D. We define the *period isomorphism*

$$\mathrm{per} : H^\bullet_{\mathrm{dR}}(X, D) \otimes_k \mathbb{C} \to H^\bullet_{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}) \otimes_{\mathbb{Q}} \mathbb{C}$$

as follows: let $D_\bullet \to X_\bullet$ be smooth proper hypercovers of $D \to X$ as in Definition 3.3.6. Let

$$C_\bullet = \mathrm{Cone} C(D_\bullet) \to C(X_\bullet)) \in C^-(\mathbb{Z}[\mathrm{Sm}]) \ .$$

Then $H^\bullet(R\Gamma(C_\bullet))$ consists of

$$(H^\bullet_{\mathrm{dR}}(X,D), H^\bullet_{\mathrm{sing}}(X,D), \mathrm{per})\ .$$

In detail: per is given by the composition of the isomorphisms

$$H^\bullet_{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}, \mathbb{Q})) \otimes_{\mathbb{Q}} \mathbb{C} \to H^\bullet(R\Gamma_{\mathrm{sing}}(C_\bullet)$$

with

$$H^\bullet\phi_{\mathrm{sing},\mathrm{dR}^{\mathrm{an}}}(C_\bullet)^{-1} \circ H^\bullet\phi_{\mathrm{dR},\mathrm{dR}^{\mathrm{an}}}(C_\bullet)\ .$$

We define the *period pairing*

$$\mathrm{per} : H^\bullet_{\mathrm{dR}}(X,D) \times H^{\mathrm{sing}}_\bullet(X^{\mathrm{an}}, D^{\mathrm{an}}) \to \mathbb{C}$$

to be the map

$$(\omega,\gamma) \mapsto \gamma(\mathrm{per}(\omega))$$

where we view classes in relative singular homology as linear forms on relative singular cohomology.

Lemma 5.5.5. per *is well-defined, compatible with products and long exact sequences for relative cohomology.*

Proof. By definition of relative algebraic de Rham cohomology (see Definition 3.3.6), the morphism takes values in $H^\bullet_{\mathrm{dR}}(X,D) \otimes_k \mathbb{C}$. The first map is an isomorphism by proper descent in singular cohomology, see Theorem 2.7.6.

Compatibility with long exact sequences and multiplication comes from the definition. □

We make this explicit in the case of a divisor with normal crossings. Recall the description of relative de Rham cohomology in this case in Proposition 3.3.19.

Theorem 5.5.6. *Let X be a smooth affine variety of dimension d and $D \subset X$ a simple divisor with normal crossings. Let $\omega \in \Omega^d_X(X)$ with associated cohomology class $[\omega] \in H^d_{\mathrm{dR}}(X,D)$. Let $\sum a_j\gamma_j$ with $a_j \in \mathbb{Q}$ and $\gamma_j : \Delta_i \to X^{\mathrm{an}}$ be a differentiable singular cchain as in Definition 2.2.2 with boundary in D^{an}. Then*

$$\mathrm{per}([\omega], c) = \sum a_j \int_{\Delta_i} \gamma^*(\omega)\ .$$

Proof. Let $D_\bullet$ as in Section 3.3.6. We apply the considerations of the proof of Theorem 5.3.3 to X and the components of $D_\bullet$. Note that $\omega|_{D_I} = 0$ for dimension reasons. □

Part II

Nori Motives

Chapter 6

Nori's diagram category

We explain Nori's construction of an abelian category attached to the representation of a diagram and establish some properties for it. The construction is completely formal. It mimicks the standard construction of the Tannakian dual of a rigid tensor category with a fibre functor. Only, we do not have a tensor product or even a category but only what we should think of as the fibre functor.

The results are due to Nori. Notes from some of his talks are available [N, N1]. There is a also a sketch in Levine's survey [L1] §5.3. In the proofs of the main results we follow closely the diploma thesis of von Wangenheim in [vW].

6.1 Main results

6.1.1 Diagrams and representations

Let R be a noetherian, commutative ring with unit.

Definition 6.1.1. A *diagram* D is a directed graph on a set of vertices $V(D)$ and edges $E(D)$. A *diagram* with *identities* is a diagram with a choice of a distinguished edge $\mathrm{id}_v : v \to v$ for every $v \in D$. A diagram is called *finite* if it has only finitely many vertices. A *finite full subdiagram* of a diagram D is a diagram containing a finite subset of vertices of D and all edges (in D) between them.

By abuse of notation we often write $v \in D$ instead of $v \in V(D)$. The set of all directed edges between $p, q \in D$ is denoted by $D(p, q)$.

Remark 6.1.2. One may view a diagram as a category where composition of morphisms is not defined. The notion of a diagram with identity edges is not standard. The notion is useful later when we consider multiplicative structures.

Example 6.1.3. Let $\mathcal{C}$ be a small category. Then we can associate a diagram $D(\mathcal{C})$ with vertices the set of objects in $\mathcal{C}$ and edges given by morphisms. It is even a diagram with identities. By abuse of notation we usually also write $\mathcal{C}$ for the diagram.

Definition 6.1.4. A *representation* T of a diagram D in a small category $\mathcal{C}$ is a map T of directed graphs from D to $D(\mathcal{C})$. A *representation* T of a diagram D with identities is a representation such that id is mapped to id.

For $p, q \in D$ and every edge m from p to q we denote their images simply by Tp, Tq and $Tm : Tp \to Tq$ (mostly without brackets).

Remark 6.1.5. Alternatively, a representation is defined as a functor from the *path category* $\mathcal{P}(D)$ to $\mathcal{C}$. Recall that the objects of the path category are the vertices of D, and the morphisms are sequences of directed edges $e_1 e_2 \dots e_n$ for $n \geq 0$ with the edge e_i starting in the end point of e_{i-1} for $i = 2, \dots, n$. Morphisms are composed by concatenating edges.

We are particularly interested in representations in categories of modules.

Definition 6.1.6. Let R be a noetherian commutative ring with unit. By $R-$Mod we denote the category of finitely generated R-modules. By $R-$Proj we denote the subcategory of finitely generated projective R-modules.

Note that these categories are essentially small by passing to isomorphic objects, so we will not worry about smallness from now on.

Definition 6.1.7. Let S be a commutative unital R-algebra and $T : D \to R-\mathrm{Mod}$ a representation. We denote T_S the representation

$$D \xrightarrow{T} R-\mathrm{Mod} \xrightarrow{\otimes_R S} S-\mathrm{Mod} .$$

Definition 6.1.8. Let T be a representation of D in $R-$Mod. We define the *ring of endomorphisms* of T by

$$\mathrm{End}(T) := \left\{ (e_p)_{p \in D} \in \prod_{p \in D} \mathrm{End}_R(Tp) | e_q \circ Tm = Tm \circ e_p \ \forall p, q \in D \ \forall m \in D(p,q) \right\}.$$

Remark 6.1.9. In other words, an element of $\mathrm{End}(T)$ consists of tuples $(e_p)_{p \in V(D)}$ of endomorphisms of Tp, such that all diagrams of the following form commute:

$$\begin{array}{ccc} Tp & \xrightarrow{Tm} & Tq \\ \downarrow{\scriptstyle e_p} & & \downarrow{\scriptstyle e_q} \\ Tp & \xrightarrow{Tm} & Tq \end{array}$$

Note that the ring of endomorphisms does not change when we replace D by the path category $\mathcal{P}(D)$.

6.1.2 Explicit construction of the diagram category

The diagram category can be characterized by a universal property, but it also has a simple explicit description that we give first.

Definition 6.1.10 (Nori)**.** Let R be a noetherian commutative ring with unit. Let T be a representation of D in $R-$Mod.

1. Assume D is finite. Then we put
$$\mathcal{C}(D,T) = \mathrm{End}(T)-\mathrm{Mod}$$
the category of finitely generated R-modules equipped with an R-linear operation of the algebra $\mathrm{End}(T)$.

2. In general let
$$\mathcal{C}(D,T) = 2-\mathrm{colim}_F \mathcal{C}(F, T|_F)$$
where F runs through the system of finite subdiagrams of D.

   More explicitly: the objects of $\mathcal{C}(D,T)$ are the objects of $\mathcal{C}(F,T|_F)$ for some finite subdiagram F. For $X \in \mathcal{C}(F,T|_F)$ and $F \subset F'$ we write $X_{F'}$ for the image of X in $\mathcal{C}(F', T|_{F'})$. For objects $X, Y \in \mathcal{C}(D,T)$, we put
$$\mathrm{Mor}_{\mathcal{C}(D,T)}(X,Y) = \varinjlim_{F} \mathrm{Mor}_{\mathcal{C}(F,T|_F)}(X_F, Y_F)\ .$$

The category $\mathcal{C}(D,T)$ is called the *diagram category*. With
$$f_T : \mathcal{C}(D,T) \longrightarrow R-\mathrm{Mod}$$
we denote the forgetful functor.

Remark 6.1.11. The representation $T : D \longrightarrow \mathcal{C}(D,T)$ extends to a functor on the path category $\mathcal{P}(D)$. By construction the diagram categories $\mathcal{C}(D,T)$ and $\mathcal{C}(\mathcal{P}(D),T)$ agree. The point of view of the path category will be useful Chapter 7, in particular in Definition 7.2.1.

In section 6.5 we will prove that under additional conditions for R, satisfied in the cases of most interest, there is the following even more direct description of $\mathcal{C}(D,T)$ as comodules over a coalgebra.

Theorem 6.1.12. *If the representation T takes values in free modules over a field or Dedekind domain R, the diagram category is equivalent to the category of finitely generated comodules (see Definition 6.5.4) over the coalgebra $A(D,T)$ where*
$$A(D,T) = \mathrm{colim}_F A(F,T) = \mathrm{colim}_F \mathrm{End}(T|_F)^\vee$$
with F running through the system of all finite subdiagrams of D and $^\vee$ the R-dual.

The proof of this theorem is given in Section 6.5.

6.1.3 Universal property: Statement

Theorem 6.1.13 (Nori). *Let D be a diagram and*

$$T : D \longrightarrow R\text{–Mod}$$

a representation of D.

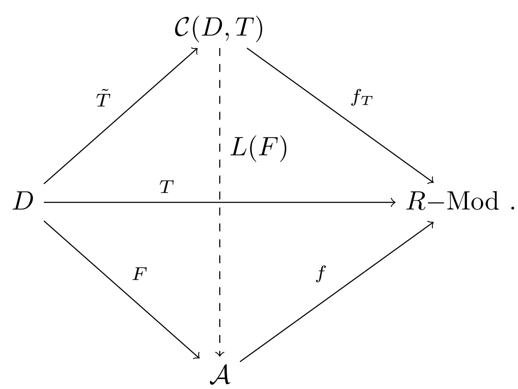
Then there exists an R-linear abelian category $\mathcal{C}(D,T)$, together with a representation

$$\tilde{T} : D \longrightarrow \mathcal{C}(D,T),$$

and a faithful, exact, R-linear functor f_T, such that:

1. *T factorizes over $D \xrightarrow{\tilde{T}} \mathcal{C}(D,T) \xrightarrow{f_T} R\text{–Mod}$.*
2. *$\tilde{T}$ satisfies the following universal property: Given*
   - (a) *another R-linear, abelian category $\mathcal{A}$,*
   - (b) *an R-linear, faithful, exact functor, $f : \mathcal{A} \to R\text{–Mod}$,*
   - (c) *another representation $F : D \to \mathcal{A}$,*

   such that $f \circ F = T$, then there exists a functor $L(F)$ - unique up to unique isomorphism of functors - such that the following diagram commutes:



The category $\mathcal{C}(D,T)$ together with $\tilde{T}$ and f_T is uniquely determined by this property up to unique equivalence of categories. It is explicitly described by the diagram category of Definition 6.1.10. It is functorial in D in the obvious sense.

The proof will be given in Section 6.4. We are going to view f_T as an extension of T from D to $\mathcal{C}(D,T)$ and sometimes write simply T instead of f_T.

The universal property generalizes easily.

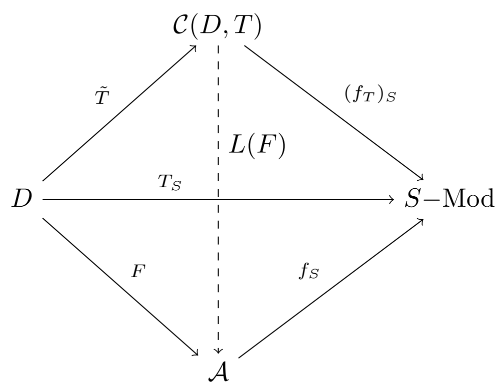
Corollary 6.1.14. *Let D, R, T be as in Theorem 6.1.19. Let $\mathcal{A}$ and f, F be as in loc.cit. 2. (a)-(c). Moreover, let S be a faithfully flat commutative unitary R-algebra S and*

$$\phi : T_S \to (f \circ F)_S$$

an isomorphism of representations into $S-\mathrm{Mod}$. Then there exists a functor $L(F) : \mathcal{C}(D,T) \to \mathcal{A}$ and an isomorphism of functors

$$\tilde{\phi} : (f_T)_S \to f_S \circ L(F)$$

such that



commutes up to ϕ and $\tilde{\phi}$. The pair $(L(F), \tilde{\phi})$ is unique up to unique isomorphism of functors.

The proof will also be given in Section 6.4.

The following properties provide a better understanding of the nature of the category $\mathcal{C}(D,T)$.

Proposition 6.1.15. 1. *As an abelian category $\mathcal{C}(D,T)$ is generated by the $\tilde{T}v$ where v runs through the set of vertices of D, i.e., it agrees with its smallest full subcategory such that the inclusion is exact containing all such $\tilde{T}v$.*

2. *Each object of $\mathcal{C}(D,T)$ is a subquotient of a finite direct sum of objects of the form $\tilde{T}v$.*

3. *If $\alpha : v \to v'$ is an edge in D such that $T\alpha$ is an isomorphism, then $\tilde{T}\alpha$ is also an isomorphism.*

Proof. Let $\mathcal{C}' \subset \mathcal{C}(D,T)$ be the subcategory generated by all $\tilde{T}v$. By definition, the representation $\tilde{T}$ factors through $\mathcal{C}'$. By the universal property of $\mathcal{C}(D,T)$, we obtain a functor $\mathcal{C}(D,T) \to \mathcal{C}'$, hence an equivalence of subcategories of $R-\mathrm{Mod}$.

The second statement follows from the first criterion since the full subcategory in $\mathcal{C}(D,T)$ of subquotients of finite direct sums is abelian, hence agrees with $\mathcal{C}(D,T)$. The assertion on morphisms follows since the functor $f_T:\mathcal{C}(D,T)\to R-\mathrm{Mod}$ is faithful and exact between abelian categories. Kernel and cokernel of $\tilde{T}\alpha$ vanish if kernel and cokernel of $T\alpha$ vanish. □

Remark 6.1.16. We will later give a direct proof, see Proposition 6.3.20. It will be used in the proof of the universal property.

The diagram category only weakly depends on T.

Corollary 6.1.17. *Let D be a diagram and $T,T':D\to R-\mathrm{Mod}$ two representations. Let S be a faithfully flat R-algebra and $\phi:T_S\to T'_S$ be an isomorphism of representations in $S-\mathrm{Mod}$. Then it induces an equivalence of categories*

$$\Phi:\mathcal{C}(D,T)\to\mathcal{C}(D,T').$$

Proof. We apply the universal property of Corollary 6.1.14 to the representation T and the abelian category $\mathcal{A}=\mathcal{C}(D,T')$. This yields a functor $\Phi:\mathcal{C}(D,T)\to\mathcal{C}(D,T')$. By interchanging the role of T and T' we also get a functor Φ' in the opposite direction. We claim that they are inverse to each other. The composition $\Phi'\circ\Phi$ can be seen as the universal functor for the representation of D in the abelian category $\mathcal{C}(D,T)$ via T. By the uniqueness part of the universal property, it is the identity. □

Corollary 6.1.18. *Let D_2 be a diagram. Let $T_2:D_2\to R-\mathrm{Mod}$ be a representation. Let*

$$D_2\xrightarrow{\tilde{T}_2}\mathcal{C}(D_2,T_2)\xrightarrow{f_{T_2}}R-\mathrm{Mod}$$

be the factorization via the diagram category.

Let $D_1\subset D_2$ be a full subdiagram. It has the representation $T_1=T_2|_{D_1}$ obtained by restricting T_2. Let

$$D_1\xrightarrow{T_1}\mathcal{C}(D_1,T_1)\xrightarrow{f_{T_1}}R-\mathrm{Mod}$$

be the factorization via the diagram category. Let $\iota:\mathcal{C}(D_1,T_1)\to\mathcal{C}(D_2,T_2)$ be the functor induced from the inclusion of diagrams. Moreover, we assume that there is a representation $F:D_2\to\mathcal{C}(D_1,T_1)$ compatible with T_2, i.e., such that there is an isomorphism of functors

$$T_2\to f_{T_2}\circ\iota\circ F=f_{T_1}\circ F\ .$$

Then ι is an equivalence of categories.

Proof. Let $T_2'=f_{T_1}\circ F:D_2\to R-\mathrm{Mod}$ and denote $T_1'=T_2'|_{D_1}:D_1\to R-\mathrm{Mod}$. Note that T_2 and T_2' and T_1 and T_1' are isomorphic by assumption.

By the universal property of the diagram category, the representation F induces a functor

$$\pi' : \mathcal{C}(D_2, T_2') \to \mathcal{C}(D_1, T_1) \ .$$

It induces $\pi : \mathcal{C}(D_2, T_2)$ by precomposition with the equivalence Φ from Corollary 6.1.17. We claim that $\iota \circ \pi$ and $\pi \circ \iota$ are isomorphic to the identity functor.

By the uniqueness part of the universal property, the composition $\iota \circ \pi' : \mathcal{C}(D_2, T_2') \to \mathcal{C}(D_2, T_2)$ is induced from the representation $\iota \circ F$ of D_2 in the abelian category $\mathcal{C}(D_2, T_2)$. By the proof of Corollary 6.1.17 this is the equivalence Φ^{-1}. In particular, $\iota \circ \pi$ is the identity.

The argument for $\pi \circ \iota$ on $\mathcal{C}(D_1, T_1)$ is analogous. □

The most important ingredient for the proof of the universal property is the following special case.

Theorem 6.1.19. *Let R be a noetherian ring and $\mathcal{A}$ an abelian, R-linear category. Let*

$$T : \mathcal{A} \longrightarrow R\text{−Mod}$$

be a faithful, exact, R-linear functor and

$$\mathcal{A} \xrightarrow{\tilde{T}} \mathcal{C}(\mathcal{A}, T) \xrightarrow{f_T} R\text{−Mod}$$

the factorization via its diagram category (see Definition 6.1.10). Then $\tilde{T}$ is an equivalence of categories.

The proof of this theorem will be given in Section 6.3.

6.1.4 Discussion of the Tannakian case

The above may be viewed as a generalization of Tannaka duality. We explain this in more detail. We are not going to use the considerations in the sequel.

Let k be a field, $\mathcal{C}$ a k-linear abelian tensor category, and

$$T : \mathcal{C} \longrightarrow k\text{−Vect}$$

a k-linear faithful tensor functor, all in the sense of [DM]. By standard Tannakian formalism (cf [Sa] and [DM]), there is a k-bialgebra A such that the category is equivalent to the category of A-comodules on finite dimensional k-vector spaces.

On the other hand, if we regard $\mathcal{C}$ as a diagram (with identities) and T as a representation into finite dimensional vector spaces, we can view the diagram category of $\mathcal{C}$ as the category $A(\mathcal{C}, T)$−Comod by Theorem 6.1.12. By Theorem 6.1.19 the category $\mathcal{C}$ is equivalent to its diagram category $A(\mathcal{C}, T)$−Comod. The construction of the two coalgebras A and $A(\mathcal{C}, T)$ coincides. Thus Nori implicitely shows that we can recover the coalgebra structure of A just by looking at the representations of $\mathcal{C}$.

The *algebra* structure on $A(\mathcal{C},T)$ is induced from the tensor product on $\mathcal{C}$ (see also Section 7.1). This defines a pro-algebraic scheme $\mathrm{Spec} A(\mathcal{C},T)$. The *coalgebra* structure turns $\mathrm{Spec} A(\mathcal{C},T)$ into a monoid scheme. We may interpret $A(\mathcal{C},T)-\mathrm{Comod}$ as the category of finite-dimensional representations of this monoid scheme.

If the tensor structure is rigid in addition, $\mathcal{C}(D,T)$ becomes what Deligne and Milne call a *neutral Tannakian category* [DM]. The rigidity structure induces an antipodal map, making $A(\mathcal{C},T)$ into a Hopf algebra. This yields the structure of a *group scheme* on $\mathrm{Spec} A(\mathcal{C},T)$, rather than only a monoid scheme.

We record the outcome of the discussion:

Theorem 6.1.20. *Let R be a field and $\mathcal{C}$ be a neutral R-linear Tannakian category with faithful exact fibre functor $T : \mathcal{C} \to R-\mathrm{Mod}$. Then $A(\mathcal{C},T)$ is equal to the Hopf algebra of the Tannakian dual.*

Proof. By construction, see [DM] Theorem 2.11 and its proof. □

A similar result holds in the case that R is a Dedekind domain and

$$T : D \longrightarrow R-\mathrm{Proj}$$

a representation into finitely generated projective R-modules. Again by Theorem 6.1.12, the diagram category $\mathcal{C}(D,T)$ equals $A(\mathcal{C},T)-\mathrm{Comod}$, where $A(\mathcal{C},T)$ is projective over R. Wedhorn shows in [Wed] that if $\mathrm{Spec} A(\mathcal{C},T)$ is a group scheme it is the same to have a representation of $\mathrm{Spec} A(\mathcal{C},T)$ on a finitely generated R-module M and to endow M with an $A(\mathcal{C},T)$-comodule structure.

6.2 First properties of the diagram category

Let R be a unitary commutative noetherian ring, D a diagram and $T : D \to R-\mathrm{Mod}$ a representation. We investigate the category $\mathcal{C}(D,T)$ introduced in Definition 6.1.10.

Lemma 6.2.1. *If D is a finite diagram, then* $\mathrm{End}(T)$ *is an R-algebra which is finitely generated as an R-module.*

Proof. For any $p \in D$ the module Tp is finitely generated. Since R is noetherian, the algebra $\mathrm{End}_R(Tp)$ then is finitely generated as R-module. Thus $\mathrm{End}(T)$ becomes a unitary subalgebra of $\prod_{p\in Ob(D)} \mathrm{End}_R(Tp)$. Since $V(D)$ is finite and R is noetherian,

$$\mathrm{End}(T) \subset \prod_{p\in Ob(D)} \mathrm{End}_R(Tp)$$

is finitely generated as R-module. □

Lemma 6.2.2. *Let D be a finite diagram and $T : D \to R-\mathrm{Mod}$ a representation. Then:*

1. *Let S be a flat R-algebra. Then:*
$$\mathrm{End}_S(T_S) = \mathrm{End}_R(T) \otimes S$$
2. *Let $F : D' \to D$ be morphism of diagrams and $T' = T \circ F$ the induced representation. Then F induces a canonical R-algebra homomorphism*
$$F^* : \mathrm{End}(T) \to \mathrm{End}(T')\ .$$

Proof. The algebra $\mathrm{End}(T)$ is defined as a limit, i.e., a kernel

$$0 \to \mathrm{End}(T) \to \prod_{p \in V(D)} \mathrm{End}_R(Tp) \xrightarrow{\phi} \prod_{m \in D(p,q)} \mathrm{Hom}_R(Tp, Tq)$$

with $\phi(p)(m) = e_q \circ Tm - Tm \circ e_p$. As S is flat over R, this remains exact after tensoring with S. As the R-module Tp is finitely presented and S flat, we have

$$\mathrm{End}_R(Tp) \otimes S = \mathrm{End}_S(T_S p)\ .$$

Hence we get

$$0 \to \mathrm{End}(T|_F) \otimes S \to \prod_{p \in V(D)} \mathrm{End}_S(T_S(p)) \xrightarrow{\phi} \prod_{m \in D(p,q)} \mathrm{Hom}_S(T_S(p), T_S(q))\ .$$

This is the defining sequence for $\mathrm{End}(T_S)$.

The morphism of diagrams $F : D' \to D$ induces a homomorphism

$$\prod_{p \in V(D)} \mathrm{End}_R(Tp) \to \prod_{p' \in V(D')} \mathrm{End}_R(T'p'),$$

by mapping $e = (e_p)_p$ to $F^*(e)$ with $(F^*(e))_{p'} = e_{f(p')}$ in $\mathrm{End}_R(T'p') = \mathrm{End}_R(Tf(p'))$. It is compatible with the induced homomorphism

$$\prod_{m \in D(p,q)} \mathrm{Hom}_R(Tp, Tq) \to \prod_{m' \in D'(p',q')} \mathrm{Hom}_R(T'p', T'q').$$

Hence it induces a homomorphism on the kernels. □

Proposition 6.2.3. *Let R be unitary commutative noetherian ring, D a finite diagram and $T : D \longrightarrow R-\mathrm{Mod}$ be a representation. For any $p \in D$ the object Tp is a natural left $\mathrm{End}(T)$-module. This induces a representation*

$$\tilde{T} : D \longrightarrow \mathrm{End}(T)-\mathrm{Mod},$$

such that T factorises via

$$D \xrightarrow{\tilde{T}} \mathcal{C}(D, T) \xrightarrow{f_T} R-\mathrm{Mod}.$$

Proof. For all $p \in D$ the projection

$$pr : \mathrm{End}(T) \to \mathrm{End}_R(Tp)$$

induces a well-defined action of $\mathrm{End}(T)$ on Tp making Tp into a left $\mathrm{End}(T)$-module. To check that $\tilde{T}$ is a representation of left $\mathrm{End}(T)$-modules, we need $Tm \in \mathrm{Hom}_R(Tp, Tq)$ to be $\mathrm{End}(T)$-linear for all $p, q \in D, m \in D(p, q)$. This corresponds directly to the commutativity of the diagram in Remark 6.1.9. □

Now let D be general. We study the system of finite subdiagrams $F \subset D$. Recall that subdiagrams are full, i.e., they have the same edges.

Corollary 6.2.4. *The finite subdiagrams of D induce a directed system of abelian categories $\big(\mathcal{C}(D, T|_F)\big)_{F\subset D \mathit{finite}}$ with exact, faithful R-linear functors as transition maps.*

Proof. The transition functors are induced from the inclusion via Lemma 6.2.2. □

Recall that we have defined $\mathcal{C}(D, T)$ as 2-colimit of this system, see Definition 6.1.10.

Proposition 6.2.5. *The 2-colimit $\mathcal{C}(D, T)$ exists. It provides an R-linear abelian category such that the composition of the induced representation with the forgetful functor*

$$\begin{array}{ccccc} D & \xrightarrow{\tilde{T}} & \mathcal{C}(D,T) & \xrightarrow{f_T} & R\text{−Mod} \\ p & \mapsto & Tp & \mapsto & Tp. \end{array}$$

yields a factorization of T. The functor f_T is R-linear, faithful and exact.

Proof. It is a straightforward calculation that the limit category inherits all structures of an R-linear abelian category. It has well-defined (co)products and (co)kernels because the transition functors are exact. It has a well-defined R-linear structure as all transition functors are R-linear. Finally, one shows that every kernel resp. cokernel is a monomorphism resp. epimorphism using the fact that all transition functors are faithful and exact.

So for every $p \in D$ the R-module Tp becomes an $\mathrm{End}(T|_F)$-module for all finite $F \subset D$ with $p \in F$. Thus it represents an object in $\mathcal{C}(D, T)$. This induces a representation

$$\begin{array}{ccc} D & \xrightarrow{\tilde{T}} & \mathcal{C}(D,T) \\ p & \mapsto & Tp. \end{array}$$

The forgetful functor is exact, faithful and R-linear. Composition with the forgetful functor f_T obviously yields the initial diagram T. □

We now consider functoriality in D.

Lemma 6.2.6. *Let D_1, D_2 be diagrams and $G : D_1 \to D_2$ a map of diagrams. Let further $T : D_2 \to R\text{-Mod}$ be a representation and*

$$D_2 \xrightarrow{\tilde{T}} \mathcal{C}(D_2, T) \xrightarrow{f_T} R\text{-Mod}$$

the factorization of T through the diagram category $\mathcal{C}(D_2, T)$ as constructed in Proposition 6.2.5. Let

$$D_1 \xrightarrow{\widetilde{T \circ G}} \mathcal{C}(D_1, T \circ G) \xrightarrow{f_{T \circ G}} R\text{-Mod}$$

be the factorization of $T \circ G$.

Then there exists a faithful R-linear, exact functor $\mathcal{G}$, such that the following diagram commutes.

$$\begin{array}{ccccc}
D_1 & & \xrightarrow{\quad G \quad} & & D_2 \\
\downarrow{\scriptstyle \widetilde{T \circ G}} & & & & \downarrow{\scriptstyle \tilde{T}} \\
\mathcal{C}(D_1, T \circ G) & & \overset{\mathcal{G}}{\dashrightarrow} & & \mathcal{C}(D_2, T) \\
& \searrow{\scriptstyle f_{T \circ G}} & & \swarrow{\scriptstyle f_T} & \\
& & R\text{-Mod} & &
\end{array}$$

Proof. Let D_1, D_2 be finite diagrams first. Let $T_1 = T \circ G|_{D_1}$ and $T_2 = T|_{D_2}$. The homomorphism

$$G^* : \mathrm{End}(T_2) \to \mathrm{End}(T_1)$$

of Lemma 6.2.2 induces by restriction of scalars a functor on diagram categories as required.

Let now D_1 be finite and D_2 arbitrary. Let E_2 be finite full subdiagram of D_2 containing $G(D_1)$. We apply the finite case to $G : D_1 \to E_2$ and obtain a functor

$$\mathcal{C}(D_1, T_1) \to \mathcal{C}(E_2, T_2)$$

which we compose with the canonical functor $\mathcal{C}(E_2, T_2) \to \mathcal{C}(D_2, T_2)$. By functoriality, it is independent of the choice of E_2.

Let now D_1 and D_2 be arbitrary. For every finite subdiagram $E_1 \subset E_1$ we have constructed

$$\mathcal{C}(E_1, T_1) \to \mathcal{C}(D_2, T_2)\ .$$

They are compatible and hence define a functor on the limit. □

Isomorphic representations have equivalent diagram categories. More precisely:

Lemma 6.2.7. *Let* $T_1, T_2 : D \to R-\mathrm{Mod}$ *be representations and* $\phi : T_1 \to T_2$ *an isomorphism of representations. Then* ϕ *induces an equivalence of categories* $\Phi : \mathcal{C}(D, T_1) \to \mathcal{C}(D, T_2)$ *together with an isomorphism of representations*

$$\tilde{\phi} : \Phi \circ \tilde{T}_1 \to \tilde{T}_2$$

such that $f_{T_2} \circ \tilde{\phi} = \phi$.

Proof. We only sketch the argument which is analogous to the proof of Lemma 6.2.6. It suffices to consider the case $D = F$ finite. The functor

$$\Phi : \mathrm{End}(T_1)-\mathrm{Mod} \to \mathrm{End}(T_2)-\mathrm{Mod}$$

is the extension of scalars for the R-algebra isomorphism $\mathrm{End}(T_1) \to \mathrm{End}(T_2)$ induced by conjugating by ϕ. □

6.3 The diagram category of an abelian category

In this section we give the proof of Theorem 6.1.19: the diagram category of the diagram category of an abelian category with respect to a representation given by an exact faithful functor is the abelian category itself.

We fix a commutative noetherian ring R with unit and an R-linear abelian category $\mathcal{A}$. By R-algebra we mean a unital R-algebra, not necessarily commutative.

6.3.1 A calculus of tensors

We start with some general constructions of functors. We fix a unital R-algebra E, finitely generated as R-module, not necessarily commutative. The most important case is $E = R$, but this is not enough for our application.

In the simpler case where R is a field, the constructions in this sections can already be found in [DMOS].

Definition 6.3.1. Let E be an R-algebra which is finitely generated as R-module. We denote $E-\mathrm{Mod}$ the category of finitely generated left E-modules.

The algebra E and the objects of $E-\mathrm{Mod}$ are noetherian because R is.

Definition 6.3.2. Let $\mathcal{A}$ be an R-linear abelian category and p be an object of $\mathcal{A}$. Let E be a not necessarily commutative R algebra and

$$E^{op} \xrightarrow{f} \mathrm{End}_{\mathcal{A}}(p)$$

be a morphism of R-algebras. We say that p is a *right* E*-module in* $\mathcal{A}$.

Example 6.3.3. Let $\mathcal{A}$ be the category of left R'-modules for some R-algebra R'. Then a right E-module in $\mathcal{A}$ is the same thing as an (R', E)-bimodule, i.e., a left R'-module with the structure of a right E-module.

Lemma 6.3.4. *Let $\mathcal{A}$ be an R-linear abelian category and p be an object of $\mathcal{A}$. Let E be a not necessarily commutative R-algebra and p a right E-module in $\mathcal{A}$. Then*

$$\mathrm{Hom}_{\mathcal{A}}(p, \_) : \mathcal{A} \to R\mathrm{-Mod}$$

can naturally be viewed as a functor to $E-\mathrm{Mod}$.

Proof. For every $q \in \mathcal{A}$, the algebra E operates on $\mathrm{Hom}_{\mathcal{A}}(p, q)$ via functoriality. □

Proposition 6.3.5. *Let $\mathcal{A}$ be an R-linear abelian category and p be an object of $\mathcal{A}$. Let E be a not necessarily commutative R algebra and p a right E-module in $\mathcal{A}$. Then the functor*

$$\mathrm{Hom}_{\mathcal{A}}(p, \_) : \mathcal{A} \longrightarrow E\mathrm{-Mod}$$

has an R-linear left adjoint

$$p \otimes_E \_ : E\mathrm{-Mod} \longrightarrow \mathcal{A}.$$

It is right exact. It satisfies

$$p \otimes_E E = p,$$

and on endomorphisms of E we have (using $\mathrm{End}_E(E) \cong E^{op}$*)*

$$\begin{array}{cccc} p \otimes_E \_ : & \mathrm{End}_E(E) & \longrightarrow & \mathrm{End}_{\mathcal{A}}(p) \\ & a & \longmapsto & f(a). \end{array}$$

Proof. Right exactness of $p \otimes_E \_$ follows from the universal property. For every $M \in E-\mathrm{Mod}$, the value of $p \otimes_E M$ is uniquely determined by the universal property. In the case of $M = E$, it is satisfied by p itself because we have for all $q \in \mathcal{A}$

$$\mathrm{Hom}_{\mathcal{A}}(p, q) = \mathrm{Hom}_E(E, \mathrm{Hom}_{\mathcal{A}}(p, q)).$$

This identification also implies the formula on endomorphisms of $M = E$.

By compatibility with direct sums, this implies that $p \otimes_E E^n = \bigoplus_{i=1}^n p$ for free E-modules. For the same reason, morphisms $E^m \xrightarrow{(a_{ij})_{ij}} E^n$ between free E-modules must be mapped to $\bigoplus_{i=1}^m p \xrightarrow{f(a_{ij})_{ij}} \bigoplus_{i=1}^n p$.

Let M be a finitely presented left E-module. We fix a finite presentation

$$E^{m_1} \xrightarrow{(a_{ij})_{ij}} E^{m_0} \overset{\pi_a}{\twoheadrightarrow} M \to 0.$$

Since $p \otimes_E \_$ preserves cokernels (if it exists), we need to define

$$p \otimes_E M := \mathrm{Coker}(p^{m_1} \xrightarrow{\tilde{A} := f(a_{ij})_{ij}} p^{m_0}).$$

We check that it satisfies the universal property. Indeed, for all $q \in \mathcal{A}$, we have a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Hom}_{\mathcal{A}}(p \otimes E^{m_1}, q) & \longleftarrow & \mathrm{Hom}_{\mathcal{A}}(p \otimes E^{m_0}, q) & \longleftarrow & \mathrm{Hom}_{\mathcal{A}}(p \otimes M, q) & \longleftarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow & & \\
\mathrm{Hom}_E(E^{m_1}, \mathrm{Hom}_{\mathcal{A}}(p,q)) & \longleftarrow & \mathrm{Hom}_E(E^{m_0}, \mathrm{Hom}_{\mathcal{A}}(p,q)) & \longleftarrow & \mathrm{Hom}_E(M, \mathrm{Hom}_{\mathcal{A}}(p,q)) & \longleftarrow & 0
\end{array}$$

Hence the dashed arrow exists and is an isomorphism.

The universal property implies that $p \otimes_E M$ is independent of the choice of presentation and functorial. We can also make this explicit. For a morphism between arbitrary modules $\varphi : M \to N$ we choose lifts

$$\begin{array}{ccccccc}
E^{m_1} & \xrightarrow{A} & E^{m_0} & \xrightarrow{\pi_A} & M & \longrightarrow & 0 \\
\downarrow \varphi^1 & & \downarrow \varphi^0 & & \downarrow \varphi & & \\
E^{n_1} & \xrightarrow{B} & E^{n_0} & \xrightarrow{\pi_B} & N & \longrightarrow & 0.
\end{array}$$

The respective diagram in $\mathcal{A}$,

$$\begin{array}{ccccccc}
p^{m_1} & \xrightarrow{\tilde{A}} & p^{m_0} & \xrightarrow{\pi_{\tilde{A}}} & \mathrm{Coker}(\tilde{A}) & \longrightarrow & 0 \\
\downarrow \tilde{\varphi^1} & & \downarrow \tilde{\varphi^0} & & \downarrow \exists! & & \\
p^{n_1} & \xrightarrow{\tilde{B}} & p^{n_0} & \xrightarrow{\pi_{\tilde{B}}} & \mathrm{Coker}(\tilde{B}) & \longrightarrow & 0.
\end{array}$$

induces a unique morphism $p \otimes_E \_(\varphi) : p \otimes_E M \to p \otimes_E N$ that keeps the diagram commutative. It is independent of the choice of lifts as different lifts of projective resolutions are homotopic. This finishes the construction. $\square$

Corollary 6.3.6. *Let E be an R-algebra finitely generated as R-module and $\mathcal{A}$ an R-linear abelian category. Let*

$$T : \mathcal{A} \longrightarrow E\mathrm{-Mod}$$

be an exact, R-linear functor into the category of finitely generated E-modules. Further, let p be a right E-module in $\mathcal{A}$ with structure given by

$$E^{op} \xrightarrow{f} \mathrm{End}_{\mathcal{A}}(p)$$

a morphism of R-algebras. Then the composition

$$E^{op} \xrightarrow{f} \mathrm{End}_{\mathcal{A}}(p) \xrightarrow{T} \mathrm{End}_E(Tp).$$

induces a right action on Tp, making it into an E-bimodule. The composition

$$\begin{array}{ccccc} E-\mathrm{Mod} & \xrightarrow{p\otimes_E \text{-}} & \mathcal{A} & \xrightarrow{T} & E-\mathrm{Mod} \\ M & \mapsto & p\otimes_E M & \mapsto & Tp\otimes_E M \end{array}$$

becomes the usual tensor functor of E-modules.

Proof. It is obvious that the composition

$$\begin{array}{ccccc} E-\mathrm{Mod} & \xrightarrow{p\otimes_E \text{-}} & \mathcal{A} & \xrightarrow{T} & E-\mathrm{Mod} \\ E^n & \mapsto & p\otimes_E E^n & \mapsto & Tp\otimes_E E^n \end{array}$$

induces the usual tensor functor

$$Tp\otimes_E \_ : E-\mathrm{Mod} \longrightarrow E-\mathrm{Mod}$$

on free E-modules. For arbitrary finitely generated E-modules this follows from the fact that $Tp\otimes_E \_$ is right exact and T is exact. □

Remark 6.3.7. Let E be an R-algebra, let M be a right E-module and N be a left E-module. We obtain the tensor product $M\otimes_E N$ by dividing out the equivalence relation $m\cdot e\otimes n \sim m\otimes e\cdot n$ for all $m\in M, n\in N, e\in E$ of the tensor product $M\otimes_R N$ of R-modules. We will now see that a similar approach holds for the abstract tensor products $p\otimes_R M$ and $p\otimes_E M$ in $\mathcal{A}$ as defined in Proposition 6.3.5. For the easier case that R is a field, this approach has been used in [DM].

Lemma 6.3.8. *Let $\mathcal{A}$ be an R-linear, abelian category, E a not necessarily commutative R-algebra which is finitely generated as R-module and $p\in\mathcal{A}$ a right E-module in $\mathcal{A}$. Let $M\in E-\mathrm{Mod}$ and $E'\in E-\mathrm{Mod}$ be in addition a right E-module. Then $p\otimes_E E'$ is a right E-module in $\mathcal{A}$ and we have*

$$p\otimes_E (E'\otimes_E M) = (p\otimes_E E')\otimes_E M.$$

Moreover,

$$(p\otimes_E E)\otimes_R M = p\otimes_R M.$$

Proof. The right E-module structure on $p\otimes_E E'$ is defined by functoriality. The equalities are immediate from the universal property. □

Proposition 6.3.9. *Let $\mathcal{A}$ be an R-linear, abelian category. Let further E be a unital R-algebra with finite generating family $e_1,\dots,e_m$. Let p a right E-module in $\mathcal{A}$ with structure given by*

$$E^{op} \xrightarrow{f} \mathrm{End}_{\mathcal{A}}(p).$$

Let M be a left E-module.

Then $p \otimes_E M$ is isomorphic to the cokernel of the map

$$\Sigma : \bigoplus_{i=1}^{m} (p \otimes_R M) \longrightarrow p \otimes_R M$$

given by

$$\sum_{i=1}^{m} (f(e_i) \otimes \mathrm{id}_M - \mathrm{id}_p \otimes e_i \mathrm{id}_M)\, \pi_i$$

with π_i the projection to the i-summand.

More suggestively (even if not quite correct), we write

$$\Sigma : (x_i \otimes v_i)_{i=1}^m \mapsto \sum_{i=1}^{m} (f(e_i)(x_i) \otimes v_i - x_i \otimes (e_i \cdot v_i))$$

for $x_i \in p$ and $v_i \in M$.

Proof. Consider the sequence

$$\bigoplus_{i=1}^{m} E \otimes_R E \longrightarrow E \otimes E \longrightarrow E \longrightarrow 0$$

where the first map is given by

$$(x_i \otimes y_i)_{i=1}^m \mapsto \sum_{i=1}^{m} x_i e_i \otimes y_i - x_i \otimes e_i y_i$$

and the second is multiplication. We claim that it is exact. The sequence is exact in E because E is unital. The composition of the two maps is zero, hence the cokernel maps to E. The elements in the cokernel satisfy the relation $\bar{x} e_i \otimes \bar{y} = \bar{x} \otimes e_i \bar{y}$ for all $\bar{x}, \bar{y}$ and $i = 1, \dots, m$. The e_i generate E, hence $\bar{x} e \otimes \bar{y} = \bar{x} \otimes e \bar{y}$ for all $\bar{x}, \bar{y}$ and all $e \in E$. Hence the cokernel equals $E \otimes_E E$ which is E via the multiplication map.

Now we tensor the sequence from the left by p and from the right by M and obtain an exact sequence

$$\bigoplus_{i=1}^{m} p \otimes_E (E \otimes_R E) \otimes_E M \longrightarrow p \otimes_E (E \otimes_R E) \otimes_E M \longrightarrow p \otimes_E E \otimes_E M \to 0.$$

Applying the computation rules of Lemma 6.3.8, we get the sequence in the proposition. □

Similarly to Proposition 6.3.5 and Corollary 6.3.6, but less general, we construct a contravariant functor $\mathrm{Hom}_R(p, \_)$:

Proposition 6.3.10. *Let $\mathcal{A}$ be an R-linear abelian category, and p be an object of $\mathcal{A}$. Then the functor*

$$\mathrm{Hom}_{\mathcal{A}}(\_, p) : \mathcal{A}^\circ \longrightarrow R\text{--Mod}$$

has a left adjoint

$$\mathrm{Hom}_R(\_, p) : R\text{--Mod} \longrightarrow \mathcal{A}^\circ.$$

This means that for all $M \in R\text{--Mod}$ and $q \in \mathcal{A}$, we have

$$\mathrm{Hom}_{\mathcal{A}}(q, \mathrm{Hom}_R(M, p)) = \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathcal{A}}(q, p)).$$

It is left exact. It satisfies

$$\mathrm{Hom}_R(R, p) = p.$$

If

$$T : \mathcal{A} \longrightarrow R\text{--Mod}$$

is an exact, R-linear functor into the category of finitely generated R-modules then the composition

$$\begin{array}{ccccc} R\text{--Mod} & \xrightarrow{\mathrm{Hom}(\_,p)} & \mathcal{A} & \xrightarrow{T} & R\text{--Mod} \\ M & \mapsto & \mathrm{Hom}_R(M, p) & \mapsto & \mathrm{Hom}_R(M, Tp) \end{array}$$

is the usual $\mathrm{Hom}(\_, Tp)$-functor in $R\text{--Mod}$.

Proof. The arguments are the same as in the proof of Proposition 6.3.5 and Corollary 6.3.6. □

Remark 6.3.11. Let $\mathcal{A}$ be an R-linear, abelian category. The functors $\mathrm{Hom}_R(\_, p)$ as defined in Proposition 6.3.10 and $p \otimes_R \_$ as defined in Proposition 6.3.6 are also functorial in p, i.e., we have even functors

$$\mathrm{Hom}_R(\_, \_) : (R\text{--Mod})^\circ \times \mathcal{A} \longrightarrow \mathcal{A}$$

and

$$\_ \otimes_R \_ : \mathcal{A} \times R\text{--Mod} \longrightarrow \mathcal{A}.$$

We will denote the image of a morphism $p \xrightarrow{\alpha} q$ under the functor $\mathrm{Hom}_R(M, \_)$ by

$$\mathrm{Hom}_R(M, p) \xrightarrow{\alpha\circ} \mathrm{Hom}_R(M, q)$$

This notation $\alpha\circ$ is natural since by composition

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{\mathrm{Hom}(M,\_)} & \mathcal{A} & \xrightarrow{T} & R\text{--Mod} \\ p & \mapsto & \mathrm{Hom}_R(M, p) & \mapsto & \mathrm{Hom}_R(M, Tp) \end{array}$$

$T(\alpha\circ)$ becomes the usual left action of $T\alpha$ on $\mathrm{Hom}_R(M, Tp)$.

Proof. This follows from the universal property. □

We will now check that the above functors have very similar properties to usual tensor and Hom-functors in $R-\mathrm{Mod}$.

Lemma 6.3.12. *Let $\mathcal{A}$ be an R-linear, abelian category and M a finitely generated R-module. Then the functor $\mathrm{Hom}_R(M, \_)$ is right-adjoint to the functor $\_ \otimes_R M$.*

If

$$T : \mathcal{A} \longrightarrow R-\mathrm{Mod}$$

is an R-linear, exact functor into finitely generated R-modules, the composed functors $T \circ \mathrm{Hom}_R(M, \_)$ and $T \circ (\_ \otimes_R M)$ yield the usual hom-tensor adjunction in $R-\mathrm{Mod}$.

Proof. The assertion follows from the universal property and the identification $T \circ \mathrm{Hom}_R(M, \_) = \mathrm{Hom}_R(M, T\_)$ in Proposition 6.3.10 and $T \circ \_ \otimes_R M = (T\_) \otimes_R M$ in Proposition 6.3.6. □

6.3.2 Construction of the equivalence

Definition 6.3.13. Let $\mathcal{A}$ be an abelian category and $\mathcal{S}$ a not necessarily abelian subcategory. With $\langle \mathcal{S} \rangle$ we denote the smallest full abelian subcategory of $\mathcal{A}$ containing $\mathcal{S}$, i.e., the intersection of all full subcategories of $\mathcal{A}$ that are abelian, contain $\mathcal{S}$, and for which the inclusion functor is exact.

Lemma 6.3.14. *Let $\mathcal{A} = \langle F \rangle$ for a finite set of objects. Let $T : \langle F \rangle \to R-\mathrm{Mod}$ be a faithful exact functor. Then the inclusion $F \to \langle F \rangle$ induces an equivalence*

$$\mathrm{End}(T|_F)-\mathrm{Mod} \longrightarrow \mathcal{C}(\langle F \rangle, T).$$

Proof. Let $E = \mathrm{End}(T|_F)$. Its elements are tuples of endomorphisms of Tp for $p \in F$ commuting with all morphisms $p \to q$ in F.

We have to show that $E = \mathrm{End}(T)$. In other words, that any element of E defines a unique endomorphism of Tq for all objects q of $\langle F \rangle$ and commutes with all morphisms in $\langle F \rangle$.

Any object q is a subquotient of a finite direct sum of copies of objects $p \in F$. The operation of E on Tp with $p \in F$ extends uniquely to an operation on direct sums, kernels and cokernels of morphisms. It is also easy to see that the action commutes with Tf for all morphisms f between these objects. This means that it extends to all objects $\langle F \rangle$, compatible with all morphisms. □

We first concentrate on the case $\mathcal{A} = \langle p \rangle$. From now on, we abbreviate $\mathrm{End}(T|_{\{p\}})$ by $E(p)$.

Lemma 6.3.15. *Let $\langle p\rangle = \mathcal{A}$ be an abelian category. Let $\langle p\rangle \xrightarrow{T} R\text{–Mod}$ a faithful exact R-linear functor into the category of finitely generated R-modules. Let*

$$\langle p\rangle \xrightarrow{\tilde{T}} E(p)\text{–Mod} \xrightarrow{f_T} R\text{–Mod}$$

be the factorization via the diagram category of T constructed in Proposition 6.2.5. Then:

1. *There exists an object $X(p) \in Ob(\langle p\rangle)$ such that*

$$\tilde{T}(X(p)) = E(p).$$

2. *The object $X(p)$ has a right $E(p)$-module structure in $\mathcal{A}$*

$$E(p)^{op} \to \mathrm{End}_{\mathcal{A}}(X(p))$$

such that the induced $E(p)$-module structure on $E(p)$ is the product.

3. *There is an isomorphism*

$$\tau : X(p) \otimes_{E(p)} \tilde{T}p \to p$$

which is natural in $f \in \mathrm{End}_{\mathcal{A}}(p)$, i.e.,

$$\begin{array}{ccc} p & \xrightarrow{f} & p \\ \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} \\ X(p) \otimes_{E(p)} \tilde{T}p & \xrightarrow{id \otimes \tilde{T}f} & X(p) \otimes_{E(p)} \tilde{T}p \end{array}$$

An easier construction of $X(p)$ in the field case can be found in [DM], the construction for R being a noetherian ring is due to Nori [N].

Proof. We consider the object $\mathrm{Hom}_R(Tp, p) \in \mathcal{A}$. Via the contravariant functor

$$\begin{array}{ccc} R\text{–Mod} & \overset{\mathrm{Hom}(\_,p)}{\longrightarrow} & \mathcal{A} \\ Tp & \mapsto & \mathrm{Hom}_R(Tp, p) \end{array}$$

of Proposition 6.3.10 it is a right $\mathrm{End}_R(Tp)$-module in $\mathcal{A}$ which, after applying T just becomes the usual right $\mathrm{End}(Tp)$-module $\mathrm{Hom}_R(Tp, Tp)$. For each $\varphi \in \mathrm{End}(Tp)$, k we will write $\circ\varphi$ for the action on $\mathrm{Hom}(Tp, p)$ as well. By Lemma 6.3.12 the functors $\mathrm{Hom}_R(Tp, \_)$ and $\_ \otimes_R Tp$ are adjoint, so we obtain an evaluation map

$$\tilde{ev} : \mathrm{Hom}_R(Tp, p) \otimes_R Tp \longrightarrow p$$

that becomes the usual evaluation in $R-$Mod after applying T. Our aim is now to define $X(p)$ as a suitable subobject of $\mathrm{Hom}_R(Tp,p) \in \mathcal{A}$. The structures on $X(p)$ will be induced from the structures on $\mathrm{Hom}_R(Tp,p)$.

Let $M \in R-$Mod. We consider the functor

$$\begin{array}{ccc} \mathcal{A} & \stackrel{\mathrm{Hom}_R(M,\_)}{\longrightarrow} & \mathcal{A} \\ p & \mapsto & \mathrm{Hom}_R(M,p) \end{array}$$

of Remark 6.3.11. The endomorphism ring $\mathrm{End}_{\mathcal{A}}(p)) \subset \mathrm{End}_R(Tp)$ is finitely generated as R-module, since T is faithful and R is noetherian. Let $\alpha_1,...,\alpha_n$ be a generating family. Since

$$E(p) = \{\varphi \in \mathrm{End}(Tp) | T\alpha \circ \varphi = \varphi \circ T\alpha \ \forall \alpha : p \to p\},$$

we can write $E(p)$ as the kernel of

$$\begin{array}{ccc} \mathrm{Hom}(Tp,Tp) & \longrightarrow & \bigoplus_{i=1}^n \mathrm{Hom}(Tp,Tp) \\ u & \mapsto & u \circ T\alpha_i - T\alpha_i \circ u \end{array}$$

By the exactness of T, the kernel $X(p)$ of

$$\begin{array}{ccc} \mathrm{Hom}(Tp,p) & \longrightarrow & \bigoplus_{i=1}^n \mathrm{Hom}(Tp,p) \\ u & \mapsto & u \circ T\alpha_i - \alpha_i \circ u \end{array}$$

is a preimage of $E(p)$ under T in $\mathcal{A}$.

By construction, the right $\mathrm{End}_R(Tp)$-module structure on $\mathrm{Hom}_R(Tp,p)$ restricts to a right $E(p)$-module structure on $X(p)$ whose image under $\tilde{T}$ yields the natural $E(p)$ right-module structure on $E(p)$.

Now consider the evaluation map

$$\tilde{ev} : \mathrm{Hom}_R(Tp,p) \otimes_R Tp \longrightarrow p$$

mentioned at the beginning of the proof. By Proposition 6.3.9 we know that the cokernel of the map Σ defined there is isomorphic to $X(p) \otimes_{E(p)} \tilde{T}p$. The diagram

$$\begin{array}{ccccccc} \bigoplus_{i=1}^k (X(p) \otimes_R Tp) & \xrightarrow{\Sigma} & X(p) \otimes_R Tp & \xrightarrow{\mathrm{incl} \otimes id} & \mathrm{Hom}_R(Tp,p) \otimes_R Tp & \xrightarrow{\tilde{\mathrm{ev}}} & p \\ & & & \searrow^{\mathrm{Coker}(\Sigma)} & & & \\ & & & & X(p) \otimes_{E(p)} \tilde{T}p & & \end{array}$$

in $\mathcal{A}$ maps via T to the diagram

$$\begin{array}{ccccccc}
\bigoplus_{i=1}^{k}(E(p)\otimes_R Tp) & \xrightarrow{\Sigma} & E(p)\otimes_R Tp & \xrightarrow{\text{incl}\otimes id} & \text{Hom}_R(Tp,Tp)\otimes_R Tp & \xrightarrow{\text{ev}} & Tp \\
 & & & \searrow^{\text{Coker}(\Sigma)} & & & \\
 & & & & E(p)\otimes_{E(p)}\tilde{T}p & &
\end{array}$$

in $R-$Mod, where the composition of the horizontal maps becomes zero. Since T is faithful, the respective horizontal maps in $\mathcal{A}$ are zero as well and induce a map

$$\tau : X(p)\otimes_{E(p)} Tp \longrightarrow p$$

that keeps the diagram commutative. By definition of Σ in Proposition 6.3.9, the respective map

$$\tilde{T}\tau : E(p)\otimes_{E(p)}\tilde{T}p \longrightarrow \tilde{T}p$$

becomes the natural evaluation isomorphism of E-modules. Since $\tilde{T}$ is faithful, τ is an isomorphism as well.

Naturality in f holds since $\tilde{T}$ is faithful and

$$\begin{array}{ccc}
\tilde{T}p & \xrightarrow{\tilde{T}f} & \tilde{T}p \\
\downarrow{\scriptstyle \tilde{T}\tau} & & \downarrow{\scriptstyle \tilde{T}\tau} \\
E(p)\otimes_{E(p)}\tilde{T}p & \xrightarrow{id\otimes\tilde{T}f} & E(p)\otimes_{E(p)}\tilde{T}p
\end{array}$$

commutes in $E(p)-$Mod. □

Proposition 6.3.16. *Let $\langle p\rangle = \mathcal{A}$ be an R-linear, abelian category and*

$$\mathcal{A}\xrightarrow{T} R-\text{Mod}$$

be as in Theorem 6.1.19. Let

$$\mathcal{A}\xrightarrow{\tilde{T}}\mathcal{C}(\mathcal{A},T)\xrightarrow{f_T} R-\text{Mod}$$

be the factorization of T via its diagram category. Then $\tilde{T}$ is an equivalence of categories with inverse given by $X(p)\otimes_{E(p)}$ _ with $X(p)$ the object constructed in Lemma 6.3.15.

Proof. We have $\mathcal{A} = \langle p\rangle$, thus $\mathcal{C}(\mathcal{A},T) = E(p)-$Mod. Consider the object $X(p)$ of Lemma 6.3.15. It is a right $E(p)$-module in $\mathcal{A}$, in other words

$$\begin{array}{cccc}
f: & (E(p))^{op} & \longrightarrow & \text{End}_{\mathcal{A}}(X(p)) \\
 & \varphi & \longmapsto & \circ\varphi
\end{array}$$

We apply Corollary 6.3.6 to $E = E(p)$, the object $X(p)$, the above f and the functor

$$\tilde{T} : \langle p \rangle \longrightarrow E(p)\text{−Mod} .$$

It yields the functor

$$X(p) \otimes_{E(p)} \_ : E(p)\text{−Mod} \longrightarrow \langle p \rangle$$

such that the composition

$$\begin{array}{ccccc} E(p)\text{−Mod} & \xrightarrow{X(p)\otimes_{E(p)}\_} & \langle p \rangle & \xrightarrow{\tilde{T}} & E(p)\text{−Mod} \\ M & \longmapsto & X(p) \otimes_{E(p)} M & \mapsto & \tilde{T}(X(p)) \otimes_{E(p)} M = E(p) \otimes_{E(p)} M \end{array}$$

becomes the usual tensor product of $E(p)$-modules, and therefore yields the identity functor.

We want to check that $X(p) \otimes_{E(p)} \_$ is a left-inverse functor of $\tilde{T}$ as well. Thus we need to find a natural isomorphism τ, i.e., for all objects $p_1, p_2 \in \mathcal{A}$ we need isomorphisms τ_{p_1}, τ_{p_2} such that for morphisms $f : p_1 \to p_2$ the following diagram commutes:

$$\begin{array}{ccc} X(p) \otimes_{E(p)} \tilde{T}p_1 & \xrightarrow{id \otimes \tilde{T}f} & X(p) \otimes_{E(p)} \tilde{T}p_2 \\ \downarrow{\scriptstyle \tau_{p_1}} & & \downarrow{\scriptstyle \tau_{p_2}} \\ p_1 & \xrightarrow{f} & p_2 \end{array}$$

Since the functors T and f_T are faithful and exact, and we have $T = f_t \circ \tilde{T}$, we know that $\tilde{T}$ is faithful and exact as well. We have already shown that $\tilde{T} \circ X(p) \otimes_{E(p)} \_$ is the identity functor. So $X(p) \otimes_{E(p)} \_$ is faithful exact as well. Since $\mathcal{A}$ is generated by p, it suffices to find a natural isomorphism for p and its endomorphisms. This is provided by the isomorphism τ of Lemma 6.3.15. □

Proof of Theorem 6.1.19. If $\mathcal{A}$ is generated by one object p, then the functor $\tilde{T}$ is an equivalence of categories by Proposition 6.3.16. It remains to reduce to this case.

The diagram category $\mathcal{C}(\mathcal{A}, T)$ arises as a direct limit, hence we have

$$2\text{−colim}_{F \subset Ob(\mathcal{A})} \text{End}(T|_F)\text{−Mod}$$

and in the same way we have

$$\mathcal{A} = 2\text{−colim}_{F \subset Ob(\mathcal{A})} \langle F \rangle$$

with F ranging over the system of full subcategories of $\mathcal{A}$ that contain only a finite number of objects. Moreover, by Lemma 6.3.14, we have $\mathrm{End}(T|_F) = \mathrm{End}(T|_{\langle F\rangle})$. Hence it suffices to check equivalence of categories

$$\langle F\rangle \xrightarrow{\tilde{T}|_{\langle F\rangle}} \mathrm{End}(T|_F)\text{−Mod}$$

for all abelian categories that are generated by a finite number of objects.

We now claim that $\langle F\rangle \cong \langle\bigoplus_{p\in F} p\rangle$ are equivalent: indeed, since any endomorphism of $\bigoplus_{p\in F} p$ is of the form $(a_{pq})_{p,q\in F}$ with $a_{pq} : p \to q$, and since F has all finite direct sums, we know that $\langle\bigoplus_{p\in F} p\rangle$ is a full subcategory of $\langle F\rangle$. On the other hand, for any $q, q' \in F$ the inclusion $q \hookrightarrow \bigoplus_{p\in F} p$ is a kernel and the projection $\bigoplus_{p\in F} p \twoheadrightarrow q'$ is a cokernel, so for any set of morphisms $(a_{qq'})_{q,q'\in F}$, the morphism $a_{qq'} : q \to q'$ by composition

$$q \hookrightarrow \bigoplus_{p\in F} \xrightarrow{(a_{pp'})_{p,p'\in F}} \bigoplus_{p'\in F} p' \twoheadrightarrow q'$$

is contained in $\langle\bigoplus_{p\in F} p\rangle$. Thus $\langle F\rangle$ is a full subcategory of $\langle\bigoplus_{p\in F} p\rangle$.

Similarly one sees that $\mathrm{End}(T|_{\{p\}})$−Mod is equivalent to $\mathrm{End}(T|_F)$−Mod. So we can even assume that our abelian category is generated by just one object $q = \bigoplus_{p\in F} p$. □

6.3.3 Examples and applications

We work out a couple of explicit examples in order to demonstrate the strength of Theorem 6.1.19. We also use the arguments of the proof to deduce an additional property of the diagram property as a first step towards its universal property.

Throughout let R be a noetherian unital ring.

Example 6.3.17. Let $T : R\text{−Mod} \to R\text{−Mod}$ be the identity functor viewed as a representation. Note that R−Mod is generated by the object R^n. By Theorem 6.1.19 and Lemma 6.3.14, we have equivalences of categories

$$\mathrm{End}(T|_{\{R^n\}})\text{−Mod} \longrightarrow \mathcal{C}(R\text{−Mod}, T) \longrightarrow R\text{−Mod}.$$

By definition, $E = \mathrm{End}(T|_{\{R^n\}})$ consists of those elements of $\mathrm{End}_R(R^n)$ which commute with all elements of $\mathrm{End}_{\mathcal{A}}(R^n)$, i.e., the center of the matrix algebra, which is R.

This can be made more interesting by playing with the representation.

Example 6.3.18 (Morita equivalence)**.** Let R be a noetherian commutative unital ring, $\mathcal{A} = R$−Mod. Let P be a flat finitely generated R-module and

$$T : R\text{−Mod} \longrightarrow R\text{−Mod}, \quad M \mapsto M \otimes_R P.$$

It is faithful and exact, hence the assumptions of Theorem 6.1.19 are satisfied and we get an equivalence

$$\mathcal{C}(R\text{−Mod}, T) \longrightarrow R\text{−Mod} .$$

Note that $\mathcal{A} = \langle R \rangle$ and hence by Lemma 6.3.14, $\mathcal{C}(R\text{−Mod}, T) = E\text{−Mod}$ with $E = \mathrm{End}_R(T|_{\{R\}}) = \mathrm{End}_R(P)$. Hence we have shown that

$$\mathrm{End}_R(P)\text{−Mod} \to R\text{−Mod}$$

is an equivalence of categories. This is a case of Morita equivalence of categories of modules.

Example 6.3.19. Let R be a noetherian commutative unital ring and E an R-algebra finitely generated as an R-module. Let

$$T : E\text{−Mod} \to R\text{−Mod}$$

be the forgetful functor. The category E−Mod is generated by the module E. Hence by Theorem 6.1.19 and Lemma 6.3.14, we have again equivalences of categories

$$E'\text{−Mod} \longrightarrow \mathcal{C}(E\text{−Mod}, T) \longrightarrow E\text{−Mod},$$

where $E' = \mathrm{End}(T|_{\{E\}})$ is the subalgebra of $\mathrm{End}_R(E)$ of endomorphisms compatible with all E-morphisms $E \to E$. Note that $\mathrm{End}_E(E) = E^{op}$ and hence E' is the centralizer of E^{op} in $\mathrm{End}_R(E)$

$$E' = C_{\mathrm{End}_R(E)}(E^{op}) = E .$$

Hence in this case the functor $\mathcal{A} \to \mathcal{C}(\mathcal{A}, T)$ is the identity.

We deduce another consequence of the explicit description of $\mathcal{C}(D, T)$.

Proposition 6.3.20. *Let D be a diagram and $T : D \to R\text{−Mod}$ a representation. Let*

$$D \xrightarrow{\tilde{T}} \mathcal{C}(D, T) \xrightarrow{f_T} R\text{−Mod}$$

its factorization. Then the category $\mathcal{C}(D, T)$ is generated by the image of $\tilde{T}$:

$$\mathcal{C}(D, T) = \langle \tilde{T}(D) \rangle .$$

Proof. It suffices to consider the case when D is finite. Let $X = \bigoplus_{p \in D} Tp$ and $\mathbb{E} = \mathrm{End}_R(X)$. Let $S \subset \mathbb{E}$ be the R-subalgebra generated by Te for $e \in E(D)$ and the projectors $p_p : X \to T(p)$. Then

$$E = \mathrm{End}(T) = C_{\mathbb{E}}(S)$$

is the commutator of S in $\mathbb{E}$. (The endomorphisms commuting with the projectors are those respecting the decomposition. By definition, $\mathrm{End}(T)$ consists of those endomorphisms of the summands commuting with all Te.)

By construction $\mathcal{C}(D,T) = E-\mathrm{Mod}$. We claim that it is equal to

$$\mathcal{A} := \langle\{\tilde{T}p | p \in D\}\rangle = \langle\tilde{X}\rangle$$

with $\tilde{X} = \bigoplus_{p\in D} \tilde{T}p$. The category has a faithful exact representation by $f_T|_{\mathcal{A}}$. Note that $f_T(\tilde{X}) = X$. By Theorem 6.1.19, the category $\mathcal{A}$ is equivalent to its diagram category $\mathcal{C}(\langle\tilde{X}\rangle, f_T) = E'-\mathrm{Mod}$ with $E' = \mathrm{End}(f_T|_{\mathcal{A}})$. By Lemma 6.3.14, E' consists of elements of $\mathbb{E}$ commuting with all elements of $\mathrm{End}_{\mathcal{A}}(\tilde{X})$. Note that

$$\mathrm{End}_{\mathcal{A}}(\tilde{X}) = \mathrm{End}_E(X) = C_{\mathbb{E}}(E)$$

and hence

$$E' = C_{\mathbb{E}}(C_{\mathbb{E}}(E)) = C_{\mathbb{E}}(C_{\mathbb{E}}(C_{\mathbb{E}}(S)) = C_{\mathbb{E}}(S)$$

because a triple commutator equals the simple commutator. We have shown $E = E'$ and the two categories are equivalent.

□

Remark 6.3.21. This is a direct proof of Proposition 6.1.15.

6.4 Universal property of the diagram category

At the end of this section we will be able to establish the universal property of the diagram category.

Let $T : D \longrightarrow R-\mathrm{Mod}$ be a diagram and

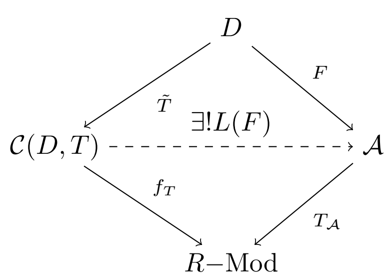
$$D \xrightarrow{\tilde{T}} \mathcal{C}(D,T) \xrightarrow{f_T} R-\mathrm{Mod}$$

the factorization of T via its diagram category. Let $\mathcal{A}$ be another R-linear abelian category, $F : D \to \mathcal{A}$ a representation, and $f : \mathcal{A} \to R-\mathrm{Mod}$ a faithful, exact, R-linear functor into the categories of finitely generated R-modules such that $f \circ F = T$.

Our aim is to deduce that there exists - uniquely up to isomorphism - an R-linear exact faithful functor

$$L(F) : \mathcal{C}(D,T) \to \mathcal{A},$$

making the following diagram commute:



Proposition 6.4.1. *There is a functor $L(F)$ making the diagram commute.*

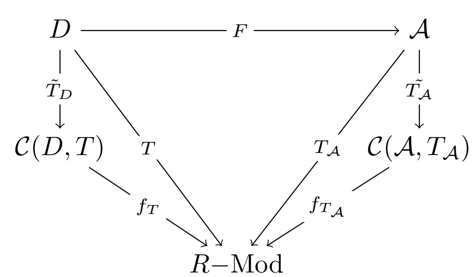
Proof. We can regard $\mathcal{A}$ as a diagram and obtain a representation

$$\mathcal{A} \xrightarrow{T_{\mathcal{A}}} R\text{--Mod},$$

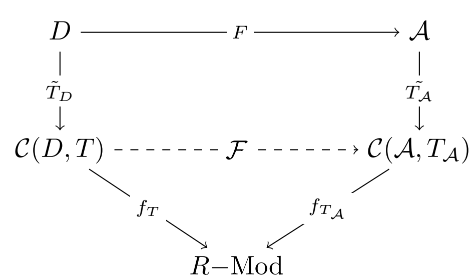
that factorizes via its diagram category

$$\mathcal{A} \xrightarrow{\tilde{T}_{\mathcal{A}}} \mathcal{C}(\mathcal{A}, T_{\mathcal{A}}) \xrightarrow{f_{T_{\mathcal{A}}}} R\text{--Mod}.$$

We obtain the following commutative diagram



By functoriality of the diagram category (see Proposition 6.2.6) there exists an R-linear faithful exact functor $\mathcal{F}$ such that the following diagram commutes:



Since $\mathcal{A}$ is R-linear, abelian, and T is faithful, exact, R-linear, we know by Proposition 6.1.19, that $\tilde{T}_{\mathcal{A}}$ is an equivalence of categories. The functor

$$L(F) : \mathcal{C}(D, T) \to \mathcal{A},$$

is given by the composition of $\mathcal{F}$ with the inverse of $\tilde{T}_{\mathcal{A}}$. Since an equivalence of R-linear categories is exact, faithful and R-linear, $L(F)$ is so as well, as it is the composition of such functors. □

Proposition 6.4.2. *The functor $L(F)$ is unique up to unique isomorphism.*

Proof. Let L' be another functor satisfying the condition in the diagram. Let $\mathcal{C}'$ be the subcategory of $\mathcal{C}(D,T)$ on which $L' = L(F)$. We claim that the inclusion is an equivalence of categories. Without loss of generality, we may assume D is finite.

Note that the subcategory is full because $f : \mathcal{A} \to R-\mathrm{Mod}$ is faithful. It contains all objects of the form $\tilde{T}p$ for $p \in D$. As the functors are additive, this implies that they also have to agree (up to canonical isomorphism) on finite direct sums of objects. As the functors are exact, they also have to agree on and all kernels and cokernels. Hence $\mathcal{C}'$ is the full abelian subcategory of $\mathcal{C}(D,T)$ generated by $\tilde{T}(D)$. By Proposition 6.3.20 this is all of $\mathcal{C}(D,T)$. □

Proof of Theorem 6.1.13. Let $T : D \to R-\mathrm{Mod}$ be a representation and $f : \mathcal{A} \to R-\mathrm{Mod}$, $F : D \to \mathcal{A}$ as in the statement. By Proposition 6.4.1 the functor $L(F)$ exists. It is unique by Proposition 6.4.2. Hence $\mathcal{C}(D,T)$ satisfies the universal property of Theorem 6.1.13.

Let $\mathcal{C}$ be another category satisfying the universal property. By the universal property for $\mathcal{C}(D,T)$ and the representation of D in $\mathcal{C}$, we get a functor $\Psi : \mathcal{C}(D,T) \to \mathcal{C}$. By interchanging their roles, we obtain a functor Ψ' in the opposite direction. Their composition $\Psi' \circ \Psi$ satisfies the universal property for $\mathcal{C}(D,T)$ and the representation $\tilde{T}$. By the uniqueness part, it is isomorphic to the identity functor. The same argument also applies to $\Psi \circ \Psi'$. Hence Ψ is an equivalence of categories.

Functoriality of $\mathcal{C}(D,T)$ in D is Lemma 6.2.6. □

The generalized universal property follows by a trick.

Proof of Corollary 6.1.14. Let $T : D \to R-\mathrm{Mod}$, $f : \mathcal{A} \to R-\mathrm{Mod}$ und $F : D \to \mathcal{A}$ be as in the corollary. Let S be a faithfully flat R-algebra and

$$\phi : T_S \to (f \circ F)_S$$

an isomorphism of representations into $S-\mathrm{Mod}$. We first show the existence of $L(F)$.

Let $\mathcal{A}'$ be the category with objects of the form (V_1, V_2, ψ) where $V_1 \in R-\mathrm{Mod}$, $V_2 \in \mathcal{A}$ and $\psi : V_1 \otimes_R S \to f(V_2) \otimes_R S$ an isomorphism. Morphisms are defined as pairs of morphisms in $R-\mathrm{Mod}$ and $\mathcal{A}$ such the obvious diagram commutes. This category is abelian because S is flat over R. Kernels and cokernels are taken componentwise. Let $f' : \mathcal{A}' \to R-\mathrm{Mod}$ be the projection to the first component. It is faithful and exact because S is faithfully flat over R.

The data T, F and ϕ define a representation $F' : D \to \mathcal{A}'$ compatible with T. By the universal property of Theorem 6.1.13, we obtain a factorization

$$F' : D \xrightarrow{\tilde{T}} \mathcal{C}(D,T) \xrightarrow{L(F')} \mathcal{A}' \ .$$

We define $L(F)$ as the composition of $L(F')$ with the projection to the second component. The transformation

$$\tilde{\phi} : (f_T)_S \to f_S \circ L(F)$$

is defined on $X \in \mathcal{C}(D,T)$ using the isomorphism ψ part of the object $L(F')(X) \in \mathcal{A}'$.

Conversely, the triple $(f, L(F), \tilde{\phi})$ satisfies the universal property of $L(F')$. By the uniqueness part of the universal property, this means that it agrees with $L(F')$. This makes $L(F)$ unique. □

6.5 The diagram category as a category of comodules

Under more restrictive assumptions on R and T, we can give a description of the diagram category of comodules, see Theorem 6.1.12.

6.5.1 Preliminary discussion

In [DM] Deligne and Milne note that if R is a field, E a finite-dimensional R-algebra, and V an E-module that is finite-dimensional as R-vector space then V has a natural structure as comodule over the coalgebra $E^\vee := \mathrm{Hom}_R(E,R)$. For an algebra E finitely generated as an R-module over an arbitrary noetherian ring R, the R-dual $E^\vee$ does not even necessarily carry a natural structure of an R-coalgebra. The problem is that the dual map to the algebra multiplication

$$E^\vee \xrightarrow{\mu^*} (E \otimes_R E)^\vee$$

does not generally define a comultiplication because the canonical map

$$\rho : E^\vee \otimes_R E^\vee \to \mathrm{Hom}(E, E^\vee) \cong (E \otimes_R E)^\vee$$

fails to be an isomorphism in general. In this chapter we will see that this isomorphism holds true for the R-algebras $\mathrm{End}(T_F)$ if we assume that R is a Dedekind domain or field. We will then show that by

$$\begin{aligned} \mathcal{C}(D,T) &= 2\text{−}\mathrm{colim}_{F\subset D}(\mathrm{End}(T_F)\text{−Mod}) \\ &= 2\text{−}\mathrm{colim}_{F\subset D}(\mathrm{End}(T_F)^\vee\text{−Comod}) = (2\text{−}\mathrm{colim}_{F\subset D}\mathrm{End}(T_F)^\vee)\text{−Comod} \end{aligned}$$

we can view the diagram category $\mathcal{C}(D,T)$ as the category of finitely generated comodules over the coalgebra $2\text{−}\mathrm{colim}_{F\subset D}\mathrm{End}(T_F)^\vee$.

Remark 6.5.1. Note that the category of comodules over an arbitrary coalgebra C is not abelian in general, since the tensor product $X \otimes_R -$ is right exact, but in general not left exact. If C is flat as R-algebra (e.g. free), then the category of C-comodules is abelian [MM, pg. 219].

6.5.2 Coalgebras and comodules

Let R be a noetherian ring with unit.

Proposition 6.5.2. *Let E be an R-algebra which is finitely generated as R-module. Then the canonical map*

$$\begin{array}{rccc} \rho: & E^\vee \otimes_R M & \to & \mathrm{Hom}(E, M) \\ & \varphi \otimes m & \mapsto & (n \mapsto \varphi(n) \cdot m) \end{array}$$

becomes an isomorphism for all R-modules M if and only if E is projective.

Proof. [Str, Proposition 5.2] □

Lemma 6.5.3. *Let E be an R-algebra which is finitely generated and projective as an R-module.*

1. *The R-dual module $E^\vee$ carries a natural structure of a counital coalgebra.*
2. *Any left E-module that is finitely generated as R-module carries a natural structure as left $E^\vee$-comodule.*
3. *We obtain an equivalence of categories between the category of finitely generated left E-modules and the category of finitely generated left $E^\vee$-comodules.*

Proof. By the repeated application of Proposition 6.5.2, this becomes a straightforward calculation. We will sketch the main steps of the proof.

1. If we dualize the associativity constraint of E we obtain a commutative diagram of the form

$$\begin{array}{ccc} (E \otimes_R E \otimes_R E)^\vee & \xleftarrow[(\mu \otimes id)^*]{} & (E \otimes_R E)^\vee \\ \uparrow{\scriptstyle (id \otimes \mu)^*} & & \uparrow{\scriptstyle \mu^*} \\ (E \otimes_R E)^\vee & \xleftarrow[\mu^*]{} & E^\vee. \end{array}$$

By the use of the isomorphism in Propostion 6.5.2 and Hom-Tensor adjunction we obtain the commutative diagram

$$\begin{array}{ccc} E^\vee \otimes_R E^\vee \otimes_R E^\vee & \xleftarrow[\mu^* \otimes id^*]{} & E^\vee \otimes_R E^\vee \\ \uparrow{\scriptstyle id^* \otimes \mu^*} & & \uparrow{\scriptstyle \mu^*} \\ E^\vee \otimes_R E^\vee & \xleftarrow[\mu^*]{} & E^\vee, \end{array}$$

which induces a cocommutative comultiplication on $E^\vee$. Similarly we obtain the counit diagram, so $E^\vee$ naturally gets a coalgebra structure.

2. For an E-module M we analogously dualize the respective diagram

$$\begin{array}{ccc} M & \xleftarrow[\;m\;]{} & E \otimes_R M \\ {\scriptstyle m}\uparrow & & \uparrow{\scriptstyle id \otimes m} \\ E \otimes_R M & \xleftarrow[\mu \otimes id]{} & E \otimes_R E \otimes_R M \end{array}$$

and use Proposition 6.5.2 and Hom-Tensor adjunction to see that the E-multiplication induces a well-defined $E^\vee$-comultiplication

$$\begin{array}{ccc} M & \xrightarrow{\hat{m}} & E^\vee \otimes_R M \\ {\scriptstyle \hat{m}}\downarrow & & \downarrow{\scriptstyle id \otimes \hat{m}} \\ E^\vee \otimes_R M & \xrightarrow{\mu^* \otimes id} & E^\vee \otimes_R E^\vee \otimes_R M \end{array}$$

on M.

3. For any homomorphism $f : M \longrightarrow N$ of left E-modules, the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle \mu}\uparrow & & \uparrow{\scriptstyle \mu} \\ E \otimes_R M & \xrightarrow{id \otimes f} & E \otimes_R N \end{array}$$

induces a commutative diagram

$$\begin{array}{ccc} E^\vee \otimes_R M & \xrightarrow{id \otimes f} & E^\vee \otimes_R N, \\ {\scriptstyle \hat{\mu}}\uparrow & & \uparrow{\scriptstyle \hat{\mu}} \\ M & \xrightarrow{f} & N \end{array}$$

thus f is a homomorphism of left $E^\vee$-comodules.

4. Conversely, we can dualize the $E^\vee$-comodule structure to obtain a $(E^\vee)^\vee = E$-module structure. The two constructions are inverse to each other.

□

Definition 6.5.4. Let A be a coalgebra over R. Then we denote by $A-$Comod the category of comodules over A that are finitely generated as a R-modules.

Recall that $R-$Proj denotes the category of finitely generated projective R-modules.

Corollary 6.5.5. *Let R be a field or Dedekind domain, D a diagram and*

$$T : D \longrightarrow R-\mathrm{Proj}$$

a representation. Set $A(D,T) := \varinjlim_{F\subset D \text{ finite}} \mathrm{End}(T_F)^\vee$. Then $A(D,T)$ has the structure of a coalgebra and the diagram category of T is the abelian category $A(D,T)-$Comod.

Proof. For any finite subset $F \subset D$ the algebra $\mathrm{End}(T_F)$ is a submodule of the finitely generated projective R-module $\prod_{p\in F}\mathrm{End}(T_p)$. Since R is a field or Dedekind domain, for a finitely generated module to be projective is equivalent to being torsion free. Hence the submodule $\mathrm{End}(T_F)$ is also finitely generated and torsion-free, or equivalently, projective. By the previous lemma, $\mathrm{End}(T_F)^\vee$ is an R-coalgebra and $\mathrm{End}(T_F)-\mathrm{Mod} \cong \mathrm{End}(T_F)^\vee-\mathrm{Comod}$. From now on, we denote $\mathrm{End}(T_F)^\vee$ with $A(F,T)$. Taking limits over the direct system of finite subdiagrams as in Definition 6.1.10, we obtain

$$\begin{aligned}\mathcal{C}(D,T) := 2-\mathrm{colim}_{F\subset D \text{ finite}}\mathrm{End}(T_F)-\mathrm{Mod} \\ = 2-\mathrm{colim}_{F\subset D \text{ finite}}A(F,T)-\mathrm{Comod}.\end{aligned}$$

Since the category of coalgebras is cocomplete, $A(D,T) = \varinjlim_{F\subset D} A(F,T)$ is a coalgebra as well.

We now need to show that the categories $2-\mathrm{colim}_{F\subset D \text{ finite}}(A(F,T)-\mathrm{Comod})$ and $A(D,T)-$Comod are equivalent. For any finite F the canonical map $A(F,T) \longrightarrow A(D,T)$ via restriction of scalars induces a functor

$$\phi_F : A(F,T)-\mathrm{Comod} \longrightarrow A(D,T)-\mathrm{Comod}$$

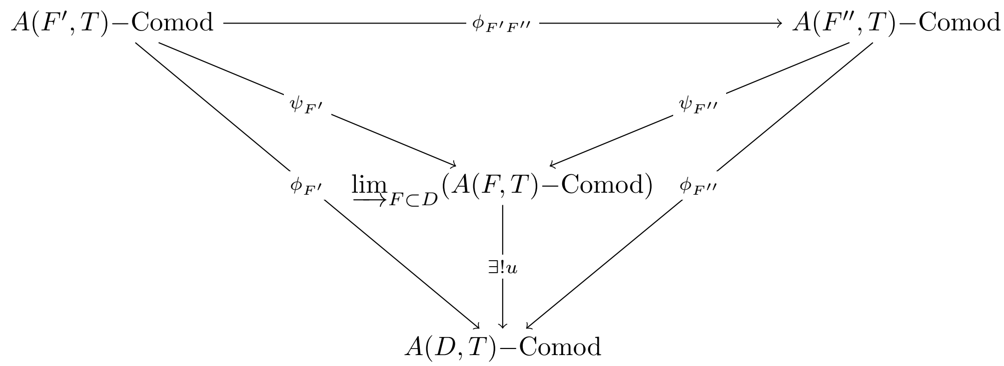
and therefore by the universal property a unique functor

$$u : \varinjlim A(F,T)-\mathrm{Comod} \longrightarrow A(D,T)-\mathrm{Comod}.$$

such that for all finite $F', F'' \subset D$ with $F' \subset F''$ and the canonical functors

$$\psi_F : A(F',T)-\mathrm{Comod} \longrightarrow \varinjlim_{F\subset D} A(F,T)-\mathrm{Comod}$$

the following diagram commutes:



We construct an inverse map to u: Let M be an $A(D,T)$-comodule and

$$m : M \to M \otimes_R A(D,T)$$

be the comultiplication. Let $M = \langle x_1, .., x_n \rangle_R$. Then $m(x_i) = \sum_{k=1}^n a_{ki} \otimes x_k$ for certain $a_{ki} \in A(D,T)$. Every a_{ki} is already contained in an $A(F,T)$ for sufficiently large F. By taking the union of these finitely many F, we can assume that all a_{ki} are contained in one coalgebra $A(F,T)$. Since $x_1, .., x_n$ generate M as R-module, m defines a comultiplication

$$\tilde{m} : M \to M \otimes_R A(F,T).$$

So M is an $A(F,T)$-comodule in a natural way, thus via ψ_F an object of $2-\text{colim}_I(A_i-\text{Comod})$. □

We also need to understand the behavior of $A(D,T)$ under base-change.

Lemma 6.5.6 (Base change)**.** *Let R be a field or a Dedekind domain and $T : D \to R-\text{Proj}$ a representation. Let $R \to S$ be flat. Then*

$$A(D,T_S) = A(D,T) \otimes_R S\ .$$

Proof. Let $F \subset D$ be a finite subdiagram. Recall that

$$A(F,T) = \text{Hom}_R(\text{End}(T|_F), R)\ .$$

Both R and $\text{End}_R(T|_F)$ are projective because R is a field or a Dedekind domain. Hence by Lemma 6.2.2

$$\text{Hom}_R(\text{End}_R(T|_F), R) \otimes S \cong \text{Hom}_S(\text{End}_R(T|_F) \otimes S, S) \cong \text{Hom}_S(\text{End}_S((T_S)|_F), S).$$

This is nothing but $A(F,T_S)$. Tensor products commute with direct limits, hence the statement for $A(D,T)$ follows immediately. □

Chapter 7

More on diagrams

We study additional structures on a diagram and a representation that lead to the construction of a tensor product on the diagram category. The aim is then to turn it into a rigid tensor category with a faithful exact functor to a category of R-modules. The chapter is formal, but the assumptions are tailored to the application to Nori motives.

A particularly puzzling and subtle question is how the question of graded commutativity of the Künneth formula is dealt with.

We continue to work in the setting of Chapter 6.

7.1 Multiplicative structure

Let R a fixed noetherian unital commutative ring.

Recall that $R-\mathrm{Proj}$ is the category of projective R-modules of finite type over R. We only consider representations $T : D \longrightarrow R-\mathrm{Proj}$ where D is a diagram with identities, see Definition 6.1.1.

Definition 7.1.1. Let D_1, D_2 be diagrams with identities. Then $D_1 \times D_2$ is defined as the diagram with vertices of the form (v, w) for v a vertex of D_1, w a vertex of D_2, and with edges of the form (α, id) and (id, β) for α an edge of D_1 and β an edge of D_2 and with $\mathrm{id} = (\mathrm{id}, \mathrm{id})$.

Remark 7.1.2. Levine in [L1] p.466 seems to define $D_1 \times D_2$ by taking the product of the graphs in the ordinary sense. He claims (in the notation of loc. cit.) a map of diagrams

$$H_*\mathrm{Sch}' \times H_*\mathrm{Sch}' \to H_*\mathrm{Sch}'.$$

It is not clear to us how this is defined on general pairs of edges. If α, β are edges corresponding to boundary maps and hence lower the degree by 1, then we would expect $\alpha \times \beta$ to lower the degree by 2. However, there are no such edges in $H_*\mathrm{Sch}'$.

Our restricted version of products of diagrams is enough to get the implications we want.

In order to control signs in the Künneth formula, we need to work in a graded commutative setting.

Definition 7.1.3. A *graded diagram* is a diagram D with identities together with a map

$$|\cdot| : \{\text{vertices of } D\ \} \to \mathbb{Z}/2\mathbb{Z}\ .$$

For an edge $\gamma : v \to v'$ we put $|\gamma| = |v| - |v'|$. If D is a graded diagram, $D \times D$ is equipped with the grading $|(v, w)| = |v| + |w|$.

A *commutative product structure* on a graded D is a map of graded diagrams

$$\times : D \times D \to D$$

together with choices of edges

$$\begin{aligned} \alpha_{v,w} &: v \times w \to w \times v \\ \beta_{v,w,u} &: v \times (w \times u) \to (v \times w) \times u \\ \beta'_{v,w,u} &: (v \times w) \times u \to v \times (w \times u) \end{aligned}$$

for all vertices v, w, u of D.

A *graded multiplicative representation* T of a graded diagram with commutative product structure is a representation of T in $R-$Proj together with a choice of isomorphism

$$\tau_{(v,w)} : T(v \times w) \to T(v) \otimes T(w)$$

such that:

1. The composition

$$T(v) \otimes T(w) \xrightarrow{\tau^{-1}_{(v,w)}} T(v \times w) \xrightarrow{T(\alpha_{v,w})} T(w \times v) \xrightarrow{\tau_{(w,v)}} T(w) \otimes T(v)$$

   is $(-1)^{|v||w|}$ times the natural map of R-modules.

2. If $\gamma : v \to v'$ is an edge, then the diagram

$$\begin{array}{ccc} T(v \times w) & \xrightarrow{T(\gamma \times \mathrm{id})} & T(v' \times w) \\ \tau \downarrow & & \downarrow \tau \\ T(v) \otimes T(w) & \xrightarrow{(-1)^{|\gamma||w|} T(\gamma) \otimes \mathrm{id}} & T(v') \otimes T(w) \end{array}$$

   commutes.

3. If $\gamma : v \to v'$ is an edge, then the diagram

$$\begin{array}{ccc} T(w \times v) & \xrightarrow{T(\mathrm{id}\times\gamma)} & T(w \times v') \\ \tau\downarrow & & \downarrow\tau \\ T(w) \otimes T(v) & \xrightarrow{\mathrm{id}\otimes T(\gamma)} & T(w) \otimes T(v') \end{array}$$

commutes.

4. The diagram

$$\begin{array}{ccc} T(v \times (w \times u)) & \xrightarrow{T(\beta_{v,w,u})} & T((v \times w) \times u) \\ \downarrow & & \downarrow \\ T(v) \otimes T(w \times u) & & T(v \times w) \otimes T(u) \\ \downarrow & & \downarrow \\ T(v) \otimes (T(w) \otimes T(u)) & \longrightarrow & (T(v) \otimes T(w)) \otimes T(u) \end{array}$$

commutes under the standard identification

$$T(v) \otimes (T(w) \otimes T(u)) \cong (T(v) \otimes T(w)) \otimes T(u).$$

The maps $T(\beta_{v,w,u})$ and $T(\beta'_{v,w,u})$ are inverse to each other.

A *unit* for a graded diagram with commutative product structure D is a vertex $\mathbf{1}$ of degree 0 together with a choice of edges

$$u_v : v \to \mathbf{1} \times v$$

for all vertices of v. A graded multiplicative representation is *unital* if $T(\mathbf{1})$ is free of rank 1 and $T(u_v)$ is an isomorphism for all vertices v satisfying the following condition: Let $R \to T(\mathbf{1})$ be the isomorphism determined by

$$T(u_{\mathbf{1}}) : T(\mathbf{1}) \to T(\mathbf{1}) \otimes T(\mathbf{1}).$$

Under this identification $T(u_v)$ identifies with the natural isomorphism

$$T(v) \to R \otimes T(v).$$

Remark 7.1.4. 1. In particular, $T(\alpha_{v,w})$ and $T(\beta_{v,w,u})$ are isomorphisms. If $v = w$ then $T(\alpha_{v,v}) = (-1)^{|v|}$.

2. If $\mathbf{1}$ is a unit, then $T(u_{\mathbf{1}})$ defines a distinguished isomorphism $T(\mathbf{1}) \to T(\mathbf{1}) \otimes T(\mathbf{1})$. Hence it is either 0 or a free R-module of rank 1. The definition excluded the first case.

3. Note that the first and the second factor are *not* treated symmetrically. There is a choice of sign convention involved. The convention above is chosen to be consistent with the one of Section 1.3. Eventually, we want to view relative singular cohomology as graded multiplicative representation in the above sense.

4. For the purposes immediately at hand, the choice of $\beta'_{v,w,u}$ is not needed. However, it is needed later on in the definition of the product structure on the localized diagram, see Remark 7.2.2.

Let $T : D \longrightarrow R-\mathrm{Proj}$ be a representation of a diagram with identities. Recall that we defined its diagram category $\mathcal{C}(D,T)$ (see Definition 6.1.10). It R is a field or a Dedekind domain, then $\mathcal{C}(D,T)$ can be described as the category of $A(D,T)$-comodules of finite type over R for the coalgebra $A(D,T)$ defined in Theorem 6.1.12.

Proposition 7.1.5. *Let D be a graded diagram with commutative product structure with unit and T a unital graded multiplicative representation of D in $R-\mathrm{Proj}$*

$$T : D \longrightarrow R-\mathrm{Proj}.$$

1. *Then $\mathcal{C}(D,T)$ carries the structure of a commutative and associative tensor category with unit and $T : \mathcal{C}(D,T) \to R-\mathrm{Mod}$ is a tensor functor. On the generators $\tilde{T}(v)$ of $\mathcal{C}(D,T)$ the associativity constraint is induced by the edges $\beta_{v,w,u}$, the commutativity constraint is induced by the edges $\alpha_{v,w}$, the unit object is $\tilde{\mathbf{1}}$ with unital maps induced from the edges u_v.*

2. *If, in addition, R is a field or a Dedekind domain, the coalgebra $A(D,T)$ carries a natural structure of commutative bialgebra (with unit and counit).*

The unit object is going to be denoted $\mathbf{1}$.

Proof. We consider finite diagrams F and F' such that

$$\{v \times w | v, w \in F\} \subset F' \ .$$

We are going to define natural maps

$$\mu_F^* : \mathrm{End}(T|_{F'}) \to \mathrm{End}(T|_F) \otimes \mathrm{End}(T|_F).$$

Assume this for a moment. Let $X, Y \in \mathcal{C}(D,T)$. We want to define $X \otimes Y$ in $\mathcal{C}(D,T) = 2-\mathrm{colim}_F \mathcal{C}(F,T)$. Let F such that $X, Y \in \mathcal{C}(F,T)$. This means that X and Y are finitely generated R-modules with an action of $\mathrm{End}(T|_F)$. We equip the R-module $X \otimes Y$ with a structure of $\mathrm{End}(T|_{F'})$-module. It is given by

$$\mathrm{End}(T|_{F'}) \otimes X \otimes Y \to \mathrm{End}(T|_F) \otimes \mathrm{End}(T|_F) \otimes X \otimes Y \to X \otimes Y$$

where we have used the comultiplication map μ_F^* and the module structures of X and Y. This will be independent of the choice of F and F'. Properties of $\otimes$ on $\mathcal{C}(D,T)$ follow from properties of μ_F^*.

If R is a field or a Dedekind domain, let

$$\mu_F : A(F,T) \otimes A(F,T) \to A(F',T)$$

be dual to μ_F^*. Passing to the direct limit defines a multiplication μ on $A(D,T)$.

We now turn to the construction of μ_F^*. Let $a \in \mathrm{End}(T|_{F'})$, i.e., a compatible system of endomorphisms $a_v \in \mathrm{End}(T(v))$ for $v \in F'$. We describe its image $\mu_F^*(a)$. Let $(v,w) \in F \times F$. The isomorphism

$$\tau : T(v \times w) \to T(v) \otimes T(w)$$

induces an isomorphism

$$\mathrm{End}(T(v \times w)) \cong \mathrm{End}(T(v)) \otimes \mathrm{End}(T(w)).$$

We define the (v,w)-component of $\mu^*(a)$ by the image of $a_{v\times w}$ under this isomorphism.

In order to show that this is a well-defined element of $\mathrm{End}(T|_F) \otimes \mathrm{End}(T|_F)$, we need to check that diagrams of the form

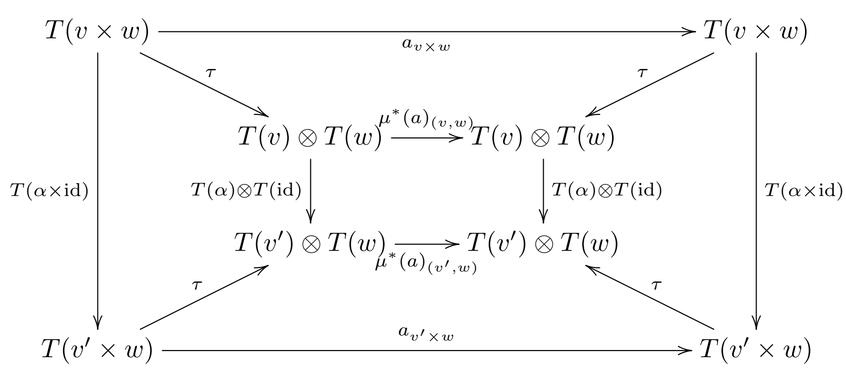
$$\begin{array}{ccc} T(v)\otimes T(w) & \xrightarrow{\mu^*(a)_{(v,w)}} & T(v)\otimes T(w) \\ \downarrow{\scriptstyle T(\alpha)\otimes T(\beta)} & & \downarrow{\scriptstyle T(\alpha)\otimes T(\beta)} \\ T(v')\otimes T(w') & \xrightarrow[\mu^*(a)_{(v',w')}]{} & T(v')\otimes T(w') \end{array}$$

commute for all edges $\alpha : v \to v'$, $\beta : w \to w'$ in F. We factor

$$T(\alpha) \otimes T(\beta) = (T(\mathrm{id}) \otimes T(\beta)) \circ (T(\alpha) \circ T(\mathrm{id}))$$

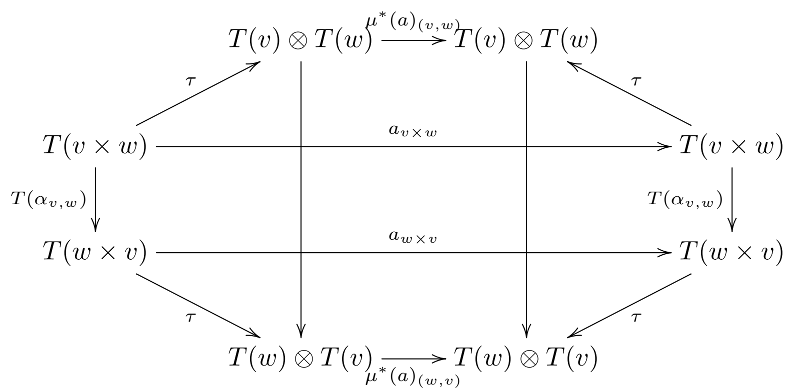
and check the factors separately.

Consider the diagram



The outer square commutes because a is a diagram endomorphism. Top and bottom commute by definition of $\mu^*(a)$. Left and right commute by property (3) up to the same sign $(-1)^{|w||\alpha|}$. Hence the middle square commutes without signs. The analogous diagram for $\mathrm{id} \times \beta$ commutes on the nose. Hence $\mu^*(a)$ is well-defined.

We now want to compare the (v,w)-component to the (w,v)-component. Recall that there is a distinguished edge $\alpha_{v,w} : v \times w \to w \times v$. Consider the diagram



By the construction of $\mu^*(a)_{(v,w)}$ (resp. $\mu^*(a)_{(w,v)}$), the upper (resp. lower) tilted square commutes. By naturality, the middle rectangle with $\alpha_{v,w}$ commutes. By property (1) of a representation of a graded diagram with commutative product, the left and right faces commute where the vertical maps are $(-1)^{|v||w|}$ times the natural commutativity of tensor products of T-modules. Hence the inner square also commutes without the sign factors. This is cocommutativity of μ^*.

The associativity assumption (3) for representations of diagrams with product structure implies the coassociativity of μ^*.

The compatibility of multiplication and comultiplication is built into the definition.

In order to define a unit object in $\mathcal{C}(D,T)$ it suffices to define a counit for $\mathrm{End}(T|_F)$. Assume $\mathbf{1} \in F$. The counit

$$u^* : \mathrm{End}(T|_F) \subset \prod_{v \in F} \mathrm{End}(T(v)) \to \mathrm{End}(T(\mathbf{1})) = R$$

is the natural projection. The assumption on unitality of T allows to check that the required diagrams commute.

This finishes the argument for the tensor category and its properties. If R is a field or a Dedekind domain, we have shown that $A(D,T)$ has a multiplication and a comultiplication. The unit element $1 \in A(D,T)$ is induced from the canonical element $1 \in A(\{\mathbf{1}\},T) = \mathrm{End}_R(T(\mathbf{1}))^\vee = R$ (Note that the last identification is indeed canonical, independent of the choice of basis vector in

$T(\mathbf{1}) \cong R$.) It remains to show that $1 \neq 0$ in $A(D,T)$ or equivalently its image is non-zero in all $A(F,T)$ with F a finite diagram containing $\mathbf{1}$. We can view 1 as map

$$\mathrm{End}(T|_F) \to R\ .$$

It is non-zero because it maps id to 1. □

Remark 7.1.6. The proof of Proposition 7.1.5 works without any changes in the arguments when we weaken the assumptions as follows: in Definition 7.1.3 replace $\times$ by a map of diagrams

$$\times : D \times D \to \mathcal{P}(D)$$

where $\mathcal{P}(D)$ is the path category of D: objects are the vertices of D and morphisms the paths. A representation T of D extends canonically to a functor on $\mathcal{P}(D)$.

Example 7.1.7. Let $D = \mathbb{N}_0$. We impose a minimal set of edges which allows for the definition of a commutative product structure such that $n \mapsto V^{\otimes n}$ for fixed vector space V becomes A multiplicative representation. The only edges are self-edges. We denote them suggestively

$$\mathrm{id}_a \times \alpha_{v,w} \times \mathrm{id}_b : a+v+w+b \to a+w+v+b$$

with $a,b,v,w \in \mathbb{N}_0$. We identify $\mathrm{id}_a \times \alpha_{0,0} \times \mathrm{id}_b = \mathrm{id}_{a+b}$ and abbreviate $\mathrm{id}_0 \times \alpha_{v,w} \times \mathrm{id}_0 = \alpha_{v,w}$. We turn it into a graded diagram via the trivial grading $|n| = 0$ fr all $n \in \mathbb{N}$.

The summation map

$$\mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0 \qquad (n,m) \mapsto n+m$$

defined a commutative product structure on $\mathbb{N}_0$ in the sense of Definition 7.1.3. The definition on edges is the obvious one. All edges $\beta_{v,w,u}$, $\beta'_{v,w,u}$ are given by the identity. The edges $\alpha_{v,w}$ are the ones specified before. The unit $\mathbf{1}$ is given by the vertex 0, the edges u_v are given by the identity.

Let V be a finite dimensional k-vector space for some field k. We define a unital graded multiplicative representation

$$T = T_V : \mathbb{N}_0 \to k\text{−Mod}, \qquad n \mapsto V^{\otimes n}$$

The morphisms

$$\tau_{(v,w)} : T(v \times w) = V^{\otimes(n+m)} \to T(v) \otimes T(w)$$

are the natural ones. All conditions are satisfied. We have in particular $T(0) = k$.

By Proposition 7.1.5, the coalgebra $A = A(\mathbb{N}_0, T)$ is a commutative bialgebra. Indeed, $\mathrm{Spec}A = \mathrm{End}(V)$ viewed as algebraic monoid over k. In more detail: The commutative algebra A is generated freely by

$$A(\{1\}, T) = \mathrm{End}_k(V)^\vee.$$

Let $v_1, \dots, v_n$ be a basis of v. Then

$$A(\mathbb{N}_0, T) = k[X_{ij}]_{i,j=1}^n$$

with X_{ij} the element dual to $E_{ij} : V \to V$ with $E_{ij}(v_s) = \delta i s v_j$. The comultiplication A is determined on its value on the X_{ij} where it is induced from multiplication of the E_{ij}. Hence

$$\Delta(X_{ij}) = \sum_{s=1}^n X_{is} X_{sj} \ .$$

7.2 Localization

The purpose of this section is to give a diagram version of the localization of a tensor category with respect to one object, i.e., a distinguished object X becomes invertible with respect to tensor product. This is the standard construction used to pass e.g. from effective motives to all motives.

We restrict to the case when R is a field or a Dedekind domain and all representations of diagrams take values in $R-$Proj.

Definition 7.2.1 (Localization of diagrams)**.** Let D^{eff} be a graded diagram with a commutative product structure with unit $\mathbf{1}$. Let $v_0 \in D^{\text{eff}}$ be a vertex. The *localized diagram* D has vertices and edges as follows:

1. for every v a vertex of D^{eff} and $n \in \mathbb{Z}$ a vertex denoted $v(n)$;
2. for every edge $\alpha : v \to w$ in D^{eff} and every $n \in \mathbb{Z}$, an edge denoted $\alpha(n) : v(n) \to w(n)$ in D;
3. for every vertex v in D^{eff} and every $n \in \mathbb{Z}$ an edge denoted $(v \times v_0)(n) \to v(n+1)$.

Put $|v(n)| = |v|$.

We equip D with a weak commutative product structure in the sense of Remark 7.1.6

$$\times : D \times D \to \mathcal{P}(D) \qquad v(n) \times w(m) \mapsto (v \times w)(n+m)$$

together with

$$\begin{aligned} \alpha_{v(n),w(m)} &= \alpha_{v,w}(n+m) \\ \beta_{v(n),w(m),u(r)} &= \beta_{v,w,u}(n+m+r) \\ \beta'_{v(n),w(m),u(r)} &= \beta'_{v,w,u}(n+m+r) \end{aligned}$$

Let $\mathbf{1}(0)$ together with

$$u_{v(n)} = u_v(n)$$

be the unit.

Note that there is a natural inclusion of multiplicative diagrams $D^{\mathrm{eff}} \to D$ which maps a vertex v to $v(0)$.

Remark 7.2.2. The above definition does not spell out $\times$ on edges. It is induced from the product structure on D^{eff} for edges of type (2). For edges of type (3) there is an obvious sequence of edges. We take their composition in $\mathcal{P}(D)$. E.g. for $\gamma_{v,n} : (v \times v_0)(n) \to v(n+1)$ and $\mathrm{id}_{w(m)} = \mathrm{id}_w(m) : w(m) \to w(m)$ we have

$$\gamma_{v,n} \times \mathrm{id}(m) : (v \times v_0)(n) \times w(m) \to v(n+1) \times w(m)$$

via

$$\begin{aligned}
(v \times v_0)(n) \times w(m) &= ((v \times v_0) \times w)(n+m) \\
&\xrightarrow{\beta'_{v,v_0,w}(n+m)} (v \times (v_0 \times w))(n+m) \\
&\xrightarrow{\mathrm{id}\times\alpha_{v_0,w}(n+m)} (v \times (w \times v_0))(n+m) \\
&\xrightarrow{\beta_{v,w,v_0}(n+m)} ((v \times w) \times v_0)(n+m) \\
&\xrightarrow{\gamma_{v\times w,n+m}} (v \times w)(n+m+1) = v(n+1) \times w(m)\ .
\end{aligned}$$

Assumption 7.2.3. Let R be a field or a Dedekind domain. Let T be a multiplicative unital representation of D^{eff} with values in $R-\mathrm{Proj}$ such that $T(v_0)$ is locally free of rank 1 as R-module.

Lemma 7.2.4. *Under Assumption 7.2.3, the representation T extends uniquely to a graded multiplicative representation of D such that $T(v(n)) = T(v) \otimes T(v_0)^{\otimes n}$ for all vertices and $T(\alpha(n)) = T(\alpha) \otimes T(\mathrm{id})^{\otimes n}$ for all edges. It is multiplicative and unital with the choice*

$$\begin{array}{ccc}
T(v(n) \times w(m)) & \xrightarrow{\tau_{v(n),w(m)}} & T(v(n)) \otimes T(w(m)) \\
{\scriptstyle \tau_{v,w}(n+m)}\big\downarrow & & \big\downarrow{=} \\
T(v) \otimes T(w) \otimes T(v_0)^{\otimes n+m} & \xrightarrow{\cong} & T(v) \otimes T(v_0)^{\otimes n} \otimes T(w) \otimes T(v_0)^{\otimes m}
\end{array}$$

where the last line is the natural isomorphism.

Proof. Define T on the vertices and edges of D via the formula. It is tedious but straightforward to check the conditions. $\square$

Proposition 7.2.5. *Let D^{eff}, D and T be as above. Assume Assumption 7.2.3. Let $A(D,T)$ and $A(D^{\mathrm{eff}},T)$ be the corresponding bialgebras. Then:*

1. *$\mathcal{C}(D,T)$ is the localization of the category $\mathcal{C}(D^{\mathrm{eff}},T)$ with respect to the object $\tilde{T}(v_0)$.*

2. *Let $\chi \in \mathrm{End}(T(v_0))^\vee = A(\{v_0\}, T)$ be the dual of* $\mathrm{id} \in \mathrm{End}(T(v_0))$. *We view it in $A(D^{\mathrm{eff}}, T)$. Then $A(D, T) = A(D^{\mathrm{eff}}, T)_\chi$ (localization of algebras).*

Proof. Let $D^{\geq n} \subset D$ be the subdiagram with vertices of the form $v(n')$ with $n' \geq n$. Clearly, $D = \mathrm{colim}_n D^{\geq n}$, and hence

$$\mathcal{C}(D, T) \cong 2\text{−}\mathrm{colim}_n \mathcal{C}(D^{\geq n}, T) \ .$$

Consider the morphism of diagrams

$$D^{\geq n} \to D^{\geq n+1}, \ \ v(m) \mapsto v(m+1).$$

It is clearly an isomorphism. We equip $\mathcal{C}(D^{\geq n+1}, T)$ with a new fibre functor $f_T \otimes T(v_0)^\vee$. It is faithful exact. The map $v(m) \mapsto \tilde{T}(v(m+1))$ is a representation of $D^{\geq n}$ in the abelian category $\mathcal{C}(D^{\geq n+1}, T)$ with fibre functor $f_T \otimes T(v_0)^\vee$. By the universal property, this induces a functor

$$\mathcal{C}(D^{\geq n}, T) \to \mathcal{C}(D^{\geq n+1}, T) \ .$$

The converse functor is constructed in the same way. Hence

$$\mathcal{C}(D^{\geq n}, T) \cong \mathcal{C}(D^{\geq n+1}, T), \qquad A(D^{\geq n}, T) \cong A(D^{\geq n+1}, T).$$

The map of graded diagrams with commutative product and unit

$$D^{\mathrm{eff}} \to D^{\geq 0}$$

induces an equivalence on tensor categories. Indeed, we represent $D^{\geq 0}$ in $\mathcal{C}(D^{\mathrm{eff}}, T)$ by mapping $v(m)$ to $\tilde{T}(v) \otimes \tilde{T}(v_0)^m$. By the universal property (see Corollary 6.1.18), this implies that there is a faithful exact functor

$$\mathcal{C}(D^{\geq 0}, T) \to \mathcal{C}(D^{\mathrm{eff}}, T)$$

inverse to the obvious inclusion. Hence we also have $A(D^{\mathrm{eff}}, T) \cong A(D^{\geq 0}, T)$ as unital bialgebras.

On the level of coalgebras, this implies

$$A(D, T) = \mathrm{colim}_n A(D^{\geq n}, T) = \mathrm{colim}_n A(D^{\mathrm{eff}}, T)$$

because $A(D^{\geq n}, T)$ isomorphic to $A(D^{\mathrm{eff}}, T)$ as coalgebras. $A(D^{\mathrm{eff}}, T)$ also has a multiplication, but the $A(D^{\geq n}, T)$ for general $n \in \mathbb{Z}$ do not. However, they carry a weak $A(D^{\mathrm{eff}}, T)$-module structure analogous to Remark 7.1.6 corresponding to the map of graded diagrams

$$D^{\mathrm{eff}} \times D^{\geq n} \to \mathcal{P}(D^{\geq n}).$$

We want to describe the transition maps of the direct limit. From the point of view of $D^{\mathrm{eff}} \to D^{\mathrm{eff}}$, it is given by $v \mapsto v \times v_0$.

In order to describe the transition maps $A(D^{\text{eff}}, T) \to A(D^{\text{eff}}, T)$, it suffices to describe $\text{End}(T|_F) \to \text{End}(T|_{F'})$ where F, F' are finite subdiagrams of D^{eff} such that $v \times v_0 \in V(F')$ for all vertices $v \in V(F)$. It is induced by

$$\text{End}(T(v)) \to \text{End}(T(v \times v_0)) \xrightarrow{\tau} \text{End}(T(v)) \otimes \text{End}(T(v_0)) \quad a \mapsto a \otimes \text{id}.$$

On the level of coalgebras, this corresponds to the map

$$A(D^{\text{eff}}, T) \to A(D^{\text{eff}}, T), \quad x \mapsto x\chi.$$

Note finally, that the direct limit $\text{colim} A(D^{\text{eff}}, T)$ with transition maps given by multiplication by χ agrees with the localization $A(D^{\text{eff}}, T)_\chi$. □

7.3 Nori's Rigidity Criterion

Implicit in Nori's construction of motives is a rigidity criterion, which we are now going to formulate and prove explicitly.

Let R be a Dedekind domain or a field and $\mathcal{C}$ an R-linear tensor category. Recall that $R-$Mod is the category of finitely generated R-modules and $R-$Proj the category of finitely generated projective R-modules.

We assume that the tensor product on $\mathcal{C}$ is associative, commutative and unital. Let $\mathbf{1}$ be the unit object. Let $T : \mathcal{C} \to R-$Mod be a faithful tensor functor with values in $R-$Mod. In particular, $T(\mathbf{1}) \cong R$. By what we have shown above this implies that $\mathcal{C}$ is equivalent to the category of representations of a pro-algebraic monoid over R.

Recall:

Definition 7.3.1. Let $\mathcal{C}$ be as a above with R a field. We say that $\mathcal{C}$ is *rigid*, if every object $V \in \mathcal{C}$ has a strong dual $V^\vee$, i.e., for all $X, Y \in \mathcal{C}$

$$\begin{aligned}\text{Hom}(X \otimes V, Y) &= \text{Hom}(X, V^\vee \otimes Y),\\ \text{Hom}(X, V \otimes Y) &= \text{Hom}(X \otimes V^\vee, Y)\end{aligned}$$

By Tannaka duality this implies that the Tannaka dual of $\mathcal{C}$ is a group. We are going to show below that actually a weaker assumption suffices. Hence by abuse of terminology, we call $\mathcal{C}$ *rigid* also in the case where R is a Dedekind domain, if its Tannaka dual is a group.

We introduce an ad-hoc notion.

Definition 7.3.2. Let V be an object of $\mathcal{C}$. We say that V *admits a perfect duality* if there is morphism

$$q : V \otimes V \to \mathbf{1},$$

or

$$\mathbf{1} \to V \otimes V$$

such that $T(V)$ is projective and $T(q)$ (respectively its dual) is a non-degenerate bilinear form.

Definition 7.3.3. Let V be an object of $\mathcal{C}$. By $\langle V\rangle_\otimes$ we denote the smallest full abelian unital tensor subcategory of $\mathcal{C}$ containing V.

We start with the simplest case of the criterion.

Lemma 7.3.4. *Let V be an object such that $\mathcal{C} = \langle V\rangle_\otimes$ and such that V admits a perfect duality. Then $\mathcal{C}$ is rigid.*

Proof. By standard Tannakian formalism, $\mathcal{C}$ is the category of comodules for a bialgebra A, which is commutative and of finite type as an R-algebra. Indeed: The construction of A as a coalgebra was explained in Proposition 6.1.12. We may view $\mathcal{C}$ as graded diagram (with trivial grading) with a unital commutative product structure in the sense of Definition 7.1.3. The fibre functor T is a unital graded multiplicative representation. The algebra structure on A is the one of Proposition 7.1.5. It is easy to see that A is generated by $A(\{V\}, T,)$ as an algebra. The argument is given in more detail below.

We want to show that A is a Hopf algebra, or equivalently, that the algebraic monoid $M = \mathrm{Spec} A$ is an algebraic group.

By Lemma 7.3.7 it suffices to show that there is a closed immersion $M \to G$ of monoids into an algebraic group G. We are going to construct this group or rather its ring of regular functions. We have

$$A = \lim A_n$$

with $A_n = A(\mathcal{C}_n, T)$ for $\mathcal{C}_n = \langle \mathbf{1}, V, V^{\otimes 2}, \ldots, V^{\otimes n}\rangle$, the smallest full abelian subcategory containing $\mathbf{1}, V, \ldots, V^{\otimes n}$. By construction, there is a surjective map

$$\bigoplus_{i=0}^{n} \mathrm{End}_R((T(V)^{\otimes i})^\vee \to A_n$$

or, dually, an injective map

$$A_n^\vee \to \bigoplus_{i=0}^{n} \mathrm{End}_R(T(V)^{\otimes i})$$

where $A_n^\vee$ consists of those endomorphisms compatible with all morphisms in $\mathcal{C}_n$. In the limit, there is a surjection of bialgebras

$$\bigoplus_{i=0}^{\infty} \mathrm{End}_R((T(V)^{\otimes i})^\vee) \to A$$

and the kernel is generated by the relation defined by compatibility with morphisms in $\mathcal{C}$. One such relation is the commutativity constraint, hence the map factors via the symmetric algebra

$$S^*(\mathrm{End}(T(V)^\vee) \to A\ .$$

Note that $S^*(\mathrm{End}(T(V)^\vee)$ is canonically the ring of regular functions on the algebraic monoid $\mathrm{End}(T(V))$. Another morphism in $\mathcal{C}$ is the pairing $q : V \otimes V \to \mathbf{1}$. We want to work out the explicit equation induced by q.

We choose a basis $e_1, \ldots, e_r$ of $T(V)$. Let

$$a_{i,j} = T(q)(e_i, e_j) \in R$$

By assumption, the matrix is invertible. Let X_{st} be the matrix coefficients on $\mathrm{End}(T(V))$ corresponding to the basis e_i. Compatibility with q gives for every pair (i, j) the equation

$$\begin{aligned} a_{ij} &= q(e_i, e_j) \\ &= q((X_{rs})e_i, (X_{r's'})e_j) \\ &= q\left(\sum_r X_{ri}e_r, \sum_{r'} X_{r'j}e_{r'}\right) \\ &= \sum_{r,r'} X_{ri}X_{r'j}q(e_r, e_{r'}) \\ &= \sum_{r,r'} X_{ri}X_{r'j}a_{rr'} \end{aligned}$$

Note that the latter is the (i, j)-term in the product of matrices

$$(X_{ir})^t(a_{rr'})(X_{r'j}) \; .$$

Let $(b_{ij}) = (a_{ij})^{-1}$. With

$$(Y_{ij}) = (b_{ij})(X_{i'r})^t(a_{rr'})$$

we have the coordinates of the inverse matrix. In other words, our set of equations defines the isometry group $G(q) \subset \mathrm{End}(T(V))$. We now have expressed A as quotient of the ring of regular functions of $G(q)$.

The argument works in the same way, if we are given

$$q : \mathbf{1} \to V \otimes V$$

instead. □

Proposition 7.3.5 (Nori)**.** *Let $\mathcal{C}$ and $T : \mathcal{C} \to R-\mathrm{Mod}$ be as defined at the beginning of the section. Let $\{V_i | i \in I\}$ be a set of objects of $\mathcal{C}$ with the properties:*

1. *It generates $\mathcal{C}$ as an abelian tensor category, i.e., the smallest full abelian tensor subcategory of $\mathcal{C}$ containing all V_i is equal to $\mathcal{C}$.*
2. *For every V_i there is an object W_i and a morphism*

   $$q_i : V_i \otimes W_i \to \mathbf{1}$$

   such that $T(q_i) : T(V_i) \otimes T(W_i) \to T(\mathbf{1}) = R$ is a perfect pairing of free R-modules.

Then $\mathcal{C}$ is rigid, i.e., for every object V there is a dual object $V^\vee$ such that

$$\mathrm{Hom}(V\otimes A, B) = \mathrm{Hom}(A, V^\vee \otimes B)\ , \qquad \mathrm{Hom}(V^\vee\otimes A, B) = \mathrm{Hom}(A, V\otimes B)\ .$$

This means that the Tannakian dual of $\mathcal{C}$ is not only a monoid but a group.

Remark 7.3.6. The Proposition also holds with the dual assumption, existence of morphisms

$$q_i : \mathbf{1} \to V_i \otimes W_i$$

such that $T(q_i)^\vee : T(V)^\vee \otimes T(W_i)^\vee \to R$ is a perfect pairing.

Proof. Consider $V_i' = V_i \oplus W_i$. The pairing q_i extends to a symmetric map q_i' on $V_i' \otimes V_i'$ such that $T(q_i')$ is non-degenerate. We now replace V_i by V_i'. Without loss of generality, we can assume $V_i = W_i$.

For any finite subset $J \subset I$, let $V_J = \bigoplus_{j\in J} V_j$. Let q_J be the orthogonal sum of the q_j for $j \in J$. It is again a symmetric perfect pairing.

For every object V of $\mathcal{C}$, we write $\langle V\rangle_\otimes$ for the smallest full abelian tensor subcategory of $\mathcal{C}$ containing V. By assumption we have

$$\mathcal{C} = \bigcup_J \langle V_J\rangle_\otimes$$

We apply the standard Tannakian machinery. It attaches to every $\langle V_J\rangle_\otimes$ an R-bialgebra A_J such that $\langle V_J\rangle_\otimes$ is equivalent to the category of A_J-comodules. If we put

$$A = \lim A_J$$

then $\mathcal{C}$ will be equivalent to the category of A-comodules. It suffices to show that A_J is a Hopf-algebra. This is the case by Lemma 7.3.4. $\square$

Finally, the missing lemma on monoids.

Lemma 7.3.7. *Let R be noetherian ring, G be an algebraic group scheme of finite type over R and $M \subset G$ a closed immersion of a submonoid with $1 \in M(R)$. Then M is an algebraic group scheme over R.*

Proof. This seems to be well-known. It is appears as an exercise in [Re] 3.5.1 2. We give the argument:

Let S be any finitely generated R-algebra. We have to show that the value $S \mapsto M(S)$ is a group. We take base change of the situation to S. Hence without loss of generality, it suffices to consider $R = S$. If $g \in G(R)$, we denote the isomorphism $G \to G$ induced by left multiplication with g also by $g : G \to G$. Take any $g \in G(R)$ such that $gM \subset M$ (for example $g \in M(R)$). Then one has

$$M \supseteq gM \supseteq g^2M \supseteq \cdots$$

As G is Noetherian, this sequence stabilizes, say at $s \in \mathbb{N}$:

$$g^s M = g^{s+1} M$$

as closed subschemes of G. Since every g^s is an isomorphism, we obtain that

$$M = g^{-s} g^s M = g^{-s} g^{s+1} M = gM$$

as closed subschemes of G. So for every $g \in M(R)$ we showed that $gM = M$. Since $1 \in M(R)$, this implies that $M(R)$ is a subgroup. □

Example 7.3.8. We explain the simplest example. It is a dressed up version of Example 7.1.7 where we obtained and algebraic monoid. Let $D = \mathbb{N}_0$. We have the same self edges $\mathrm{id}_a \times \alpha_{v,w} \times \mathrm{id}_b$ as previously and in addition edges $n+2 \to n$ denoted suggestively $\mathrm{id}_a \times b \times \mathrm{id}_b : a+2+b \to a+b$.

We equip it with the trivial grading and the commutative product structure obtained by componentwise addition. The unit is given by 0 with $u_v = \mathrm{id}$.

Let k be a field and (V,b) a finite dimensional k-vector space with a non-degenerate bilinear form $V^\times V \to k$. We define a graded multiplicative representation

$$T_{V,b} : \mathbb{N}_0 \to k\text{−Mod} \qquad v \mapsto V^{\otimes v}.$$

The edge b is mapped to the linear map $\tilde{b} : V^{\otimes 2} \to k$ induced from the bilinear map b. The assumptions of the rigidity criterion in Proposition 7.3.5 are satisfied for $\mathcal{C} = \mathcal{C}(D,T)$. Indeed it is generated by the object of the form $T(1) = V$ as the an abelian tensor category. It is self-dual in the sense of the criterion in $\mathcal{C}$.

Let $v_1, \dots, v_n$ be a basis of V and B the matrix of b. The bialgebra $A = A(\mathbb{N}_0, T_{V,b})$ is generated by symbols X_{ij} as in Example 7.1.7. There is a relation coming from the edge b. It was computed in the proof of Lemma 7.3.4 as the matrix product

$$(X_{ij})_{ij} B (X_{st})_{st} = 0.$$

Hence

$$X = \mathrm{Spec} A = \mathrm{O}(b)$$

as algebraic group scheme.

7.4 Comparing fibre functors

7.4.1 The space of comparison maps

We pick up the story but with two representations instead of one.

Let R be a Dedekind domain or a field. Let R−Mod be the category of finitely generated R-modules and R−Proj the category of finitely generated projective modules. Let D be a graded diagram with commutative product structure (see Definition 7.1.3) and $T_1, T_2 : D \to R-$Proj two graded multiplicative representations. Recall that we have attached coalgebras $A_1 := A(D, T_1)$ and $A_2 := A(D, T_2)$ to these representations (see Theorem 6.1.12). They are even bialgebras by Proposition 7.1.5. The diagram categories $\mathcal{C}(D, T_1)$ and $\mathcal{C}(D, T_2)$ are defined as the categories of comodules for these coalgebras. They carry a structure of unital commutative tensor category.

Remark 7.4.1. In the case that D is the diagram defined by a rigid tensor category and T_1, T_2 faithful tensor functors, it is the classical result of Tannakua theory that not only $G_1 = \mathrm{Spec}A_1$ and $G_2 = \mathrm{Spec}A_2$ are both groups, but they are forms of each other. All tensor functors are isomorphisms and the space of all tensor functors is a torsor under G_1 and G_2. Our aim is to imitate this as much as possible for a general diagram D. As we will see, the results will be weaker.

Definition 7.4.2. Let D be a diagram, R a Dedekind domain or a field. Let T_1 and T_2 be represention of D in $R-$Proj. Let $F \subset D$ be a finite subdiagram. We define

$$\mathrm{Hom}(T_1|_F, T_2|_F) =$$
$$\left\{ (f_p)_{p\in D} \in \prod_{p\in D} \mathrm{Hom}_R(T_1p, T_2p) | f_q \circ T_1 m = T_2 m \circ f_p \ \forall p, q \in D \ \forall m \in D(p, q) \right\}.$$

Put

$$A_{1,2} = \mathrm{colim}_F \mathrm{Hom}(T_1|_F, T_2|_F)^\vee$$

where $^\vee$ denotes the R-dual and F runs through all finite subdiagrams of D.

Note that our assumptions guarantee that $\mathrm{Hom}(T_1|_F, T_2|_F)$ is a projective R-module and hence has a well-behaved R-dual.

Proposition 7.4.3. *1. The operation*

$$\mathrm{End}(T_1|_F) \times \mathrm{Hom}(T_1|_F, T_2|_F) \to \mathrm{Hom}(T_1|_F, T_2|_F)$$

induces a compatible comultiplication

$$A_1 \otimes A_{1,2} \leftarrow A_{1,2}.$$

The operation

$$\mathrm{Hom}(T_1|_F, T_2|_F) \times \mathrm{End}(T_2|_F) \to \mathrm{Hom}(T_1|_F, T_2|_F)$$

induces a compatible comultiplication

$$A_{1,2} \otimes A_2 \leftarrow A_{1,2}.$$

The composition

$$\mathrm{Hom}(T_1|_F, T_2|_F) \times \mathrm{Hom}(T_2|_F, T_1|_F) \times \mathrm{Hom}(T_1|_F, T_2|_F) \to \mathrm{Hom}(T_1|_F, T_2|_F) \to \mathrm{Hom}(T_1|_F, T_2|_F)$$

induces a natural map

$$A_{1,2} \otimes A_{2,1} \otimes A_{1,2} \leftarrow A_{1,2}.$$

2. *Assume that D carries a commutative product structure. Then $A_{1,2}$ is a faithfully flat commutative unital R-algebra with multiplication induced by the tensor structure of the diagram category (unless $A_{1,2} = 0$) and the above maps are algebra homomorphisms.*

Proof. The statement on comultiplication follows in the same way as the comultiplication on A_1 and A_2 themselves, see Theorem 6.1.12. The module $A_{1,2}$ is faithfully flat over R because it is the direct limit of locally free R-modules.

The hard part is the existence of the multiplication. This follows by going through the proof of Proposition 7.1.5, replacing $\mathrm{End}(T|_F)$ by $\mathrm{Hom}(T_1|_F, T_2|_F)$ in the appropriate places.

Recall that $u_{\mathbf{1}}$ defines a distinguished isomorphism $R \to T_i(\mathbf{1})$. The element $1 \in A_{1,2}$ is induced by the image of the map $\mathrm{Hom}(T_1(\mathbf{1}), T_2(\mathbf{1})) \to R$ dual to the distinguished basis. □

Remark 7.4.4. As in Remark 7.1.6, a weak product structure on D suffices.

Lemma 7.4.5. *Let R be a Dedekind domain or a field. Let S be a faithfully flat ring extension of R. Then the follow data are equivalent:*

1. *an R-linear map $\phi^\vee : A_{1,2} \to S$ (of R-algebras);*
2. *a morphism of representations (with unital commutative product structure). $\Phi : T_1 \otimes S \to T_2 \otimes S$;*

Moreover, every (unital tensor) functor $\Phi : \mathcal{C}(D, T_1) \to \mathcal{C}(D, T_2)$ gives rise to a morphism of representations.

Proof. By base change it suffices to consider $S = R$. This will simplyfiy notation.

We first establish the statement without multiplicative structures. By construction we can restrict to the case where the diagram D is finite.

Such a morphism of representations defines an element $\phi \in \mathrm{Hom}(T_1, T_2)$ or equvialently an R-linear map $\phi^\vee : A_{1,2} \to R$. Conversely, ϕ a morphism of representations.

Let $\Phi : \mathcal{C}(D, T_1) \to \mathcal{C}(D, T_2)$ be an S-linear functor. By composing with the universal representations $\tilde{T}_1$ and $\tilde{T}_2$ we obtain a morphism of representations $T_1 \otimes \to T_2 \otimes S$.

Finally, compatibility with product structure translates into multilicativity of the map ϕ. □

Remark 7.4.6. It does *not* follow that that a morphism of representations gives rise to functor between categories. Indeed, a linear map $V_1 \to V_2$ does *not* give rise to an algebra homomorphism $\mathrm{End}(V_2) \to \mathrm{End}(V_1)$.

We translate the statements to geometric language.

Theorem 7.4.7. *Let R be a field or a Dedekind domain. Let D be a diagram with commutative product structure, $T_1, T_2 : D \to R-\mathrm{Proj}$ two representations. Let $X_{1,2} = \mathrm{Spec}A_{1,2}$, $G_1 = \mathrm{Spec}A_1$ and $G_2 = \mathrm{Spec}A_2$. The scheme $X_{1,2}$ is faithfully flat over R unless it is empty.*

1. *The monoid G_1 operates on $X_{1,2}$ from the left*
$$\mu : G_1 \times X_{1,2} \to X_{1,2}.$$
2. *The monoid G_2 operates on $X_{1,2}$ from the right*
$$\mu : X_{1,2} \times G_2 \to X_{1,2}.$$
3. *There is a natural morphism*
$$X_{1,2} \times X_{2,1} \times X_{1,2} \to X_{1,2}.$$

Let S be a faithfully flat extension of R. The choice of a point $X_{1,2}(S)$ is equivalent to a morphism of representations $T_1 \otimes S \to T_2 \otimes S$.

Remark 7.4.8. It is possible for $X_{1,2}$ to be empty as we will see in the examples below.

Example 7.4.9. For $D = \mathrm{Pairs}$ or $D = \mathrm{Good}$ and the representations $T_1 = H^*_{\mathrm{dR}}$ (de Rham cohomology) and $T_2 = H^*$ (singular cohomology) this is going to induce the operation of the motivic Galois group $G_{\mathrm{mot}} = \mathrm{Spec}A_2$ on the torsor $X = \mathrm{Spec}A_{1,2}$.

We formulate the main result on the comparison of representations. By a torsor we will mean a torsor in the *fpqc*-topology, see Definition 1.7.3. For background on torsors, see Section 1.7.

Theorem 7.4.10. *Let $R \to S$ be faithfully flat and*
$$\varphi : T_1 \otimes_R S \to T_2 \otimes_R S$$
an isomorphism of unitary multiplicative representations.

1. *Then there is $\phi \in X_{1,2}(S)$ such that the induced maps*
$$G_{1,S} \to X_{1,2,S}, \quad g \mapsto \mu(g\phi)$$
$$G_{2,S} \to X_{1,2,S}, \quad g \mapsto \mu(\phi g)$$
are isomorphisms.

2. *This map* ϕ *induces an equivalence of unital tensor categories*

$$\Phi : \mathcal{C}(D, T_1) \to \mathcal{C}(D, T_2).$$

3. *The comparison algebra* $A_{1,2}$ *is canonically isomorphic to the comparison algebra for the category* $\mathcal{C} = \mathcal{C}(D, T_1)$ *and the fibre functors* f_{T_1} *and* $f_{T_2} \circ \Phi$.

Assume in addition that $\mathcal{C}(D, T_1)$ *is rigid. Then:*

4. $X_{1,2}$ *is a* G_1*-left torsor and a* G_2*-right torsor in the fpqc-topology.*

5. *For flat extensions* $R \to S'$*, all sections* $\psi \in X_{1,2}(S')$ *are isomorphisms of representations* $T_1 \otimes S' \to T_2 \otimes S'$*. The map* $\psi \to \psi^{-1}$ *defines an isomorphism of schemes* $\iota : X_{1,2} \to X_{2,1}$.

6. $X_{1,2}$ *is a torsor in the sense of Definition 1.7.9 with structure map given by via* ι *and Theorem 7.4.7*

$$X_{1,2}^3 \cong X_{1,2} \times X_{2,1} \times X_{1,2} \to X_{1,2} \ .$$

   Moreover, the groups attached to $X_{1,2}$ *via Proposition 1.7.10 are* G_1 *and* G_2.

Proof. The first statement over S follows directly from the definitions.

We obtain the functor and its inverse by applying the universal property of the diagram catgories in the general form of Corollary 6.1.14. They are inverse to each other by the uniqueness part of the universal property.

We use the notation $A(D, T_1, T_2)$ for the period algebra $A_{1,2}$. By definition, $A(D, T_1, T_2) = A(D, f_{T_2} \circ \Phi \circ \tilde{T}_1)$. The map of diagrams $\tilde{T}_1 : D \to \mathcal{C}$ defines an algebra homomorphism

$$A(D, T_1, T_2) \to A(\mathcal{C}, f_{T_1}, f_{T_2} \circ \Phi)$$

by the same argument as in the proof of Lemma 6.2.6. We check that it is an isomorphism after base change to S. Over S, we may use the isomorphism ϕ to replace $T_2 \otimes S$ be the isomorphic $T_1 \otimes S$. The claim now follows from the isomorphism

$$A(D, T_1 \otimes S) \to A(\mathcal{C}(D, T_1), f_{T_1})$$

which is the main content of Theorem 6.1.19 on the diagram category of an abelian category.

Now suppose in addition that $\mathcal{C}(D, T_1)$ is rigid. By the equivalence this implies that $\mathcal{C}(D, T_2)$ is rigid. This means that the monoids G_1 and G_2 are group schemes. The first property translate into $X_{1,2}$ being a G_1-left and G_2-right torsor in the *fpqc*-topology.

Let $\psi : T_1 \otimes S' \to T_2 \otimes S'$ be a morphism of representations. We claim that it is an isomorphism. This can be checked after a base change to S. Then T_2

becomes isomorphic to T_1 via φ and we may replace T_2 by T_1 in the argument. The morphism ψ can now be identified with a section $\psi \in G_1(S' \otimes S)$. This is a group, hence it has an inverse, which can be interpreted as the inverse of the morphism of representations.

Consider $X_{1,2}^3 \to X_{1,2}$ as defined in the theorem. We claim that it satisfies the torsor identities of Definition 1.7.9. This can be checked after base change to S where we can replace $X_{1,2}$ by G_1. The map is then given by

$$G_1^3 \to G_1, \qquad (a,b,c) \mapsto ab^{-1}c$$

which is the trivial torsor. In particular the left group defined by the torsor $X_{1,2}$ is nothing but G_1. The same argument also applies to G_2. □

Remark 7.4.11. See also the discussion of the Tannakian case in Section 6.1.4. In this case $X_{1,2}$ is the G-torsor of isomorphisms between the fibre functors T_1 and T_2 of [DM, Theorem 3.2], see also Theorem 7.4.18. The above theorem is more general as it starts out with a commutative diagram instead of a rigid category. However, it is also weaker as is uses the existence of a point.

7.4.2 Some examples

We make the above theory explicit in a number of simple examples. The aim is to understand conditions needed in order to ensure that $X_{1,2}$ is a torsor. It will turn out that rigidity of the diagram category is not enough.

Example 7.4.12. We consider again Example 7.1.7. Let k be a field. The diagram is $\mathbb{N}_0$ with only edges $\mathrm{id}_a \times \alpha_{v,w} \times \mathrm{id}_b$. It carries a commutative product structure as before.

Let V_1 and V_2 be finite dimensional k-vector spaces. Let $T_i : n \mapsto V_i^{\otimes n}$ be the multiplicative representations as in before. We have shown that $G_i = \mathrm{End}(V_i)$ as algebraic k-scheme. The same argument yields

$$X_{1,2} = \mathrm{Hom}(V_1, V_2)$$

as algebraic k-scheme with the natural left and right operations by G_i.

Example 7.4.13. We consider again Example 7.3.8. We have $D = \mathbb{N}_0$ with additional edges generated from an extra edge $b : 2 \to 0$. Let (V_i, b_i) be finite dimensional vector spaces with a non-degenerate bilinear form. We obtain

$$X_{1,2} = \mathrm{Isom}((V_1, b_1), (V_2, b_2))$$

the space of linear maps compatible with the forms, i.e., the space of isometries. In this case G_1 and G_2 are algebraic groups, indeed the orthogonal groups of b_1 and b_2, respectively. The diagram categories were rigid.

We claim that $X_{1,2} = \emptyset$ if $\dim V_2 < \dim V_1$. The argument can already be explained in the case $V_1 = k^2$, $V_2 = 1$ both with the standard scalar product. If

$X_{1,2} \neq \emptyset$, there would be a K-valued point for some field extension K/k. This would mean the existence of a linear map $K^2 \to K$ with matrix (a, b) such that $a^2 = 1$, $b^2 = 1$ and $ab = 0$. This is impossible. We can write down the same argument in terms of equations: the algebra $A_{1,2}$ is generated by X, Y subject to the equations $X^2 - 1, Y^2 - 1, XY$. This implies $0 = 1$ in $A_{1,2}$.

On the other hand, if $\dim V_1 < \dim V_2$, then $X_{1,2} \neq \emptyset$. Nevertheless, the groups G_1, G_2 are not isomorphic over any field extension of k. Hence $X_{1,2}$ is *not* a torsor. This contrasts starkly to the Tannakian case. Note that the points of $X_{1,2}$ do not give rise to functors - they would be tensor functors and hence invertible.

The example shows:

Corollary 7.4.14. *There is a diagram D with unital commuative product structure and a pair of unital multiplicative representations T_1, T_2 such that the resulting tensor categories are both rigid, but non-equivalent.*

Example 7.4.15. We resume the situation of Example 7.4.13, but with $\dim V_1 = \dim V_2$. The two spaces become isometric over $\bar{k}$ because any two non-degenerate bilinear forms are equivalent over the algebraic closure. By Theorem 7.4.10, $X_{1,2}$ is a torsor and the two diagram categories are equivalent. Hence the categories of representations of all orthogonal groups of the same dimension are equivalent. Note that we are considering algebraic k-representations of k-algebraic groups here.

Example 7.4.16. We consider another variant of Example 7.3.8. Let $D = \mathbb{N}_0$ with edges

$$\mathrm{id}_n \times \alpha_{v,w} \times \mathrm{id}_n : n + v + w + m \to n + v + w + m$$
$$\mathrm{id}_n \times b \times \mathrm{id}_m : n + 2 + m \to n + m$$
$$\mathrm{id}_n \times b' \times \mathrm{id}_m : n + m \to n + 2 + m$$

with identifications as before $\mathrm{id}_n \times \alpha_{0,0} \times \mathrm{id}_m = \mathrm{id}_{n+m}$. We use again the trivial grading and the obvious commutative product structure with all $\beta_{u,v,w}$ and $\beta'_{u,v,w}$ given by the identity.

Let (V, b) be a finite dimensional k-vector space with a non-degenerate bilinar form $V^{\otimes 2} \to k$. We define multiplicative representation $n \mapsto V^{\otimes n}$ which assigns the form b to the edge b and the dual of b to the edge b'.

As in the case of Example 7.3.8, the category $\mathcal{C}(D, T)$ is the category of representations of the group $O(b)$. The algebra is not changed because the additional relations for b' are automatic.

If we have two such representations attached to (V_1, b_1) and (V_2, b_2) than $X_{1,2}$ is either empty (if $\dim V_1 \neq \dim V_2$) or an $O(b_1)$-torsor (if $\dim V_1 = \dim V_2$). The additional edge b' forces any morphism of representations to be an isomorphism.

We formalize this.

Lemma 7.4.17. *Let D be graded diagram with a commutative product structure. Let $T_1, T_2 : D \to R-\mathrm{Mod}$ be multiplicative representations. Suppose that for every vertex v there is a vertex w and a pair of edges $e_v : v \times w \to \mathbf{1}$ and $e'_v : \mathbf{1} \to v \times w$ such that $T_i(e_v)$ and $T_i(e'_v)$ are a non-degenerate bilinear and its dual.*

Let $R \to S$ be faithfully flat. Then every morphism of representations

$$\phi : T_1 \otimes S \to T_2 \otimes S$$

is an isomorphism. Hence Proposition 7.4.10 applies in this case.

As Example 7.4.16 has shown, the space $X_{1,2}$ may still be empty!

Proof. Let v be an edge. Compatibility with e_v forces the map $T_1(v) \otimes S \to T_2(v) \otimes S$ to be injective. Compatibility with e'_v forces is to be surjective, hence bijective. □

This applies in particular in the Tannakian case. Moreover, in this case $X_{1,2}$ is non-empty.

Theorem 7.4.18 (The Tannakian case)**.** *Let k be a field, $\mathcal{C}$ a rigid tensor category. Let $F_1, F_2 : \mathcal{C} \to k-\mathrm{Mod}$ be two faithful fibre functors with associated groups G_1 and G_2.*

1. *Let S be a k-algebra. Let*

$$\phi : F_1 \otimes S \to F_2 \otimes S$$

be a morphism of tensor functors. Then ϕ is an isomorphism.

2. *$X_{1,2}$ is non-empty and a G_1-left and G_2-right torsor.*

This is [DM, Proposition 1.9] and [DM, Theorem 3.2]. We give the proof directly in our notation.

Proof. For the first statement simply apply Proppsition 7.4.17to the diagram defined by $\mathcal{C}$.

We now consider $X_{1,2}$ and need to show that the natural map $k \to A_{1,2}$ is injective. As in the proof of Theorem 6.1.19, we can write $\mathcal{C} = 2-\mathrm{colim}\{p\}$ where p runs through all objects of $\mathcal{C}$. Here where $\{p\}$ here means the full subcategory with only object p. (In general we would consider finite subdiagrams F, but in the abelian case we can replace F by the direct sum of its objects.) Hence

$$A_1 = \lim A(\{p\}, T_1), A_{1,2} = \lim A(\{p\}, T_1, T_2).$$

Without loss of generality we assume that $\mathbf{1}$ is a direct summand of p.

We check that injectivity on the level of $\langle p\rangle$. Let $X(p) \subset \mathrm{Hom}_R(T_1(p), p)$ be the object constructed in Lemma 6.3.15. By loc. cit. $T_1(X(p)) = \mathrm{End}(T_1|_p) = A(p, T_1)^\vee$. The same arguments show that

$$T_2(X(p)) = \mathrm{Hom}(T_1|_p, T_2|_p) = A(\langle p\rangle, T_1, T_2).$$

The splitting of p induces a morphism

$$X(p) \to \mathrm{Hom}_R(T_1(p), p) \to \mathrm{Hom}_R(T_1(\mathbf{1}), \mathbf{1}) = \mathbf{1}$$

Applying T_1 gives the map

$$A(\{p\}, T_1)^\vee \to k$$

defining the unit element of A_1. It is surjective. As T_1 is faithful, this implies that $X(p) \to \mathbf{1}$ is surjective. By applying the faithful functor T_2 we get a surjection

$$A(\{p\}, T_1, T_2)^\vee \to \mathrm{Hom}_k(T_1(\mathbf{1}), T_2(\mathbf{1})) = k.$$

This is the map defining the unit of $A_{1,2}$. Hence $k \to A_{1,2}$ is injective. □

7.4.3 The description as formal periods

For later use, we give an alternative description of the same algebra.

Definition 7.4.19. Let D be a diagram. Let $T_1, T_2 : D \to R-\mathrm{Proj}$ be representations. We define the space of *formal periods* $P_{1,2}$ as the R-module generated by symbols

$$(p, \omega, \gamma)$$

where p is a vertex of D, $\omega \in T_1 p$, $\gamma \in T_2 p^\vee$ with the following relations:

1. linearity in ω, γ;
2. (functoriality) If $f : p \to p'$ is an edge in D, $\gamma \in T_2 p'^\vee$, $\omega \in T_1 p$, then

$$(p', T_1 f(\omega), \gamma) = (p, \omega, T_2 f^\vee(\gamma)).$$

Proposition 7.4.20. *Assume D has a unital commutative product structure and T_1, T_2 are unital multipliative representation. Then $P_{1,2}$ is a commutative R-algebra with multiplication given on generators by*

$$(p, \omega, \gamma)(p', \omega', \gamma') = (p \times p', \omega \otimes \omega', \gamma \otimes \gamma')$$

Proof. It is obvious that the relations of $P_{1,2}$ are respected by the formula. □

There is a natural transformation

$$\Psi : P_{1,2} \to A_{1,2}$$

defined as follows: let $(p,\omega,\gamma)\in P_{1,2}$. Let F be a finite diagram containing p. Then

$$\Psi(p,\omega,\gamma)\in A_{1,2}(F)=\mathrm{Hom}(T_1|_F,T_2|_F)^\vee,$$

is the map

$$\mathrm{Hom}(T_1|_F,T_2|_F)\to\mathbb{Q}$$

which maps $\phi\in\mathrm{Hom}(T_1|_F,T_2|_F)$ to $\gamma(\phi(p)(\omega))$. Clearly this is independent of F and respects relations of $P_{1,2}$.

Theorem 7.4.21. *Let D be diagram. Then the above map*

$$\Psi: P_{1,2}\to A_{1,2}$$

is an isomorphism. If D carries a commutative product structure and T_1,T_2 are graded multiplicative representations, then it is an isomorphism of R-algebras.

Proof. For a finite subdiagram $F\subset D$ let $P_{1,2}(F)$ be the space of periods. By definition $P=\mathrm{colim}_F P(F)$. The statement is compatible with these direct limits. Hence without loss of generality $D=F$ is finite.

By definition, $P_{1,2}(F)$ is the submodule of

$$\prod_{p\in D} T_1p\otimes T_2p^\vee$$

of elements satisfying the relations induced by edges of D. By definition, $A_{1,2}(F)$ is the submodule of

$$\prod_{p\in D}\mathrm{Hom}(T_1p,T_2p)^\vee$$

of elements satisfying the relations induced by edges of D. As all T_ip are locally free and of finite rank, this is the same thing.

The compatibility with products is easy to see. □

Remark 7.4.22. The theorem is also of interest in the case $T=T_1=T_2$. It then gives an explicit description of Nori's coalgebra by generators and relations. We have implictly used the description in some of the examples.

Let p be a vertex of D. We choose a basis $\omega_1,\dots,\omega_n$ of T_1v and a basis $\gamma_1,\dots,\gamma_n$ of $(T_2p)^\vee$. We call

$$P_{ij}=((p,\omega_i,\gamma_j))_{i,j}$$

the *formal period matrix* at p. Will later discuss this point of view systematically.

Proposition 7.4.23. *Let D be a diagram with a unital commutative product structure. Assume that there is a faithfully flat extension $R\to S$ and an isomorphism of representations $\varphi: T_1\otimes S\to T_2\otimes S$. Moreover, assume that $\mathcal{C}(D,T_1)$ is rigid. Then $X_{1,2}=\mathrm{Spec}P_{1,2}$ becomes a torsor in the sense of Definition 1.7.9 with structure map*

$$P_{1,2}\to P_{1,2}^{\otimes 3}$$

given by

$$P_{ij} \mapsto \sum_{k,\ell} P_{ik} \otimes P_{k\ell}^{-1} \otimes P_{\ell j}$$

Proof. We translate Theorem 7.4.10 into the alternative description. □

Chapter 8

Nori motives

We explain Nori's construction of an abelian category of motives. It is defined as the diagram category (see Chapters 6 and 7) of a certain diagram. It is universal for all cohomology theories that can be compared with singular cohomology.

8.1 Essentials of Nori Motives

As before, we denote $\mathbb{Z}-$Mod the category of finitely generated $\mathbb{Z}$-modules and $\mathbb{Z}-$Proj the category of finitely generated free $\mathbb{Z}$-modules.

8.1.1 Definition

Let k be a subfield of $\mathbb{C}$. For a variety X over k, we define singular cohomology as singular cohomology of $X(\mathbb{C}) = X \times_k \mathbb{C}$. As in Chapter 2.1, we denote it simply by $H^i(X, \mathbb{Z})$.

Definition 8.1.1. Let k be a subfield of $\mathbb{C}$. The diagram $\mathrm{Pairs}^{\mathrm{eff}}$ of *effective pairs* consists of triples (X, Y, i) with X a k-variety, $Y \subset X$ a closed subvariety and an integer i. There are two types of edges between effective pairs:

1. (functoriality) For every morphism $f : X \to X'$ with $f(Y) \subset Y'$ an edge
$$f^* : (X', Y', i) \to (X, Y, i) \ .$$
2. (coboundary) For every chain $X \supset Y \supset Z$ of closed k-subschemes of X an edge
$$\partial : (Y, Z, i) \to (X, Y, i+1) \ .$$

The diagram has identities (see Definition 6.1.1) given by the identity morphism. The diagram is graded (see Definition 7.1.3) by $|(X, Y, i)| = i$.

Proposition 8.1.2. *The assignment*

$$H^* : \mathrm{Pairs}^{\mathrm{eff}} \to \mathbb{Z}{-}\mathrm{Mod}$$

which maps to (X, Y, i) relative singular cohomology $H^i(X(\mathbb{C}), Y(\mathbb{C}), \mathbb{Z})$ is a representation in the sense of Definition 6.1.4. It maps $(\mathbb{G}_m, \{1\}, 1)$ to $\mathbb{Z}$.

Proof. Relative singular cohomology was defined in 2.1.1. By definition, it is contraviantly functorial. This defines H^* on edges of type 1. The connecting morphism for triples, see Corollary 2.1.4, defines the representation on edges of type 2. We compute $H^1(\mathbb{G}_m, \{1\}, \mathbb{Z})$ via the sequence for relative cohomology

$$H^0(\mathbb{C}^*, \mathbb{Z}) \to H^0(\{1\}, \mathbb{Z}) \to H^1(\mathbb{C}^*, \{1\}, \mathbb{Z}) \to H^1(\mathbb{C}^*, \mathbb{Z}) \to H^1(\{1\}, \mathbb{Z})$$

The first map is an isomorphism. The last group vanishes for dimension reasons. Finally, $H^1(\mathbb{C}^*, \mathbb{Z}) \cong \mathbb{Z}$ because $\mathbb{C}^*$ is homotopy equivalent to the unit circle. □

Definition 8.1.3.

1. The category of effective *mixed Nori motives* $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}} = \mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}(k)$ is defined as the diagram category $\mathcal{C}(\mathrm{Pairs}^{\mathrm{eff}}, H^*)$ from Theorem 6.1.13.

2. For an effective pair (X, Y, i), we write $H^i_{\mathrm{Nori}}(X, Y)$ for the corresponding object in $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$. We put

   $$\mathbf{1}(-1) = H^1_{\mathrm{Nori}}(\mathbb{G}_m, \{1\}) \in \mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}} \ ,$$

   the *Lefschetz motive*.

3. The category $\mathcal{MM}_{\mathrm{Nori}} = \mathcal{MM}_{\mathrm{Nori}}(k)$ of *Nori motives* is defined as the localization of $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$ with respect to $\mathbb{Z}(-1)$.

4. We also write H^* for the extension of H^* to $\mathcal{MM}_{\mathrm{Nori}}$.

Remark 8.1.4. This is equivalent to Nori's orginal definition by Theorem 8.3.4.

8.1.2 Main results

Theorem 8.1.5 (Nori)**.**

1. *$\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$ has a natural structure of commutative tensor category with unit such that H^* is a tensor functor.*

2. *$\mathcal{MM}_{\mathrm{Nori}}$ is a rigid tensor category.*

3. *$\mathcal{MM}_{\mathrm{Nori}}$ is equivalent to the category of representations of a pro-algebraic group scheme $G_{\mathrm{mot}}(k, \mathbb{Z})$ over $\mathbb{Z}$.*

For the proof see Section 8.3.1.

Definition 8.1.6. The group scheme $G_{\mathrm{mot}}(k, \mathbb{Z})$ is called the *motivic Galois group* in the sense of Nori.

Remark 8.1.7. The first statement also holds with the coefficient ring $\mathbb{Z}$ replaced by any noetherian ring R. The other two hold if R is a Dedekind ring R or field. Of particular interest is the case $R = \mathbb{Q}$.

The proof of this theorem will take the rest of the chapter. We now explain the key ideas. In order to define the tensor structure, we would like to apply the abstract machine developed in Section 7.1. However, the shape of the Künneth formula

$$H^n(X \times Y, \mathbb{Q}) \cong \bigoplus_{i+j=n} H^i(X, \mathbb{Q}) \otimes H^i(Y, \mathbb{Q})$$

is not of the required kind. Nori introduces a subdiagram of *good pairs* where relative cohomology is concentrated in a single degree and free, so that the Künneth formula simplifies. The key insight now becomes that it is possible to recover *all* pairs from good pairs. This is done via an algebraic skeletal filtration constructed from the Basic Lemma as discussed in Section 2.5. As a byproduct, we will also know that $\mathcal{MM}^{\text{eff}}_{\text{Nori}}$ and $\mathcal{MM}_{\text{Nori}}$ are given as representations of an algebra monoid. In the next step, we have to verify rigidity, i.e., we have to show that the monoid is an algebraic group. We do this by verifying the abstract criterion of Section 7.3.

On the way, we need to establish a general "motivic" property of Nori motives.

Theorem 8.1.8. *There is a natural contravariant triangulated functor*

$$R : K_b(\mathbb{Z}[\text{Var}]) \to D^b(\mathcal{MM}^{\text{eff}}_{\text{Nori}})$$

on the homotopy category of bounded homological complexes in $\mathbb{Z}[\text{Var}]$ *such that for every effective pair* (X, Y, i) *we have*

$$H^i(R(\text{Cone}(Y \to X)) = H^i_{\text{Nori}}(X, Y).$$

For the proof see Section 8.3.1. The theorem allows, for example, to define motives of simplicial varieties or motives with support.

The category of motives is supposed to be the universal abelian category such that all cohomology theories with suitable properties factor via the category of motives. We do not yet have such a theory, even though it is reasonable to conjecture that $\mathcal{MM}_{\text{Nori}}$ is the correct description. In any case, it does have a universal property which is good enough for many applications.

Theorem 8.1.9 (Universal property)**.** *Let* $\mathcal{A}$ *be an abelian category with a faithful exact functor* $f : \mathcal{A} \to R{-}\text{Mod}$ *for a noetherian ring* R*. Let*

$$H'^* : \text{Pairs}^{\text{eff}} \to \mathcal{A}$$

be a representation. Assume that there is an extension $R \to S$ *such that* S *is faithfully flat over* R *and* $\mathbb{Z}$ *and an isomorphism of representations*

$$\Phi : H'^*_S \to (f \circ H'^*)_S.$$

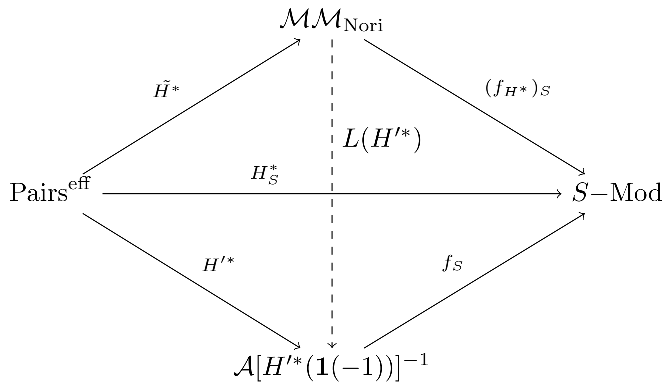
Then H'^* extends to $\mathcal{MM}_{\mathrm{Nori}}$:

$$\mathrm{Pairs}^{\mathrm{eff}} \to \mathcal{MM}_{\mathrm{Nori}} \to \mathcal{A}[H'^*(\mathbf{1}(-1))]^{-1}.$$

More precisely, there exists a functor $L(H'^*) : \mathcal{MM}_{\mathrm{Nori}} \to \mathcal{A}[\mathbf{1}(-1)]^{-1}$ and an isomorphism of functors

$$\tilde{\Phi} : (f_{H^*})_S \to f_S \circ L(H'^*)$$

such that



commutes up to ϕ and $\tilde{\phi}$. The pair $(L(H'^*), \tilde{\phi})$ is unique up to unique isomorphism of functors.

If, moreover, $\mathcal{A}$ is a tensor category, f a tensor functor and H'^* a graded multiplicative representation on $\mathrm{Good}^{\mathrm{eff}}$*, then $L(H'^*)$ is a tensor functor and $\tilde{\phi}$ is an isomorphism of tensor functors.*

For the proof see Section 8.3.1. This means that $\mathcal{MM}_{\mathrm{Nori}}$ is universal for all cohomology theories with a comparison isomorphism to singular cohomology. Actually, it suffice to have a representation of $\mathrm{Good}^{\mathrm{eff}}$ or $\mathrm{VGood}^{\mathrm{eff}}$, see Definition 8.2.1.

Example 8.1.10. Let $R = k$, $\mathcal{A} = k-\mathrm{Mod}$, H'^* algebraic de Rham cohomology see Chapter 3. Let $S = \mathbb{C}$, and let the comparison isomorphism Φ be the period isomorphism of Chapter 5. By the universal property, de Rham cohomology extends to $\mathcal{MM}_{\mathrm{Nori}}$. We will study this example in a lot more detail in Part III in order to understand the period algebra.

Example 8.1.11. Let $R = \mathbb{Z}$, $\mathcal{A}$ the category of mixed $\mathbb{Z}$-Hodge structures, H'^* the functor assigning a mixed Hodge structure to a variety or a pair. Then $S = \mathbb{Z}$ and Φ is the functor mapping a Hodge structure to the underlying $\mathbb{Z}$-module. By the universal property, H'^* factors canonically via $\mathcal{MM}_{\mathrm{Nori}}$. In other words, motives define mixed Hodge structures.

Example 8.1.12. Let ℓ be a prime, $R = \mathbb{Z}_\ell$, and $\mathcal{A}$ the category of finitely generated $\mathbb{Z}_\ell$-modules with a continuous operation of $\mathrm{Gal}(\bar{k}/k)$. Let H'^* be ℓ-adic cohomology over $\bar{k}$. For X a variety and $Y \subset X$ a closed subvariety with open complement $j : U \to X$, we have

$$(X, Y, i) \mapsto H^i_{et}(X_{\bar{k}}, j_!\mathbb{Z}_\ell).$$

In this case, we let $S = \mathbb{Z}_l$ and use the comparison isomorphism between ℓ-adic and singular cohomology.

Corollary 8.1.13. *The category $\mathcal{MM}_{\mathrm{Nori}}$ is independent of the choice of embedding $\sigma : k \to \mathbb{C}$. More precisely, $\sigma' : k \to \mathbb{C}$ be another embedding. Let H'^* be singular cohomology with respect to this embedding. Then there is an equivalence of categories*

$$\mathcal{MM}_{\mathrm{Nori}}(\sigma) \to \mathcal{MM}_{\mathrm{Nori}}(\sigma').$$

Proof. Use $S = \mathbb{Z}_\ell$ and the comparison isomorphism given by comparing both singular cohomology functors with ℓ-adic cohomology. This induces the functor. □

Remark 8.1.14. Note that the equivalence is *not* canonical. In the argument above it depends on the choice of embeddings of $\bar{k}$ into $\mathbb{C}$ extending σ and σ', respectively. If we are willing to work with rational coefficients instead, we can compare both singular cohomologies with algebraic de Rham cohomology (with $S = k$). This gives a compatible system of comparison equivalences.

8.2 Yoga of good pairs

We now turn to alternative descriptions of $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$ better suited to the tensor structure.

8.2.1 Good pairs and good filtrations

Definition 8.2.1. Let k be a subfield of $\mathbb{C}$.

1. The diagram $\mathrm{Good}^{\mathrm{eff}}$ of *effective good pairs* is the full subdiagram of $\mathrm{Pairs}^{\mathrm{eff}}$ with vertices the triples (X, Y, i) such that singular cohomology satisfies
$$H^j(X(\mathbb{C}), Y(\mathbb{C}); \mathbb{Z}) = 0, \text{ unless } j = i.$$
and is free for $j = i$.

2. The diagram $\mathrm{VGood}^{\mathrm{eff}}$ of *effective very good pairs* is the full subdiagram of those effective good pairs (X, Y, i) with X affine, $X \smallsetminus Y$ smooth and either X of dimension i and Y of dimension $i-1$, or $X = Y$ of dimension less than i.

We will later (see Definition 8.3.2) also introduce the diagrams Pairs of *pairs*, Good of *good pairs* and VGood of *very good pairs* as localization (see Definition 7.2.1) with respect to $(\mathbb{G}_m, \{1\}, 1)$.

Good pairs exist in abundance by the basic lemma, see Theorem 2.5.2.

Our first aim is to show that the diagram categories attached to $\mathrm{Pairs}^{\mathrm{eff}}$, $\mathrm{Good}^{\mathrm{eff}}$ and $\mathrm{VGood}^{\mathrm{eff}}$ are equivalent. By the general principles of diagram categories this means that we have to represent the diagram $\mathrm{Pairs}^{\mathrm{eff}}$ in $\mathcal{C}(\mathrm{VGood}^{\mathrm{eff}}, H^*)$. We do this in two steps: a general variety is replaced by the Čech complex attached to an affine cover; affine varieties are replaced by complexes of very good pairs using the key idea of Nori. The construction proceeds in a complicated way because both steps involve choices which have to be made in a compatible way. We handle this problem in the same way as in [Hu3].

We start in the affine case. Using induction, one gets from the Basic Lemma 2.5.2:

Proposition 8.2.2. *Every affine variety X has a filtration*

$$\emptyset = F_{-1}X \subset F_0X \subset \cdots \subset F_{n-1}X \subset F_nX = X,$$

such that $(F_jX, F_{j-1}X, j)$ is very good.

Filtrations of the above type are called *very good filtrations.*

Proof. Let $\dim X = n$. Put $F_nX = X$. Choose a subvariety of dimension $n-1$ which contains all singular points of X. By the Basic Lemma 2.5.2, there is a subvariety $F_{n-1}X$ of dimension $n-1$ such that $(F_nX, F_{n-1}X, n)$ is good. By construction $F_{n-1}X \smallsetminus F_{n-1}X$ is smooth and hence the pair is very good. We continue by induction. □

Corollary 8.2.3. *Let X be an affine variety. The inductive system of all very good filtrations of X is filtered and functorial.*

Proof. Let F_*X and F'_*X be two very good filtrations of X. $F_{n-1}X \cup F'_{n-1}X$ has dimension $n-1$. By the Basic Lemma 2.5.2, there is subvariety $G_{n-1}X \subset X$ of dimension $n-1$ such that $(X, G_{n-1}X, n)$ is a good pair. It is automatically very good. We continue by induction.

Consider a morphism $f : X \to X'$. Let F_*X be a very good filtration. Then $f(F_iX)$ has dimension at most i. As in the proof of Corollary 8.2.2, we construct a very good filtration F_*X' with the additional property $f(F_iX) \subset F_iX'$. □

Remark 8.2.4. This allows to construct a functor from the category of affine varieties to the diagram category $\mathcal{C}(\mathrm{VGood}^{\mathrm{eff}}, H^*)$ as follows: Given an affine variety X, let F_*X be a very good filtration. The boundary maps of the triples $F_{i-1}X \subset F_iX \subset F_{i+1}X$ define a complex in $\mathcal{C}(\mathrm{VGood}^{\mathrm{eff}}, H^*)$

$$\cdots \to H^i_{\mathrm{Nori}}(F_iX, F_{i-1}X) \to H^{i+1}_{\mathrm{Nori}}(F_{i+1}X, F_iX) \to \ldots$$

Taking i-th cohomology of this complex defines an object in $\mathcal{C}(\text{VGood}^{\text{eff}}, H^*)$ whose underlying $\mathbb{Z}$-module is nothing but singular cohomology $H^i(X, \mathbb{Z})$. Up to isomorphism it is independent of the choice of filtration. In particular, it is functorial.

We are going to refine the above construction such that is also applies to complexes of varieties.

8.2.2 Čech complexes

The next step is to replace arbitrary varieties by affine ones. The idea for the following construction is from the case of étale coverings, see [F] Definition 4.2.

Definition 8.2.5. Let X a variety. A *rigidified* affine cover is a finite open affine covering $\{U_i\}_{i\in I}$ together with a choice of an index i_x for every closed point $x \in X$ such that $x \in U_{i_x}$. We also assume that in the covering every index $i \in I$ occurs as i_x for some $x \in X$.

Let $f : X \to Y$ be a morphism of varieties, $\{U_i\}_{i\in I}$ a rigidified open cover of X and $\{V_j\}_{j\in J}$ a rigidified open cover of Y. A *morphism* of rigidified covers (over f)

$$\phi : \{U_i\}_{i\in I} \to \{V_j\}_{j\in J}$$

is a map of sets $\phi : I \to J$ such that $f(U_i) \subset V_{\phi(i)}$ and for all $x \in X$ we have $\phi(i_x) = j_{f(x)}$.

Remark 8.2.6. The rigidification makes ϕ unique if it exists.

Lemma 8.2.7. *The projective system of rigidified affine covers is filtered and strictly functorial, i.e., if $f : X \to Y$ is a morphism of varieties, pull-back defines a map of projective systems.*

Proof. Any two covers have their intersection as common refinement with index set the product of the index sets. The rigidification extends in the obvious way. Preimages of rigidified covers are rigidified open covers. □

We need to generalize this to complexes of varieties. Recall from Definition 1.1.1 the additive categories $\mathbb{Z}[\text{Aff}]$ and $\mathbb{Z}[\text{Var}]$ with objects (affine) varieties and morphisms roughly $\mathbb{Z}$-linear combinations of morphisms of varieties. The support of a morphism in $\mathbb{Z}[\text{Var}]$ is the set of morphisms occuring in the linear combination.

Definition 8.2.8. Let X_* be a homological complex of varieties, i.e., an object in $C_b(\mathbb{Z}[\text{Var}])$. An *affine cover* of X_* is a complex of rigidified affine covers, i.e., for every X_n the choice of a rigidified open cover $\tilde{U}_{X_n}$ and for every $g : X_n \to X_{n-1}$ in the support of the differential $X_n \to X_{n-1}$ in the complex X_* a morphism of rigidified covers $\tilde{g} : \tilde{U}_{X_n} \to \tilde{U}_{X_{n-1}}$ over g.

Let $F_* : X_* \to Y_*$ be a morphism in $C_b(\mathbb{Z}[\mathrm{Var}])$ and $\tilde{U}_{X_*}$, $\tilde{U}_{Y_*}$ affine covers of X_* and Y_*. A morphism of affine covers over F_* is a morphism of rigidified affine covers $f_n : \tilde{U}_{X_n} \to \tilde{U}_{Y_n}$ over every morphism in the support of F_n.

Lemma 8.2.9. *Let $X_* \in C_b(\mathbb{Z}[\mathrm{Var}])$. Then the projective system of rigidified affine covers of X_* is non-empty, filtered and functorial, i.e., if $f_* : X_* \to Y_*$ is a morphism of complexes and $\tilde{U}_{X_*}$ an affine cover of X_*, then there is an affine cover $\tilde{U}_{Y_*}$ and a morphism of complexes of rigidified affine covers. Any two choices are compatible in the projective system of covers.*

Proof. Let n be minimal with $X_n \neq \emptyset$. Choose a rigidified cover of X_n. The support of $X_{n+1} \to X_n$ has only finitely many elements. Choose a rigidified cover of X_{n+1} compatible with all of them. Continue inductively.

Similar constructions show the rest of the assertion. □

Definition 8.2.10. Let X be a variety and $\tilde{U}_X = \{U_i\}_{i\in I}$ a rigidified affine cover of X. We put

$$C_\star(\tilde{U}_X) \in C_-(\mathbb{Z}[\mathrm{Aff}]),$$

the Čech complex associated to the cover, i.e.,

$$C_n(\tilde{U}_X) = \coprod_{\underline{i}\in I_n} \bigcap_{i\in\underline{i}} U_i,$$

where I_n is the set of tuples $(i_0, \dots, i_n)$. The boundary maps are the ones obtained by taking the alternating sum of the boundary maps of the simplicial scheme.

If $X_* \in C_b(\mathbb{Z}[\mathrm{Var}])$ is a complex, and $\tilde{U}_{X_*}$ a rigidified affine cover, let

$$C_\star(\tilde{U}_{X_*}) \in C_{-,b}(\mathbb{Z}[\mathrm{Aff}])$$

be the double complex $C_i(\tilde{U}_{X_j})$.

Note that all components of $C_\star(\tilde{U}_{X_*})$ are affine. The projective system of these complexes is filtered and functorial.

Definition 8.2.11. Let X be a variety, $\{U_i\}_{i\in I}$ a rigidified affine cover of X. A *very good filtration* on $\tilde{U}_X$ is the choice of very good filtrations for

$$\bigcap_{i\in J} U_i$$

for all $J \subset I$ compatible with all inclusions between these.

Let $f : X \to Y$ be a morphism of varieties, $\phi : \{U_i\}_{i\in I} \to \{V_j\}_{j\in J}$ a morphism of rigidified affine covers above f . Fix very good filtrations on both covers. The morphism ϕ is called *filtered*, if for all $J \subset I$ the induced map

$$\bigcap_{i\in I'} U_i \to \bigcap_{i\in I'} V_{\phi(i)}$$

is compatible with the filtrations.

Let $X_* \in C_b(\mathbb{Z}[\mathrm{Var}])$ be a bounded complex of varieties, $\tilde{U}_{X_*}$ an affine cover of X_*. A *very good filtration* on $\tilde{U}_{X_*}$ is a very good filtration on all $\tilde{U}_{X_n}$ compatible with all morphisms in the support of the boundary maps.

Note that the Čech complex associated to a rigidified affine cover with very good filtration is also filtered in the sense that there is a very good filtration on all $C_n(\tilde{U}_X)$ and all morphisms in the support of the differential are compatible with the filtrations.

Lemma 8.2.12. *Let X be a variety, $\tilde{U}_X$ a rigidified affine cover. Then the inductive system of very good filtrations on $\tilde{U}_X$ is non-empty, filtered and functorial.*

The same statement also holds for a complex of varieties $X_* \in C_b(\mathbb{Z}[\mathrm{Var}])$.

Proof. Let $\tilde{U}_X = \{U_i\}_{i\in I}$ be the affine cover. We choose recursively very good filtrations on $\bigcap_{i\in J} U_i$ with decreasing order of J, compatible with the inclusions.

We extend the construction inductively to complexes, starting with the highest term of the complex. □

Definition 8.2.13. Let $X_* \in C_-(\mathbb{Z}[\mathrm{Aff}])$. A *very good filtration* of X_* is given by a very good filtration $F_\cdot X_n$ for all n which is compatible with all morphisms in the support of the differentials of X_*.

Lemma 8.2.14. *Let $X_* \in C_b(\mathbb{Z}[\mathrm{Var}])$ and $\tilde{U}_{X_*}$ an affine cover of X_* with a very good filtration. Then the total complex of $C_\star(\tilde{U}_{X_*})$ carries a very good filtration.*

Proof. Clear by construction. □

8.2.3 Putting things together

Let $\mathcal{A}$ be an abelian category with a faithful forgetful functor $f : \mathcal{A} \to R{-}\mathrm{Mod}$ with R noetherian. Let $T : \mathrm{VGood}^{\mathrm{eff}} \to \mathcal{A}$ be a representation of the diagram of very good pairs.

Definition 8.2.15. Let $F_\bullet X$ be an affine variety X together with a very good filtration $F_\bullet$. We put $\tilde{R}(F_\bullet X) \in C^b(\mathcal{A})$

$$\cdots \to T(F_j X_*, F_{j-1}X_*) \to T(F_{j+1}X_*, F_j X_*) \to \dots$$

Let $F_\bullet X_*$ be a very good filtration of a complex $X_* \in C_-(\mathbb{Z}[\mathrm{Aff}])$. We put $\tilde{R}(F_\bullet X_*) \in C^+(\mathcal{A})$ the total complex of the double complex $\tilde{R}(F_\cdot X_n)_{n\in\mathbb{Z}}$.

Proposition 8.2.16. *Let $\mathcal{A}$ be an R-linear abelian category with a faithful forgetful functor f to $R{-}\mathrm{Mod}$. Let $T : \mathrm{VGood}^{\mathrm{eff}} \to \mathcal{A}$ be a representation such* *that $f \circ T$ is singular cohomology with R-coefficients. Then there is a natural contravariant triangulated functor*

$$R : C_b(\mathbb{Z}[\mathrm{Var}]) \to D^b(\mathcal{A})$$

on the category of bounded homological complexes in $\mathbb{Z}[\mathrm{Var}]$ *such that for every good pair (X, Y, i) we have*

$$H^j(R(\mathrm{Cone}(Y \to X)) = \begin{cases} 0 & j \neq i, \\ T(X, Y, i) & j = i. \end{cases}$$

Moreover, the image of $R(X)$ in $D^b(R\mathrm{-Mod})$ *computes singular cohomology of $X(\mathbb{C})$.*

Proof. We first define $R : C^b(\mathbb{Z}[\mathrm{Var}]) \to D^b(\mathcal{A})$ on objects. Let $X_* \in C_b(\mathbb{Z}[\mathrm{Var}])$. Choose a rigidified affine cover $\tilde{U}_{X_*}$ of X_*. This is possible by Lemma 8.2.9. Choose a very good filtration on the cover. This is possible by 8.2.12. It induces a very good filtration on $\mathrm{Tot}C_\star(\tilde{U}_{X_*})$. Put

$$R(X_*) = \tilde{R}(\mathrm{Tot}C_\star(\tilde{U}_{X_*})).$$

Note that any other choice yields a complex isomorphic to this one in $D^+(\mathcal{A})$ because f is faithful and exact and the image of $R(X_*)$ in $D^+(R\mathrm{-Mod})$ computes singular cohomology with R-coefficients.

Let $f : X_* \to Y_*$ be a morphism. Choose a refinement $\tilde{U}'_{X_*}$ of $\tilde{U}_{X_*}$ which maps to $\tilde{U}_{Y_*}$ and a very good filtration on $\tilde{U}'_{X_*}$. Choose a refinement of the filtrations on $\tilde{U}_{X_*}$ and $\tilde{U}_{Y_*}$ compatible with the filtration on $\tilde{U}'_{X_*}$. This gives a little diagram of morphisms of complexes $\tilde{R}$ which defines $R(f)$ in $D^+(\mathcal{A})$. □

Remark 8.2.17. Nori suggests working with Ind-objects (or rather pro-object in our dual setting) in order to get functorial complexes attached to affine varieties. However, the mixing between inductive and projective systems in our construction does not make it obvious if this works out for the result we needed. In order to avoid this situation, it might, however, be possible to do the construction in two steps. This approach is used in Harrer's generalization to complexes of smooth correspondences, [Ha], which completely avoids discussing Čech complexes.

As a corollary of the construction in the proof, we also get:

Corollary 8.2.18. *Let X be a variety, $\tilde{U}_X$ a rigidified affine cover with Čech complex $C_\star(\tilde{U}_X)$. Then*

$$R(X) \to R(C_\star(\tilde{U}_X))$$

is an isomorphism in $D^+(\mathcal{A})$.

We are mostly interested in two explicit examples of complexes.

Definition 8.2.19. Consider the situation of Proposition 8.2.16. Let $Y \subset X$ be a closed subvariety with open complement U, $i \in \mathbb{Z}$. Then we put

$$R(X,Y) = R(\mathrm{Cone}(Y \to X)), \quad R_Y(X) = R(\mathrm{Cone}(U \to X)) \in D^b(\mathcal{A})$$
$$H(X,Y,i) = H^i(R(X,Y)), \quad H_Y(X,i) = H^i(R_Y(X)) \in \mathcal{A}$$

$H(X,Y,i)$ is called *relative cohomology.* $H_Y(X,i)$ is called *cohomology with support.*

8.2.4 Comparing diagram categories

We are now ready to proof the first key theorems.

Theorem 8.2.20. *The diagram categories* $\mathcal{C}(\mathrm{Pairs}^{\mathrm{eff}}, H^*)$, $\mathcal{C}(\mathrm{Good}^{\mathrm{eff}}, H^*)$ *and* $\mathcal{C}(\mathrm{VGood}^{\mathrm{eff}}, H^*)$ *are equivalent.*

Proof. The inclusion of diagrams induces faithful functors

$$i : \mathcal{C}(\mathrm{VGoodeff}, H^*) \to \mathcal{C}(\mathrm{Good}^{\mathrm{eff}}, H^*) \to \mathcal{C}(\mathrm{Pairs}^{\mathrm{eff}}, H^*).$$

We want to apply Corollary 6.1.18. Hence it suffices to represent the diagram $\mathrm{Pairs}^{\mathrm{eff}}$ in $\mathcal{C}(\mathrm{VGood}^{\mathrm{eff}}, H^*)$ such that the restriction of the representation to $\mathrm{VGood}^{\mathrm{eff}}$ gives back H^* (up to natural isomorphism).

We turn to the construction of the representation of $\mathrm{Pairs}^{\mathrm{eff}}$ in $\mathcal{C}(\mathrm{VGood}^{\mathrm{eff}}, H^*)$. We apply Proposition 8.2.16 to

$$H^* : \mathrm{VGood}^{\mathrm{eff}} \to \mathcal{C}(\mathrm{VGood}^{\mathrm{eff}}, H^*)$$

and get a functor

$$R : C_b(\mathbb{Z}[\mathrm{Var}]) \to D^b(\mathcal{C}(\mathrm{VGood}^{\mathrm{eff}}, H^*)).$$

Consider an effective pair (X,Y,i) in D. It is represented by

$$H(X,Y,i) = H^i(R(X,Y)) \in \mathcal{C}(\mathrm{VGood}^{\mathrm{eff}}, H^*)$$

where

$$R(X,Y) = R(\mathrm{Cone}(Y \to X)) \ .$$

The construction is functorial for morphisms of pairs. This allows to represent edges of type f^*.

Finally, we need to consider edges corresponding to coboundary maps for triples $X \supset Y \supset Z$. In this case, it follows from the construction of R that there is a natural exact triangle

$$R(X,Y) \to R(X,Z) \to R(Y,Z).$$

We use the connecting morphism in cohomology to represent the edge $(Y,Z,i) \to (X,Y,i+1)$. □

For further use, we record a number of corollaries.

Corollary 8.2.21. *Every object of $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$ is a subquotient of a direct sum of objects of the form $H^i_{\mathrm{Nori}}(X,Y)$ for a good pair (X,Y,i) where $X = W \smallsetminus W_\infty$ and $Y = W_0 \smallsetminus (W_0 \cap W_\infty)$ with W smooth projective, $W_\infty \cup W_0$ a divisor with normal crossings.*

Proof. By Proposition 6.1.15, every object in the diagram category of $\mathrm{VGood}^{\mathrm{eff}}$ (and hence $\mathcal{MM}_{\mathrm{Nori}}$) is a subquotient of a direct sum of some $H^i_{\mathrm{Nori}}(X,Y)$ with (X,Y,i) very good. In particular, $X \smallsetminus Y$ can be assumed smooth.

We follow Nori: By resolution of singularities, there is a smooth projective variety W and a normal crossing divisor $W_0 \cup W_\infty \subset W$ together with a proper, surjective morphism $\pi : W \smallsetminus W_\infty \to X$ such that one has $\pi^{-1}(Y) = W_0 \smallsetminus W_\infty$ and $\pi : W \smallsetminus \pi^{-1}(Y) \to X \smallsetminus Y$ is an isomorphism. This implies that

$$H^*_{\mathrm{Nori}}(W \smallsetminus W_\infty, W_0 \smallsetminus (W_0 \cap W_\infty)) \to H^*_{\mathrm{Nori}}(X,Y)$$

is also an isomorphism by proper base change, i.e., excision. □

Remark 8.2.22. Note that the pair $(W \smallsetminus W_\infty, W_0 \smallsetminus (W_0 \cap W_\infty)$ is good, but not very good in general. Replacing Y by a larger closed subset Z, one may, however, assume that $W_0 \smallsetminus (W_0 \cap W_\infty)$ is affine. Therefore, by Lemma 8.3.8, the dual of each generator can be assumed to be very good.

It is not clear to us if it suffices to construct Nori's category using the diagram of (X,Y,i) with X smooth, Y a divisor with normal crossings. The corollary says that the diagram category has the right "generators", but there might be too few "relations".

Corollary 8.2.23. *Let $Z \subset X$ be a closed immersion. Then there is a natural object $H^i_Z(X)$ in $\mathcal{MM}_{\mathrm{Nori}}$ representing cohomology with supports. There is a natural long exact sequence*

$$\cdots \to H^i_Z(X) \to H^i_{\mathrm{Nori}}(X) \to H^i_{\mathrm{Nori}}(X \smallsetminus Z) \to H^{i+1}_Z(X) \to \cdots$$

Proof. Let $U = X \smallsetminus Z$. Put

$$R_Z(X) = R(\mathrm{Cone}(U \to X)), \quad H^i_Z(X) = H^i(R_Z(X)) .$$

□

8.3 Tensor structure

We now introduce the tensor structure using the formal set-up developed in Section 7.1. Recall that $\mathrm{Pairs}^{\mathrm{eff}}$, $\mathrm{Good}^{\mathrm{eff}}$ and $\mathrm{VGood}^{\mathrm{eff}}$ are graded diagrams with $|(X,Y,i)| = i$.

Proposition 8.3.1. *The graded diagrams* Good *and* $\mathrm{VGood}^{\mathrm{eff}}$ *carry a weak commutative product structure (see Definition 7.1.3) defined as follows: for all vertices* $(X, Y, i), (X', Y', i')$

$$(X, Y, i) \times (X', Y', i') = (X \times X', X \times Y' \cup Y \times X', i + i').$$

with the obvious definition on edges. Let also

$$\alpha : (X, Y, i) \times (X', Y', i') \to (X', Y', i') \times (X, Y, i)$$
$$\beta : (X, Y, i) \times ((X', Y', i') \times (X'', Y'', i'')) \to ((X, Y, i) \times (X', Y', i')) \times (X'', Y'', i'')$$
$$\beta' : ((X, Y, i) \times (X', Y', i')) \times (X'', Y'', i'') \to (X, Y, i) \times ((X', Y', i') \times (X'', Y'', i''))$$

be the edges given by the natural isomorphisms of varieties.

There is a unit given by $(\mathrm{Spec}\ k, \emptyset, 0)$ *and*

$$u : (X, Y, i) \to (\mathrm{Spec}\ k, \emptyset, 0) \times (X, Y, i) = (\mathrm{Spec}\ k \times X, \mathrm{Spec}\ k \times Y, i)$$

be given by the natural isomorphism of varieties.

Moreover, H^* *is a weak graded multiplicative representation (see Definition 7.1.3, Remark [**?**]) with*

$$\tau : H^{i+i'}(X \times X', X \times Y' \cup X' \times Y, \mathbb{Z}) \to H^i(X, Y, \mathbb{Z}) \otimes H^{i'}(X', Y', \mathbb{Z})$$

the Künneth isomorphism (see Theorem 2.4.1).

Proof. If (X, Y, i) and (X', Y', i') are good pairs, then by the Küennth formula so is $(X \times X', X \times Y' \cup Y \times X', i + i')$. If they are even very good, then so is their product. Hence $\times$ is well-defined on vertices. Recall that edges id $\mathrm{Good}^{\mathrm{eff}} \times \mathrm{Good}^{\mathrm{eff}}$ are of the form $\gamma \times \mathrm{id}$ or $\mathrm{id} \times \gamma$ for an edge γ of $\mathrm{Good}^{\mathrm{eff}}$. The definition of $\times$ on these edges is the natural one. We explain the case $\delta \times \mathrm{id}$ in detail. Let $X \supset Y \subset Z$ and $A \supset B$. We compose the functoriality edge for

$$(Y \times A, Z \times A \cup Y \times B) \to (Y \times A \cup X \times B, Z \times A \cup Y \times B)$$

with the boundary edge for

$$X \times A \supset Y \times A \cup X \times B \supset Z \times A \cup Y \times B$$

and obtain

$$\begin{aligned}\delta \times \mathrm{id} : (Y, Z, n) \times (A, B, m) &= (Y \times A, Z \times A \cup Y \times B, n + m)\\ &\to (X \times A, Y \times A \cup X \times B, n + m + 1) = (X, Y, n + 1) \times (A, B, m)\end{aligned}$$

as a morphism in the path category $\mathcal{P}(\mathrm{Good}^{\mathrm{eff}})$.

We need to check that H^* satisfies the conditions of Definition 7.1.3. This is tedious, but straightforward from the properties of the Künneth formula, see in particular Proposition 2.4.3 for compatibility with edges of type ∂ changing the degree.

Associativity and graded commutativity are stated in Proposition 2.4.2. □

Definition 8.3.2. Let Good and VGood be the localizations (see Definition 7.2.1) of Good^{eff} and $\text{VGood}^{\text{eff}}$, respectively, with respect to the vertex $\mathbf{1}(-1) = (\mathbb{G}_m, \{1\}, 1)$.

Proposition 8.3.3. Good *and* VGood *are graded diagrams with a weak commutative product structure (see Remark 7.1.6). Moreover,* H^* *is a graded multiplicative representation of* Good *and* VGood*.*

Proof. This follows formally from the effective case and Lemma 7.2.4. The Assumption 7.2.3 that $H^*(\mathbf{1}(-1)) = \mathbb{Z}$ is satisfied by Proposition 8.1.2. □

Theorem 8.3.4. *1. This definition of* $\mathcal{MM}_{\text{Nori}}$ *is equivalent to Nori's original definition.*

2. $\mathcal{MM}^{\text{eff}}_{\text{Nori}} \subset \mathcal{MM}_{\text{Nori}}$ *are commutative tensor categories with a faithful fiber functor* H^**.*

3. $\mathcal{MM}_{\text{Nori}}$ *is equivalent to the digram categories* $\mathcal{C}(\text{Good}, H^*)$ *and* $\mathcal{C}(\text{VGood}, H^*)$*.*

Proof. We already know by Theorem 8.2.20 that

$$\mathcal{C}(\text{VGood}^{\text{eff}}, H^*) \to \mathcal{C}(\text{Good}^{\text{eff}}, H^*) \to \mathcal{C}(\text{Pairs}^{\text{eff}}, H^*) = \mathcal{MM}^{\text{eff}}_{\text{Nori}}$$

are equivalent. Moreover, this agrees with Nori's definition using either Good^{eff} or $\text{Pairs}^{\text{eff}}$.

By Proposition 8.3.1, the diagrams $\text{VGood}^{\text{eff}}$ and Good^{eff} carry a mulitplicative structure. Hence by Proposition 7.1.5, the category $\mathcal{MM}^{\text{eff}}_{\text{Nori}}$ carries a tensor structure.

By Proposition 7.2.5, the diagram categories of the localized diagrams Good and VGood also have tensor structure and can be equivalently defined as the localization with respect to he Lefschetz object $\mathbf{1}(-1)$.

In [L1], the category of Nori motives is defined as the category of comodules of finite type over $\mathbb{Z}$ for the localization of the ring A^{eff} with respect to the element $\chi \in A(\mathbf{1}(-1))$ considered in Proposition 7.2.5. By this same Proposition, the category of A^{eff}_{χ}-comodules agrees with $\mathcal{MM}_{\text{Nori}}$. □

Our next aim is to establish rigidity using the criterion of Section 7.3. Hence we need to check that Poincaré duality is motivic, at least in a weak sense.

Remark 8.3.5. An alternative argument using Harrer's realization functor from geometric motives (see Theorem 10.3.5) is explained in [Ha].

Definition 8.3.6. Let $\mathbf{1}(-1) = H^1_{\text{Nori}}(\mathbb{G}_m)$ and $\mathbf{1}(-n) = \mathbf{1}(-1)^{\otimes n}$.

Lemma 8.3.7. *1.* $H^{2n}_{\text{Nori}}(\mathbb{P}^N) = \mathbf{1}(-n)$ *for* $N \geq n \geq 0$*.*

2. Let Z *be a projective variety of dimension* n*. Then* $H^{2n}_{\text{Nori}}(Z) \cong \mathbf{1}(-n)$*.*

3. *Let X be a smooth variety, $Z \subset X$ a smooth, irreducible, closed subvariety of pure codimension n. Then the motive with support of Corollary 8.2.23 satisfies*
$$H^{2n}_Z(X) \cong \mathbf{1}(-n).$$

Proof. 1. Embedding projective spaces linearly into higher dimensional projective spaces induces isomorphisms on cohomology and hence motives. Hence it suffices to check the top cohomology of $\mathbb{P}^N$.

We start with $\mathbb{P}^1$. Consider the standard cover of $\mathbb{P}^1$ by $U_1 = \mathbb{A}^1$ and $U_2 = \mathbb{P}^1 \smallsetminus \{0\}$. We have $U_1 \cap U_2 = \mathbb{G}_m$. By Corollary 8.2.18,

$$R(\mathbb{P}^1) \to \mathrm{Cone}\Big(R(U_1) \oplus R(U_2) \to R(\mathbb{G}_m)\Big)[-1]$$

is an isomorphism in the derived category. This induces the isomorphism $H^2_{\mathrm{Nori}}(\mathbb{P}^1) \to H^1_{\mathrm{Nori}}(\mathbb{G}_m)$. Similarly, the Čech complex (see Definition 8.2.10) for the standard affine cover of $\mathbb{P}^N$ relates $H^{2N}_{\mathrm{Nori}}(\mathbb{P}^N)$ with $H^N_{\mathrm{Nori}}(\mathbb{G}_m^N)$.

2. Let $Z \subset \mathbb{P}^N$ be a closed immersion with N large enough. Then $H^{2n}_{\mathrm{Nori}}(Z) \to H^{2n}_{\mathrm{Nori}}(\mathbb{P}^N)$ is an isomorphism in $\mathcal{MM}_{\mathrm{Nori}}$ because it is in singular cohomology.

3. We note first that under our assumptions 3. holds in singular cohomology by the Gysin isomorphism 2.1.8

$$H^0(Z) \xrightarrow{\cong} H^{2n}_Z(X).$$

For the embedding $Z \subset X$ one has the deformation to the normal cone [Fu, Sec. 5.1], i.e., a smooth scheme $D(X, Z)$ together with a morphism to $\mathbb{A}^1$ such that the fiber over 0 is given by the normal bundle $N_Z X$ of Z in X, and the other fibers by X. The product $Z \times \mathbb{A}^1$ can be embedded into $D(X, Z)$ as a closed subvariety of codimension n, inducing the embeddings of $Z \subset X$ as well as the embedding of the zero section $Z \subset N_Z X$ over 0. Hence, using the three Gysin isomorphisms and homotopy invariance, it follows that there are isomorphisms

$$H^{2n}_Z(X) \leftarrow H^{2n}_{Z \times \mathbb{A}^1}(D(X, Z)) \to H^{2n}_Z(N_Z X)$$

in singular cohomology and hence in our category. Thus, we have reduced the problem to the embedding of the zero section $Z \hookrightarrow N_Z X$. However, the normal bundle $\pi : N_Z X \to Z$ trivializes on some dense open subset $U \subset Z$. This induces an isomorphism

$$H^{2n}_Z(N_Z X) \to H^{2n}_U(\pi^{-1}(U)),$$

and we may assume that the normal bundle $N_Z X$ is trivial. In this case, we have
$$N_Z(X) = N_{Z \times \{0\}}(Z \times \mathbb{A}^n) = N_{\{0\}}(\mathbb{A}^n),$$
so that we have reached the case of $Z = \{0\} \subset \mathbb{A}^n$. Using the Künneth formula with supports and induction on n, it suffices to consider $H^2_{\{0\}}(\mathbb{A}^1)$ which is isomorphic to $H^1(\mathbb{G}_m) = \mathbf{1}(-1)$ by Corollary 8.2.23. □

The following lemma (more precisely, its dual) is formulated implicitly in [N] in order to establish rigidity of $\mathcal{MM}_{\mathrm{Nori}}$.

Lemma 8.3.8. *Let W be a smooth projective variety of dimension i, $W_0, W_\infty \subset W$ divisors such that $W_0 \cup W_\infty$ is a normal crossing divisor. Let*

$$\begin{aligned} X &= W \smallsetminus W_\infty \\ Y &= W_0 \smallsetminus W_0 \cap W_\infty \\ X' &= W \smallsetminus W_0 \\ Y' &= W_\infty \smallsetminus W_0 \cap W_\infty \end{aligned}$$

We assume that (X, Y) is a very good pair.

Then there is a morphism in $\mathcal{MM}_{\mathrm{Nori}}$

$$q : \mathbf{1} \to H^i_{\mathrm{Nori}}(X, Y) \otimes H^i_{\mathrm{Nori}}(X', Y')(i)$$

such that the dual of $H^*(q)$ is a perfect pairing.

Proof. We follow Nori's construction. The two pairs (X, Y) and (X', Y') are Poincaré dual to each other in singular cohomology, see Proposition 2.4.5 for the proof. This implies that they are both good pairs. Hence

$$H^i_{\mathrm{Nori}}(X, Y) \otimes H^i_{\mathrm{Nori}}(X', Y') \to H^{2i}_{\mathrm{Nori}}(X \times X', X \times Y' \cup Y \times X')$$

is an isomorphism. Let $\Delta = \Delta(W \smallsetminus (W_0 \cup W_\infty))$ via the diagonal map. Note that

$$X \times Y' \cup X' \times Y \subset X \times X' \smallsetminus \Delta.$$

Hence, by functoriality and the definition of cohomology with support, there is a map

$$H^{2i}_{\mathrm{Nori}}(X \times X', X \times Y' \cup Y \times X') \leftarrow H^{2i}_{\Delta}(X \times X').$$

Again, by functoriality, there is a map

$$H^{2i}_{\Delta}(X \times X') \leftarrow H^{2i}_{\bar{\Delta}}(W \times W)$$

with $\bar{\Delta} = \Delta(W)$. By Lemma 8.3.7, it is isomorphic to $\mathbf{1}(-i)$. The map q is defined by twisting the composition by (i). The dual of this map realizes Poincaré duality, hence it is a perfect pairing. □

Theorem 8.3.9 (Nori)**.** *$\mathcal{MM}_{\mathrm{Nori}}$ is rigid, hence a neutral Tannakian category. Its Tannaka dual is given by $G_{\mathrm{mot}} = \mathrm{Spec}(A(\mathrm{Good}, H^*))$.*

Proof. By Corollary 8.2.21, every object of $\mathcal{MM}_{\mathrm{Nori}}$ is a subquotient of $M = H^i_{\mathrm{Nori}}(X, Y)(j)$ for a good pair (X, Y, i) of the particular form occurring in Lemma 8.3.8. By this Lemma, they all admit a perfect pairing.

By Proposition 7.3.5, the category $\mathcal{MM}_{\mathrm{Nori}}$ is neutral Tannakian. The Hopf algebra of its Tannaka dual agrees with Nori's algebra by Theorem 6.1.20. □

8.3.1 Collection of proofs

We go through the list of theorems of Section 8.1 and give the missing proofs.

Proof of Theorem 8.1.5. By Theorem 8.3.4, the categories $\mathcal{MM}_{\mathrm{Nori}}^{\mathrm{eff}}$ and $\mathcal{MM}_{\mathrm{Nori}}$ are tensor categories. By construction, H^* is a tensor functor. The category $\mathcal{MM}_{\mathrm{Nori}}$ is rigid by Theorem 8.3.9. By loc. cit., we have a description of its Tannaka dual. □

Proof of Theorem 8.1.8. We apply Proposition 8.2.16 with $\mathcal{A} = \mathcal{MM}_{\mathrm{Nori}}^{\mathrm{eff}}$ and $T = H^*$, $R = \mathbb{Z}$. □

Proof of Theorem 8.1.9. We apply the universal property of the diagram category (see Corollary 6.1.14) to the diagram $\mathrm{Good}^{\mathrm{eff}}$, $T = H^*$ and $F = H'^*$. This gives the universal property for $\mathcal{MM}_{\mathrm{Nori}}^{\mathrm{eff}}$.

Note that $H'^*(\mathbf{1}(-1)) \cong R$ by comparison with singular cohomology. Hence everything extends to $\mathcal{MM}_{\mathrm{Nori}}$ by localizing the categories.

If $\mathcal{A}$ is a tensor category and $H'*$ a graded multiplicative representation, then all functors are tensor functors by construction. □

Part III

Periods

Chapter 9

Periods of varieties

A period, or more precisely, a period number may be thought of as the value of an integral that occurs in a geometric context. In their papers [K1] and [KZ], Kontsevich and Zagier list various ways of how to define a period.

It is stated in their papers without reference that all these variants give the same definition. We give a proof of this statement in the Period Theorem 11.2.1.

9.1 First definition

We start with the simplest definition. In this section, let $k \subset \mathbb{C}$ be a subfield.

For this definition the following data is needed:

- X a smooth algebraic variety of dimension d, defined over k,
- D a divisor on X with normal crossings, also defined over k,
- $\omega \in \Gamma(X, \Omega^d_{X/k})$ an algebraic differential form of top degree,
- Γ a rational d-dimensional C^∞-chain on X^{an} with $\partial\Gamma$ on D^{an}, i.e.,

$$\Gamma = \sum_{i=1}^{n} \alpha_i \gamma_i$$

  with $\alpha_i \in \mathbb{Q}$, $\gamma_i : \Delta_d \to X^{\mathrm{an}}$ a C^∞-map for all i and $\partial\Gamma$ a chain on D^{an} as in Definition 2.2.2.

As before, we denote by X^{an} the analytic space attached to $X(\mathbb{C})$.

Definition 9.1.1. Let $k \subset \mathbb{C}$ be a subfield.

1. Let (X, D, ω, Γ) as above. We will call the complex number
$$\int_\Gamma \omega = \sum_{i=1}^n \alpha_i \int_{\Delta_d} f_i^* \omega \ .$$
the *period (number)* of the quadruple (X, D, ω, Γ).

2. The *algebra of effective periods* $\mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}} = \mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}(k)$ *over* k is the set of all period numbers for all (X, D, ω, Γ) defined over k.

3. The *period algebra* $\mathbb{P}_{\mathrm{nc}} = \mathbb{P}_{\mathrm{nc}}(k)$ *over* k is the set of numbers of the form $(2\pi i)^n\alpha$ with $n \in \mathbb{Z}$ and $\alpha \in \mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}$.

Remark 9.1.2. 1. The subscript nc refers to the normal crossing divisor D in the above definition.

2. We will show a bit later (see Proposition 9.1.7) that $\mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}(k)$ is indeed an algebra.

3. Moreover, we will see in the next example that $2\pi i \in \mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}$. This means that $\mathbb{P}_{\mathrm{nc}}$ is nothing but the localization
$$\mathbb{P}_{\mathrm{nc}} = \mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}\left[\frac{1}{2\pi i}\right] \ .$$

4. This definition was motivated by Kontsevich's discussion of formal effective periods [K1, def. 20, p. 62]. For an extensive discussion of formal periods and their precise relation to periods see Chapter 12.

Example 9.1.3. Let $X = \mathbb{A}^1_{\mathbb{Q}}$ be the affine line, $\omega = \mathrm{d}t \in \Omega^1$. Let $D = V(t^3 - 2t)$. Let $\gamma : [0, 1] \to \mathbb{A}^1_{\mathbb{Q}}(\mathbb{C}) = \mathbb{C}$ be the line from 0 to $\sqrt{2}$. This is a singular chain with boundary in $D(\mathbb{C}) = \{0, \pm\sqrt{2}\}$. Hence it defines a class in $H_1^{\mathrm{sing}}(\mathbb{A}^1(\mathbb{C})^{\mathrm{an}}, D^{\mathrm{an}}, \mathbb{Q})$. We obtain the period
$$\int_\gamma \omega = \int_0^{\sqrt{2}} \mathrm{d}t = \sqrt{2} \ .$$
The same method works for all algebraic numbers.

Example 9.1.4. Let $X = \mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$, $D = \emptyset$ and $\omega = \frac{1}{t}\mathrm{d}t$. We choose $\gamma : S^1 \to \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$ the unit circle. It defines a class in $H_1^{\mathrm{sing}}(\mathbb{C}^*, \mathbb{Q})$. We obtain the period
$$\int_{S^1} t^{-1}\mathrm{d}t = 2\pi i \ .$$
In particular, $\pi \in \mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}(k)$ for all k.

Example 9.1.5. Let $X = \mathbb{G}_m$, $D = V((t-2)(t-1))$, $\omega = t^{-1}\mathrm{d}t$, and γ the line from 1 to 2. We obtain the period

$$\int_1^2 t^{-1}\mathrm{d}t = \log(2)\ .$$

For more advanced examples, see Part IV.

Lemma 9.1.6. *Let (X, D, ω, Γ) as before. The period number $\int_\Gamma \omega$ depends only on the cohomology classes of ω in relative de Rham cohomology and of Γ in relative singular homology.*

Proof. The restriction of ω to the analytification D_j^{an} of some irreducible component D^j of D is a holomorphic d-form on a complex manifold of dimension $d-1$, hence zero. Therefore the integral $\int_\triangle \omega$ evaluates to zero for smooth singular simplices $\triangle$ that are supported on D. Now if Γ', Γ'' are two representatives of the same relative homology class, we have

$$\Gamma'_d - \Gamma''_d \sim \partial(\Gamma_{d+1})$$

modulo simplices living on some D_I^{an} for a smooth singular chain Γ of dimension $d+1$

$$\Gamma \in \mathrm{C}^\infty_{d+1}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q}).$$

Using Stokes' theorem, we get

$$\int_{\Gamma'_d} \omega - \int_{\Gamma''_d} \omega = \int_{\partial(\Gamma_{d+1})} \omega = \int_{\Gamma_{d+1}} d\omega = 0,$$

since ω is closed. □

In the course of the chapter, we are also going to show the converse: every pair of relative cohomology classes gives rise to a period number.

Proposition 9.1.7. *The sets $\mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}(k)$ and $\mathbb{P}_{\mathrm{nc}}(k)$ are k-algebras. Moreover, $\mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}(K) = \mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}(k)$ if K/k is algebraic.*

Proof. Let (X, D, ω, Γ) and $(X', D', \omega', \Gamma')$ be two quadruples as in the definition of normal crossing periods.

By multiplying ω by an element of k, we obtain k-multiples of periods.

The product of the two periods is realized by the quadruple $(X \times X', D \times X' \cup X \times D', \omega \otimes \omega', \Gamma \times \Gamma')$.

Note that the quadruple $(\mathbb{A}^1, \{0, 1\}, \underset{\cdot}{\mathrm{t}}, [0, 1])$ has period 1. By multiplying with this factor, we do not change the period number of a quadruple, but we change its dimension. Hence we can assume that X and X' have the same dimension. The sum of their periods is then realized on the disjoint union $(X \cup X', D \cup D', \omega + \omega', \Gamma + \Gamma')$.

If K/k is finite algebraic, then we obviously have $\mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}(k) \subset \mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}(K)$. For the converse, consider a quadruple (X, D, ω, Γ) over K. We may also can view X as k-variety and write X_k for distinction. By Lemma 3.1.13 or more precisely its proof, ω can also be viewed as a differential form on X_k/k. The complex points $Y_k(\mathbb{C})$ consist of $[K : k]$ copies of the complex points $Y(\mathbb{C})$. Let Γ_k be the cycle Γ on one of them. Then the period of (X, D, ω, Γ) is the same as the period of $(X_k, D_k, \omega, \Gamma_k)$. This gives the converse inclusion.

If K/k is infinite, but algebraic, we obviously have $\mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}(K) = \bigcup_L \mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}(L)$ with L running through all fields $K \supset L \supset k$ finite over k. Hence, equality also holds in the general case. □

9.2 Periods for the category $(k, \mathbb{Q})-\mathrm{Vect}$

For a clean development of the theory of period numbers, it is of advantage to formalize the data. Recall from Section 5.1 the category $(k, \mathbb{Q})-\mathrm{Vect}$. Its objects are a pair of k-vector space V_k and $\mathbb{Q}$-vector space $V_{\mathbb{Q}}$ linked by an isomorphism $\phi_{\mathbb{C}} : V_k \otimes_k \mathbb{C} \to V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$. This is precisely what we need in order to define periods abstractly.

Definition 9.2.1. 1. Let $V = (V_k, V_{\mathbb{Q}}, \phi_{\mathbb{C}})$ be an object of $(k, \mathbb{Q})-\mathrm{Vect}$. The *period matrix* of V is the matrix of $\phi_{\mathbb{C}}$ in a choice of bases $v_1, \dots, v_n$ of V_k and $w_1, \dots, w_n$ of $V_{\mathbb{Q}}$, respectively. A complex number is a *period* of V if it is an entry of a period matrix of V for some choice of bases. The set of periods of V together with the number 0 is denoted $\mathbb{P}(V)$. We denote by $\mathbb{P}\langle V\rangle$ the k-subvector space of $\mathbb{C}$ generated by the entries of the period matrix.

2. Let $\mathcal{C} \subset (k, \mathbb{Q})-\mathrm{Vect}$ be a subcategory. We denote by $\mathbb{P}(\mathcal{C})$ the set of periods for all objects in $\mathcal{C}$.

Remark 9.2.2. 1. The object $V = (V_k, V_{\mathbb{Q}}, \phi_{\mathbb{C}})$ gives rise to a bilinear map

$$V_k \times V_{\mathbb{Q}}^{\vee} \to \mathbb{C} , \quad (v, \lambda) \mapsto \lambda(\phi_{\mathbb{C}}^{-1}(v)) ,$$

where we have extended $\lambda : V_{\mathbb{Q}} \to \mathbb{Q}$ $\mathbb{C}$-linearly to $V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} \to \mathbb{C}$. The periods of V are the numbers in its image. Note that this image is a set, not a vector space in general. The period matrix depends on the choice of bases, but the vector space $\mathbb{P}\langle V\rangle$ does not.

2. The definition of $\mathbb{P}(\mathcal{C})$ does not depend on the morphisms. If the category has only one object, the second definition specializes to the first.

Lemma 9.2.3. *Let $\mathcal{C} \subset (k, \mathbb{Q})-\mathrm{Vect}$ be a subcategory.*

1. *$\mathbb{P}(\mathcal{C})$ is closed under multiplication by k.*
2. *If $\mathcal{C}$ is additive, then $\mathbb{P}(\mathcal{C})$ is a k-vector space.*

3. *If $\mathcal{C}$ is a tensor subcategory, then $\mathbb{P}(\mathcal{C})$ is a k-algbra.*

Proof. Multiplying a basis element w_i by an element α in k multiplies the periods by α. Hence the set is closed under multiplication by elements of k^*.

Let p be a period of V and p' a period of V'. Then $p+p'$ is a period of $V\oplus V'$. If $\mathcal{C}$ is additive, then $V,V'\in\mathcal{C}$ implies $V\oplus V'\in\mathcal{C}$. Moreover, pp' is a period of $V\otimes V'$. If $\mathcal{C}$ is a tensor subcategory of $(k,\mathbb{Q})-$Vect, then $V\otimes V'$ is also in $\mathcal{C}$. □

Proposition 9.2.4. *Let $\mathcal{C}\subset(k,\mathbb{Q})-\mathrm{Vect}$ be a subcategory.*

1. *Let $\langle\mathcal{C}\rangle$ be the smallest full abelian subcategory of $(k,\mathbb{Q})-$Vect closed under subquotients and containing $\mathcal{C}$. Then $\mathbb{P}(\langle\mathcal{C}\rangle)$ is the abelian subgroup of $\mathbb{C}$ generated by $\mathbb{P}(\mathcal{C})$.*

2. *Let $\langle\mathcal{C}\rangle^{\otimes}$ be the smallest full abelian subcategory of $(k,\mathbb{Q})-$Vect closed under subquotients and tensor products and containing $\mathcal{C}$. Then $\mathbb{P}(\langle\mathcal{C}\rangle^{\otimes})$ is the (possibly non-unital) subring of $\mathbb{C}$ generated by $\mathbb{P}(\mathcal{C})$.*

Proof. The period algebra $\mathbb{P}(\mathcal{C})$ only depends on objects. Hence we can replace $\mathcal{C}$ by the full subscategory with the same objects without changing the period algebra.

Moreover, if $V\in\mathcal{C}$ and $V'\subset V$ in $(k,\mathbb{Q})-$Vect, then we can extend any basis for V' to a basis to V. In this form, the period matrix for V is block triangular with one of the blocks the period matrix of V'. This implies

$$\mathbb{P}(V')\subset\mathbb{P}(V)\ .$$

Hence, $\mathbb{P}(\mathcal{C})$ does not change, if we close it up under subobjects in $(k,\mathbb{Q})-$Vect. The same argument also implies that $\mathbb{P}(\mathcal{C})$ does not change if we close it up under quotients in $(k,\mathbb{Q})-$Vect.

After these reductions, the only thing missing to make $\mathcal{C}$ additive is closing it up under direct sums in $(k,\mathbb{Q})-$Vect. If V and V' are objects of $\mathcal{C}$, then the periods of $V\oplus V'$ are sums of periods of V and periods of V' (this is most easily seen in the pairing point of view in Remark 9.2.2). Hence closing the category up under direct sums amounts to passing from $\mathbb{P}(\mathcal{C})$ to the abelian group generated by it. It is automatically a k-vector space.

If V and V' are objects of $\mathcal{C}$, then the periods of $V\otimes V'$ are sums of products of periods of V and periods of V' (this is again most easily seen in the pairing point of view in Remark 9.2.2). Hence closing $\mathcal{C}$ up under tensor products (and their subquotients) amounts to passing to the ring generated by $\mathbb{P}(\mathcal{C})$. □

So far, we fixed the ground field k. We now want to study the behaviour under change of fields.

Definition 9.2.5. Let K/k be a finite extension of subfields of $\mathbb{C}$. Let

$$\otimes_k K : (k, \mathbb{Q})\mathrm{-Vect} \to (K, \mathbb{Q})\mathrm{-Vect}\ , (V_k, V_\mathbb{Q}, \phi_\mathbb{C}) \mapsto (V_k \otimes_k K, V_\mathbb{Q}, \phi_\mathbb{C})$$

be the *extension of scalars*.

Lemma 9.2.6. *Let K/k be a finite extension of subfields of $\mathbb{C}$. Let $V \in (k, \mathbb{Q})\mathrm{-Vect}$. Then*

$$\mathbb{P}\langle V \otimes_k K\rangle = \mathbb{P}\langle V\rangle \otimes_k K\ .$$

Proof. The period matrix for V agrees with the period matrix for $V \otimes_k K$. On the left hand side, we pass to the K-vector space generated by its entries. On the right hand side, we first pass to the k-vector space generated by its entries, and then extend scalars. □

Conversely, there is a *restriction of scalars* where we view a K-vector space V_K as a k-vector space.

Lemma 9.2.7. *Let K/k be a finite extension of subfields of $\mathbb{C}$. Then the functor $\otimes_k K$ has a right adjoint*

$$R_{K/k} : (K, \mathbb{Q})\mathrm{-Vect} \to (k, \mathbb{Q})\mathrm{-Vect}$$

For $W \in (K, \mathbb{Q})\mathrm{-Vect}$ we have

$$\mathbb{P}\langle W\rangle = \mathbb{P}\langle R_{K/k} W\rangle\ .$$

Proof. Choose a k-basis $e_1, \dots, e_n$ of K. We put

$$R_{K/k} : (K, \mathbb{Q})\mathrm{-Vect} \to (k, \mathbb{Q})\mathrm{-Vect}\ ,\ (W_K, W_\mathbb{Q}, \phi_\mathbb{C}) \mapsto (W_K, W_\mathbb{Q}^{[K:k]}, \psi_\mathbb{C})\ ,$$

where

$$\psi_\mathbb{C} : W_K \otimes_k \mathbb{C} = W_K \otimes_k K \otimes_K \mathbb{C} \cong (W_K \otimes_K \mathbb{C})^{[K:k]} \to (W_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{C})^{[K:k]}$$

maps elements of the form $w \otimes e_i$ to $\phi_\mathbb{C}(w \otimes e_i)$ in the i-component.

It is easy to check the universal property. We describe the unit and the counit. The natural map

$$V \to R_{K/k}(V \otimes_k K)$$

is given on the component V_k by the natural inclusion $V_k \to V_k \otimes K$. In order to describe it on the $\mathbb{Q}$-component, decompose $1 = \sum_{i=1}^n a_i e_i$ in K and put

$$V_\mathbb{Q} \to V_\mathbb{Q}^n \qquad v \mapsto (a_i v)_{i=1}^n\ .$$

The natural map

$$(R_{K/k} W) \otimes_k K \to W$$

is given on the K-component as the multiplication map

$$W_K \otimes_k K \to W_K$$

and on the $\mathbb{Q}$-component

$$W_{\mathbb{Q}}^n \to W_{\mathbb{Q}}$$

by summation.

This shows existence of the right adjoint. In particular, $R_{K/k}W$ is functorial and independent of the choice of basis.

In order to compute periods, we have to choose bases. Fix a $\mathbb{Q}$-basis $x_1, \ldots, x_n$ of $W_{\mathbb{Q}}$. This also defines a $\mathbb{Q}$-basis for $W_{\mathbb{Q}}^n$ in the obvious way. Fix a K-basis $y_1, \ldots, y_n$ of W_K. Multiplying by $e_1, \ldots, e_n$, we obtain a k-basis of W_K. The entries of the period matrix of W are the coefficients of $\phi_{\mathbb{C}}(y_j)$ in the basis x_l The entries of the period matrix of $R_{K/k}W$ are the coefficients of $\phi_{\mathbb{C}}(e_i y_j) =$ in the basis x_l. Hence, the K-linear span of the former agrees with the k-linear span of the latter. □

Recall from Example 5.1.4 the object $L(\alpha) \in (k, \mathbb{Q})-\mathrm{Vect}$ for a complex number $\alpha \in \mathbb{C}^*$. It is given by the data $(k, \mathbb{Q}, \alpha)$. It is invertible for the tensor structure.

Definition 9.2.8. Let $L(\alpha) \in (k, \mathbb{Q})-\mathrm{Vect}$ be invertible. We call a pairing in $(k, \mathbb{Q})-\mathrm{Vect}$

$$V \times W \to L(\alpha)$$

perfect, if it is non-degenerate in the k- and $\mathbb{Q}$-components. Equivalently, the pairing induces an isomorphism

$$V \cong W^\vee \otimes L(\alpha)$$

where $\cdot^\vee$ denotes the dual in $(k, \mathbb{Q})-\mathrm{Vect}$.

Lemma 9.2.9. *Assume that*

$$V \times W \to L(\alpha)$$

is a perfect pairing. Then

$$\mathbb{P}\langle V, W, V^\vee, W^\vee\rangle^{\oplus,\otimes} \subset \mathbb{P}\langle V, W\rangle^{\oplus,\otimes}[\alpha^{-1}] \ .$$

Proof. The left hand side is the ring generated by $\mathbb{P}(V)$, $\mathbb{P}(W)$, $\mathbb{P}(V^\vee)$ and $\mathbb{P}(W^\vee)$. Hence we need to show that $\mathbb{P}(V^\vee)$ and $\mathbb{P}(W^\vee)$ are contained in the right hand side. This is true because $W^\vee \cong V \otimes L(\alpha^{-1})$ and $\mathbb{P}(V \otimes L(\alpha^{-1}) = \alpha^{-1}\mathbb{P}(V)$ □

9.3 Periods of algebraic varieties

9.3.1 Definition

Recall from Definition 8.1.1 the directed graph of effective pairs $\mathrm{Pairs}^{\mathrm{eff}}$. Its vertices are triples (X, D, j) with X a variety, D a closed subvariety and j an integer. The edges are not of importance for the consideration of periods.

Definition 9.3.1. Let (X, D, j) be a vertex of the diagram $\mathrm{Pairs}^{\mathrm{eff}}$.

1. The *set of periods* $\mathbb{P}(X, D, j)$ is the image of the period paring (see Definition and 5.3.1 and 5.5.4

$$\mathrm{per} : H^j_{\mathrm{dR}}(X, D) \times H^{\mathrm{sing}}_j(X^{\mathrm{an}}, D^{\mathrm{an}}) \to \mathbb{C}\ .$$

2. In the same situation, the *space of periods* $\mathbb{P}\langle X, D, j\rangle$ is the $\mathbb{Q}$-vector space generated by $\mathbb{P}(X, D, j)$.

3. Let S be a set of vertices in $\mathrm{Pairs}^{\mathrm{eff}}(k)$. We define the *set of periods* $\mathbb{P}(S)$ as the union of the $\mathbb{P}(X, D, j)$ for (X, D, j) in S and the k-*space of periods* $\mathbb{P}\langle S\rangle$ as the sum of the $\mathbb{P}\langle X, D\rangle$ for $(X, D, j) \in S$.

4. The *effective period algebra* $\mathbb{P}^{\mathrm{eff}}(k)$ of k is defined as $\mathbb{P}(S)$ for S the set of (isomorphism classes of) all vertices (X, D, j).

5. The *period algebra* $\mathbb{P}(k)$ of k is defined as the set of complex numbers of the form $(2\pi i)^n\alpha$ with $n \in \mathbb{Z}$ and $\alpha \in \mathbb{P}^{\mathrm{eff}}(k)$.

Remark 9.3.2. Note that $\mathbb{P}(X, D, j)$ is closed under multiplication by elements in k but not under addition. However, $\mathbb{P}^{\mathrm{eff}}(k)$ is indeed an algebra by Corollary 9.3.5. This means that $\mathbb{P}(k)$ is nothing but the localization

$$\mathbb{P}(k) = \mathbb{P}^{\mathrm{eff}}(k)\left[\frac{1}{2\pi i}\right]\ .$$

Passing to this localization is very natural from the point of view of motives: it corresponds to passing from periods of effective motives to periods of all mixed motives. For more details, see Chapter 10.

Example 9.3.3. Let $X = \mathbb{P}^n_k$. Then $(\mathbb{P}^n_k, \emptyset, 2j)$ has period set $(2\pi i)^j k^*$. The easiest way to see this is by computing the motive of $\mathbb{P}^n_k$, e.g., in Lemma 8.3.7. It is given by $\mathbf{1}(-j)$. By compatibility with tensor product, it suffices to consider the case $j = 1$ where the same motive can be defined from the pair $(\mathbb{G}_m, \emptyset, 1)$. It has the period $2\pi i$ by Example 9.1.4. The factor k^* appears because we may multiply the basis vector in de Rham cohomology by a factor in k^*.

Recall from Theorem 5.3.3 and Theorem 5.5.6 that we have an explicit description of the period isomorphism by integration.

Lemma 9.3.4. *There are natural inclusions* $\mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}(k) \subset \mathbb{P}^{\mathrm{eff}}(k)$ *and* $\mathbb{P}_{\mathrm{nc}}(k) \subset \mathbb{P}(k)$.

Proof. By definition, it suffices to consider the effective case. By Lemma 9.1.6, the period in $\mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}(k)$ only depends on the cohomology class. By Theorem 3.3.19, the period in $\mathbb{P}^{\mathrm{eff}}(k)$ is defined by integration, i.e., by the formula in the definition of $\mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}(k)$. □

The converse inclusion is deeper, see Theorem 9.4.2.

9.3.2 First properties

Recall from Definition 5.4.2 that there is a functor

$$\mathrm{H} : \mathrm{Pairs}^{\mathrm{eff}} \to (k, \mathbb{Q})-\mathrm{Vect}$$

where the category $(k, \mathbb{Q})-\mathrm{Vect}$ was introduced in Section 5.1. By construction, we have

$$\begin{aligned} \mathbb{P}(X, D, j) &= \mathbb{P}(\mathrm{H}(X, D, j)), \\ \mathbb{P}\langle X, D, j\rangle &= \mathbb{P}\langle \mathrm{H}(X, D, j)\rangle, \\ \mathbb{P}^{\mathrm{eff}}(k) &= \mathbb{P}(\mathrm{H}(\mathrm{Pairs}^{\mathrm{eff}})) \ . \end{aligned}$$

This means that we can apply the abstract considerations of Section 5.1 to our periods algebras.

Corollary 9.3.5. *1. $\mathbb{P}^{\mathrm{eff}}(k)$ and $\mathbb{P}(k)$ are k-subalgebras of $\mathbb{C}$.*

2. If K/k is an algebraic extension of subfields of K, then $\mathbb{P}^{\mathrm{eff}}(K) = \mathbb{P}^{\mathrm{eff}}(k)$ and $\mathbb{P}(K) = \mathbb{P}(k)$.

3. If k is countable, then so is $\mathbb{P}(k)$.

Proof. It suffices to consider the effective case. The image of H is closed under direct sums because direct sums are realized by disjoint unions of effective pairs. As in the proof of Proposition 9.1.7, we can use $(\mathbb{A}^1, \{0, 1\}, 1)$ in order to shift the cohomological degree without changing the periods.

The image of H is also closed under tensor product. Hence its period set is a k-algebra by Lemma 9.2.3.

Let K/k be finite. For (X, D, i) over k, we have the base change (X_K, D_K, i) over K. By compatibility of the de Rham realization with base change (see Lemma 3.2.14), we have

$$\mathrm{H}(X, D, i) \otimes K = \mathrm{H}(X_K, D_K, j) \ .$$

By Lemma 9.2.6, this implies that the periods of (X, D, j) are contained in the periods of the base change. Hence $\mathbb{P}^{\mathrm{eff}}(k) \subset \mathbb{P}^{\mathrm{eff}}(K)$.

Conversely, if (Y, E, m) is defined over K, we may view it as defined over k via the map $\mathrm{Spec}K \to \mathrm{Spec}k$. We write (Y_k, E_k, m) in order to avoid confusion. Note that $Y_k(\mathbb{C})$ consists of $[K : k]$ many copies of $Y(\mathbb{C})$. Moreover, by Lemma 3.2.15, de Rham cohomology of Y/K agrees with de Rham cohomology of Y_k/k. Hence

$$\mathrm{H}(Y_k, E_k, m) = R_{K/k}\mathrm{H}(Y, E, m)$$

and their period sets agree by Lemma 9.2.7. Hence, we also have $\mathbb{P}^{\mathrm{eff}}(K) \subset \mathbb{P}^{\mathrm{eff}}(k)$.

Let k be countable. For each triple (X,D,j), the cohomologies $H^j_{\mathrm{dR}}(X)$ and $H_j^{\mathrm{sing}}(X^{\mathrm{an}},D^{\mathrm{an}},\mathbb{Q})$ are countable. Hence, the image of period pairing is also countable. There are only countably many isomorphism classes of pairs (X,D,j), hence the set $\mathbb{P}^{\mathrm{eff}}(k)$ is countable. □

9.4 The comparison theorem

We introduce two more variants of period algebras. Recall from Corollary 5.5.2 the functor

$$R\Gamma : K^-(\mathbb{Z}\mathrm{Sm}) \to D^+_{(k,\mathbb{Q})}$$

and

$$\mathrm{H}^i : K^-(\mathbb{Z}\mathrm{Sm}) \to (k,\mathbb{Q})\text{−Vect} .$$

Definition 9.4.1.
- Let $\mathcal{C}(\mathrm{Sm})$ be the full abelian subcategory of $(k,\mathbb{Q})$−Vect closed under subquotients generated by $\mathrm{H}^i(X_\bullet)$ for $X_\bullet \in K^-(\mathbb{Z}\mathrm{Sm})$. Let $\mathbb{P}_{\mathrm{Sm}}(k) = \mathbb{P}(\mathcal{C}(\mathrm{Sm}))$ be the *algebra of periods of complexes of smooth varieties.*
- Let $\mathcal{C}(\mathrm{SmAff})$ be the full abelian subcategory of $(k,\mathbb{Q})$−Vect closed under subquotients and generated by $\mathrm{H}^i(X_\bullet)$ for $X_\bullet \in K^-(\mathbb{Z}\mathrm{SmAff})$ with SmAff the category of smooth affine varieties over k. Let $\mathbb{P}_{\mathrm{SmAff}}(k) = \mathbb{P}(\mathcal{C}(\mathrm{SmAff}))$ be the *algebra of periods of complexes of smooth affine varieties.*

Theorem 9.4.2. *Let $k \subset \mathbb{C}$ be a subfield. Then all definitions of period algebras given so far agree:*

$$\mathbb{P}^{\mathrm{eff}}(k) = \mathbb{P}_{\mathrm{Sm}}(k) = \mathbb{P}_{\mathrm{SmAff}}(k)$$

and

$$\mathbb{P}(k) = \mathbb{P}_{\mathrm{nc}}(k) .$$

Remark 9.4.3. This is a simple corollary of Theorem 8.2.20 and Corollary 8.2.21, once we will have discussed the formal period algebra, see Corollary 12.1.9. However, the argument does not use the full force of Nori's machine, hence we give the argument directly. Note that the key input is the same as the key input into Nori's construction: the existence of good filtrations.

Remark 9.4.4. We do not know whether $\mathbb{P}^{\mathrm{eff}}(k) = \mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}(k)$. The concrete definition of $\mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}(k)$ only admits de Rham classes which are represented by a global differential form. This is true for all classes in the affine case, but not in general.

Proof. We are going to prove the identities on periods by showing that the subcategories of $(k,\mathbb{Q})$−Vect appearing in their definitions are the same.

Let $\mathcal{C}(\mathrm{Pairs}^{\mathrm{eff}})$ be the full abelian subcategory closed under subquotients and generated by $\mathrm{H}(X,D,j)$ for $(X,D) \in \mathrm{Pairs}^{\mathrm{eff}}$. Furthermore, let $\mathcal{C}(\mathrm{nc})$ be the full abelian subcategory closed under subquotients and generated by $\mathrm{H}^d(X,D)$ with X smooth, affine of dimension d and D a divisor with normal crossings.

By definition
$$\mathcal{C}(\mathrm{nc}) \subset \mathcal{C}(\mathrm{Pairs}^{\mathrm{eff}}) \ .$$
By the construction in Definition 3.3.6, we may compute any $\mathrm{H}(X,D,j)$ as $\mathrm{H}^j(C_\bullet)$ with $C_\bullet$ in $C^-(\mathbb{Z}\mathrm{Sm})$. Actually, the degree cohomology only depends on a bounded piece of $C_\bullet$. Hence

$$\mathcal{C}(\mathrm{Pairs}^{\mathrm{eff}}) \subset \mathcal{C}(\mathrm{Sm}) \ .$$

We next show that
$$\mathcal{C}(\mathrm{Sm}) \subset \mathcal{C}(\mathrm{SmAff}) \ .$$
Let $X_\bullet \in C^-(\mathbb{Z}\mathrm{Sm})$. By Lemma 8.2.9, there is a rigidified affine cover $\tilde{U}_{X_\bullet}$ of $X_\bullet$. Let $C_\bullet = C_\bullet(\tilde{U}_{X_\bullet})$ be the total complex of the associated complex of Čhech complexes (see Definition 8.2.10). By construction, $C_\bullet \in C^-(\mathbb{Z}\mathrm{SmAff})$. By the Mayer-Vietoris property, we have

$$\mathrm{H}(X_\bullet) = \mathrm{H}(C_\bullet).$$

We claim that $\mathcal{C}(\mathrm{SmAff}) \subset \mathcal{C}(\mathrm{Pairs}^{\mathrm{eff}})$. It suffices to consider bounded complexes because the cohomology of a bounded above complex of varieties only depends on a bounded quotient. Let X be smooth affine. Recall (see Proposition 8.2.2) that a very good filtration on X is a sequence of subvarieties

$$F_0X \subset F_1X \subset \dots F_nX = X$$

such that $F_jX \smallsetminus F_{j-1}X$ is smooth, with F_jX of pure dimension j, or $F_jX = F_{j-1}X$ of dimension less that j and the cohomology of $(F_jX, F_{j-1}X)$ being concentrated in degree j. The boundary maps for the triples $F_{j-2}X \subset F_{j-1}X \subset F_jX$ define a complex $\tilde{R}(F_\cdot X)$ in $\mathcal{C}(\mathrm{Pairs}^{\mathrm{eff}})$

$$\dots \to \mathrm{H}^{j-1}(F_{j-1}X, F_{j-2}X) \to \mathrm{H}^j(F_jX, F_{j-1}X) \to \mathrm{H}^{j+1}(F_{j+1}X, F_jX) \to \dots$$

whose cohomology agrees with $\mathrm{H}^\bullet(X)$.

Let $X_\bullet \in C^b(\mathbb{Z}\mathrm{SmAff})$. By Lemma 8.2.14, we can choose good filtrations on all X_n in a compatible way. The double complex $\tilde{R}(F_\cdot X)$ has the same cohomology as $X_\bullet$. By construction, it is a complex in $\mathcal{C}(\mathrm{Pairs}^{\mathrm{eff}})$, hence the cohomology is in $\mathcal{C}(\mathrm{Pairs}^{\mathrm{eff}})$.

Hence, we have now established that

$$\mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}(k) \subset \mathbb{P}^{\mathrm{eff}}(k) = \mathbb{P}_{\mathrm{Sm}}(k) = \mathbb{P}_{\mathrm{SmAff}}(k) \ .$$

We refine the argument in order to show that $\mathbb{P}_{\mathrm{SmAff}}(k) \subset \mathbb{P}_{\mathrm{nc}}(k)$. By the above computation, this will follow if periods of very good pairs are contained in $\mathbb{P}_{\mathrm{nc}}(k)$. We recall the construction of very good pairs (X,Y,n) by the direct proof of Nori's Basic Lemma I in Section 2.5.1. We let $\tilde{X}$, D_0 and D_∞ be as in Lemma 2.5.8. In particular, there is a proper surjective map $\tilde{X} \setminus D_\infty \to X$ and $D_0 \setminus D_0 \cap D_\infty = \pi^{-1}Y$. Hence the periods of (X, Y, n) are the same as the periods of $(\tilde{X} \setminus D_0, D_\infty \setminus D_0 \cap D_\infty, n)$. The latter cohomology is Poincaré dual to the cohomology of the pair $(X', Y', n) = (\tilde{X} \setminus D_\infty, D_0 \setminus D_0 \cap D_\infty, n)$ by Theorem 2.4.5. In particular, all three are very good pairs with cohomology concentrated in degree n and free. Indeed, there is a natural pairing in $\mathcal{C}$

$$\mathrm{H}^d(X, Y) \times \mathrm{H}^d(X', Y') \to L((2\pi i)^d).$$

This is shown by the same arguments as in the proof of Lemma 8.3.8 but with the functor H instead of H^i_{Nori}. By Lemma 9.2.9, the periods of (X, Y) agree up to multiplication by $(2\pi i)^d$ with the periods of (X', Y'). We are now in the situation where X' is smooth affine of dimension n and Y' is a divisor with normal crossings. By Proposition 3.3.19, every de Rham cohomology class in degree n is represented by a global differential form on X. Hence all cohomological periods of (X', Y', n) are normal crossing periods in the sense of Definition 9.1.1. □

Chapter 10

Categories of mixed motives

There are different candidates for the category of mixed motives over a field k of characteristic zero. The category of Nori motives of Chapter 8 is one of them. We review two more.

10.1 Geometric motives

We recall the definition of geometrical motives first introduced by Voevodsky, see [VSF] Chapter 5.

As before let $k \subset \mathbb{C}$ be a field (most of the time suppressed in the notation).

Definition 10.1.1 ([VSF] Chap. 5, Sect. 2.1)**.** The category of *finite correspondences* SmCor_k has as objects smooth k-varieties and as morphisms from X to Y the vector space of $\mathbb{Q}$-linear combinations of integral correspondences $\Gamma \subset X \times Y$ which are finite over X and dominant over a component of X.

The composition of $\Gamma : X \to Y$ and $\Gamma' : Y \to Z$ is defined by push-forward of the intersection of $\Gamma \times Z$ and $X \times \Gamma'$ in $X \times Y \times Z$ to $X \times Z$. The identity morphism is given by the diagonal. There is a natural covariant functor

$$\mathrm{Sm}_k \to \mathrm{SmCor}_k$$

which maps a smooth variety to itself and a morphism to its graph.

The category SmCor_k is additive, hence we can consider its homotopy category $K^b(\mathrm{SmCor}_k)$. The latter is triangulated.

Definition 10.1.2 ([VSF] Ch. 5, Defn. 2.1.1)**.** The category of *effective geometrical motives* $DM^{\mathrm{eff}}_{\mathrm{gm}} = DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$ is the pseudo-abelian hull of the localization of $K^b(\mathrm{SmCor}_k)$ with respect to the thick subcategory generated by objects of the form

$$[X \times \mathbb{A}^1 \xrightarrow{\mathrm{pr}} X]$$

for all smooth varieties X and

$$[U \cap V \to U \amalg V \to X]$$

for all open covers $U \cup V = X$ for all smooth varieties X.

Remark 10.1.3. We think of $DM_{\mathrm{gm}}^{\mathrm{eff}}$ as the bounded derived category of the conjectural abelian category of effective mixed motives.

We denote by

$$M : \mathrm{SmCor}_k \to DM_{\mathrm{gm}}^{\mathrm{eff}}$$

the functor which views a variety as a complex concentrated in degree 0. By [VSF] Ch. 5 Section 2.2, it extends (non-trivially!) to a functor on the category of all k-varieties.

$DM_{\mathrm{gm}}^{\mathrm{eff}}$ is tensor triangulated such that

$$M(X) \otimes M(Y) = M(X \times Y)$$

for all smooth varieties X and Y. The unit of the tensor structure is given by

$$\mathbb{Q}(0) = M(\mathrm{Spec} k) \ .$$

The *Tate motive* $\mathbb{Q}(1)$ is defined by the equation

$$M(\mathbb{P}^1) = \mathbb{Q}(0) \oplus \mathbb{Q}(1)[2] \ .$$

We write $M(n) = M \otimes \mathbb{Q}(1)^{\otimes n}$ for $n \geq 0$. By [VSF], Chap. 5 Section 2.2, the functor

$$(n) : DM_{\mathrm{gm}}^{\mathrm{eff}} \to DM_{\mathrm{gm}}^{\mathrm{eff}}$$

is fully faithful.

Definition 10.1.4. The category of *geometrical motives* DM_{gm} is the stabilization of $DM_{\mathrm{gm}}^{\mathrm{eff}}$ with respect to $\mathbb{Q}(1)$. Objects are of the form $M(n)$ for $n \in \mathbb{Z}$ with

$$\mathrm{Hom}_{DM_{\mathrm{gm}}}(M(n), M'(n')) = \mathrm{Hom}_{DM_{\mathrm{gm}}^{\mathrm{eff}}}(M(n+N), M'(n'+N)) \qquad N \gg 0 \ .$$

Remark 10.1.5. We think of DM_{gm} as the bounded derived category of the conjectural abelian category of mixed motives.

The category DM_{gm} is rigid by [VSF], Chap. 5 Section 2.2, i.e., every object M has a strong dual $M^\vee$ such that

$$\begin{gathered}\mathrm{Hom}_{DM_{\mathrm{gm}}}(A \otimes B, C) = \mathrm{Hom}_{DM_{\mathrm{gm}}}(A, B^\vee \otimes C) \\ A^\vee \otimes B^\vee = (A \otimes B)^\vee \\ (A^\vee)^\vee = A\end{gathered}$$

for all objects A, B, C.

Remark 10.1.6. Rigidity is a deep result. It depends on a moving lemma for cycles and computations in Voevodsky's category of motivic complexes.

Example 10.1.7. If X is smooth and projective of pure dimension d, then

$$M(X)^\vee = M(X)(-d)[-2d] \ .$$

10.2 Absolute Hodge motives

The notion of absolute Hodge motives was introduced by Deligne ([DMOS] Chapter II in the pure case), and independently by Jannsen in ([Ja1]). We follow the presentation of Jannsen, also used in our own extension to the triangulated setting ([Hu1]). We give a rough overview over the construction and refer to the literature for full details.

We fix a subfield $k \subset \mathbb{C}$ and an algebraic closure $\bar{k}/k$. Let $G_k = \mathrm{Gal}(\bar{k}/k)$. Let S be the set of embeddings $\sigma : k \to \mathbb{C}$ and $\bar{S}$ the set of embeddings $\bar{\sigma} : \bar{k} \to \mathbb{C}$. Restriction of fields induces a map $\bar{S} \to S$.

Definition 10.2.1 ([Hu1] Defn. 11.1.1)**.** Let $\mathcal{MR} = \mathcal{MR}(k)$ be the additive category of *mixed realizations* with objects given by the following data:

- a bifiltered k-vector space A_{dR};
- for each prime l, a filtered $\mathbb{Q}_l$-vector space A_l with a continuous operation of G_k;
- for each prime l and each $\sigma \in S$, a filtered $\mathbb{Q}_l$-vector space $A_{\sigma,l}$;
- for each $\sigma \in S$, a filtered $\mathbb{Q}$-vector space A_σ;
- for each $\sigma \in S$, a filtered $\mathbb{C}$-vector space $A_{\sigma,\mathbb{C}}$;
- for each $\sigma \in S$, a filtered isomorphism

$$I_{\mathrm{dR},\sigma} ; A_{\mathrm{dR}} \otimes_\sigma \mathbb{C} \to A_{\sigma,\mathbb{C}} \ ;$$

- for each $\sigma \in S$, a filtered isomorphism

$$I_{\sigma,\mathbb{C}} : A_\sigma \otimes_{\mathbb{Q}} \mathbb{C} \to A_{\sigma,\mathbb{C}} \ ;$$

- for each $\sigma \in S$ and each prime l, a filtered isomorphism

$$I_{\bar{\sigma},l} : A_\sigma \otimes_{\mathbb{Q}} \mathbb{Q}_l \to A_{\sigma,l} \ ;$$

- for each prime l and each $\sigma \in S$, a filtered isomorphism

$$I_{l,\sigma} : A_l \otimes_{\mathbb{Q}} \mathbb{Q}_l \to A_{\sigma,l} \ .$$

These data are subject to the following conditions:

- For each σ, the tuple $(A_\sigma, A_{\sigma,\mathbb{C}}, I_{\sigma,\mathbb{C}})$ is a mixed Hodge structure;
- For each l, the filtration on A_l is the *filtration by weights*: its graded pieces $\mathrm{gr}_n^W A_l$ extends to a model of finite type over $\mathbb{Z}$ which is pointwise pure of weight n in the sense of Deligne, i.e., for each closed point with residue field κ, the operation of Frobenius has eigenvalues $N(\kappa)^{n/2}$.

Morphisms of mixed realizations are morphisms of this data compatible with all filtrations and comparison isomorphisms.

The above has already used the notion of a Hodge structure as introduced by Deligne.

Definition 10.2.2 (Deligne [D4]). A *mixed Hodge structure* consists of the following data:

- a finite dimensional filtered $\mathbb{Q}$-vector space $(V_\mathbb{Q}, W_*)$;
- a finite dimensional bifiltered $\mathbb{C}$-vector space $(V_\mathbb{C}, W_*, F^*)$;
- a filtered isomorphism $I_\mathbb{C} : (V_\mathbb{Q}, W_*) \otimes \mathbb{C} \to (V_\mathbb{C}, W_*)$

sucht that for all $n \in \mathbb{Z}$ the induced triple $(\mathrm{gr}_n^W V_\mathbb{Q}, \mathrm{gr}_n^W V_\mathbb{C}, \mathrm{gr}_n^W I)$ satisfies

$$\mathrm{gr}_n^W V_C = \bigoplus_{p+q=n} F^p \mathrm{gr}_n^W V_\mathbb{C} \oplus \overline{F^q \mathrm{gr}_n V_\mathbb{C}}$$

with complex conjugation taken with respect to the $\mathbb{R}$-structure defined by $\mathrm{gr}_n^W V_\mathbb{Q} \otimes \mathbb{R}$.

A Hodge structure is called *pure of weight* n if W_* is concentrated in degree n. It is called *pure* if it is direct sum of pure Hodge structures of different weights.

A morphism of Hodge structures are morphisms of this data compatible with filtration and comparison isomorphism.

By [D4] this is an abelian category. All morphisms of Hodge structures are automatically strictly compatible with filtrations. This implies immediately:

Proposition 10.2.3 ([Hu1] Lemma 11.1.2). *The category $\mathcal{MR}$ is abelian. Kernels and cokernels are computed componentwise.*

The notation is suggestive. If X is a smooth variety, then there is a natural mixed realization $H = H^*_{\mathcal{MR}}(X)$ with

- $H_{\mathrm{dR}} = H^*_{\mathrm{dR}}(X)$ algebraic de Rham cohomology as in Chapter 3 Section 3.1;

- $H_l = H^*(X_{\bar{k}}, \mathbb{Q}_l)$ the l-adic cohomology with its natural Galois operation;
- $H_\sigma = H^*(X \times_\sigma \mathrm{Spec}(\mathbb{C}), \mathbb{Q})$ singular cohomology;
- $H_{\sigma,\mathbb{C}} = H_\sigma \otimes \mathbb{C}$ and $H_{\sigma,l} = H_\sigma \otimes \mathbb{Q}_l$;
- $I_{\mathrm{dR},\sigma}$ is the period isomorphism of Definition 5.3.1 .
- $I_{l,\sigma}$ is induced by the comparison isomorphism between l-adic and singular cohomology over $\mathbb{C}$.

Remark 10.2.4. If we assume the Hodge or the Tate conjecture, then the functor $H^*_{\mathcal{MR}}$ is fully faithful on the category of Grothendieck motives (with homological or, under these assumptions equivalently, numerical equivalence). Hence it gives a linear algebra description of the conjectural abelian category of pure motives.

Jannsen ([Ja1] Theorem 6.11.1) extends the definition to singular varieties. A refined version of his construction is given in [Hu1]. We sum up its properties.

Definition 10.2.5 ([Hu2] Defn. 2.2.2)**.** Let C^+ be the category with objects given by a tuple of complexes in the additive categories in Definition 10.2.1 with filtered quasi-isomorphisms between them. The category of *mixed realization complexes* $C_{\mathcal{MR}}$ is the full subcategory of complexes with strict differentials and cohomology objects in $\mathcal{MR}$. Let $D_{\mathcal{MR}}$ be the localization of the homotopy category of $C_{\mathcal{MR}}$ (see [Hu1]) with respect to quasi-isomorphisms (see [Hu1] 4.17).

By construction, there are natural cohomology functors:

$$H^i : C_{\mathcal{MR}} \to \mathcal{MR}$$

factoring over $D_{\mathcal{MR}}$.

Remark 10.2.6. One should think of $D_{\mathcal{MR}}$ as the derived category of $\mathcal{MR}$, even though this is false in a literal sense.

The main construction of [Hu1] is a functor from varieties to mixed realizations.

Theorem 10.2.7 ([Hu1] Section 11.2, [Hu2] Thm 2.3.1)**.** *Let* Sm_k *be the category of smooth varieties over* k*. There is a natural additive functor*

$$\tilde{R}_{\mathcal{MR}} : \mathrm{Sm}_k \to C_{\mathcal{MR}} \ ,$$

such that

$$H^i_{\mathcal{MR}}(X) = H^i(\tilde{R}_{\mathcal{MR}}(X)) \ .$$

This allows to extend $\tilde{R}$ to the additive category $\mathbb{Q}[\mathrm{Sm}_k]$ and even to the category of complexes $C^-(\mathbb{Q}[\mathrm{Sm}_k])$.

Remark 10.2.8. There is a subtle technical point here. The category C^+ is additive. Taking the total complex of a complex in C^+ gives again an object of C^+. That the subcategory $C_{\mathcal{MR}}$ is respected is a non-trivial statement, see [Hu2] Lemma 2.2.5.

Following Deligne and Jannsen, we can now define

Definition 10.2.9. An object $M \in \mathcal{MR}$ is called an *effective absolute Hodge motive* if it is a subquotient of an object in the image of

$$H^* \circ \tilde{R} : C^b(\mathbb{Q}[\mathrm{Sm}_k]) \to \mathcal{MR} \ .$$

Let $\mathcal{MM}_{\mathrm{AH}}^{\mathrm{eff}} = \mathcal{MM}_{\mathrm{AH}}^{\mathrm{eff}}(k) \subset \mathcal{MR}$ be the category of all effective absolute Hodge motives over k. Let $\mathcal{MM}_{\mathrm{AH}} = \mathcal{MM}_{\mathrm{AH}}(k) \subset \mathcal{MR}$ be the full abelian tensor subcategory generated by $\mathcal{MM}^{\mathrm{eff}}$ and the dual of $\mathbb{Q}(-1) = H^2_{\mathcal{MR}}(\mathbb{P}^1)$. Objects in $\mathcal{MM}_{\mathrm{AH}}$ are called *absolute Hodge motives over k*.

Remark 10.2.10. The rationale behind this definition lies in Remark 10.2.4. Every mixed motive is supposed to be an iterated extension of pure motives. The latter are conjecturally fully described by their mixed realization. Hence, it remains to specify which extensions of pure motives are mixed motives.

Jannsen ([Ja1] Definition 4.1) does not use complexes of varieties but only single smooth varietes. It is not clear whether the two definitions agree, see also the discussion in [Hu1] Section 22.3. On the other hand, in [Hu1] Definition 22.13 the varieties were allowed to be singular. This is equivalent to the above by the construction in [Hu3] Lemma B.5.3 where every complex of varieties is replaced by complex of smooth varieties with the same cohomology.

Recall the abelian category $(k,\mathbb{Q})-\mathrm{Vect}$ from Definition 5.1.1.

Fix $\iota : k \to \mathbb{C}$. The projection

$$A \mapsto (A_{\mathrm{dR}}, A_\iota, I_{\iota,\mathbb{C}}^{-1} I_{\mathrm{dR},\iota})$$

defines a faithful functor

$$\mathcal{MR} \to (k,\mathbb{Q})-\mathrm{Vect} \ .$$

Recall the triangulated category $D^+_{(k,\mathbb{Q})}$ from Definition 5.2.1. The projection

$$K \mapsto (K_{\mathrm{dR}}, K_\iota, K_{\iota,\mathbb{C}}, I_{\mathrm{dR},\iota}, I_{\iota,\mathbb{C}})$$

defines a functor

$$C_{\mathcal{MR}} \to C^+_{(k,\mathbb{Q})}$$

which induces also a triangulated functor

$$\mathrm{forget} : D_{\mathcal{MR}} \to D^+_{(k,\mathbb{Q})} \ .$$

Lemma 10.2.11. *There is a natural transformation of functors*

$$K^-(\mathbb{Z}[\mathrm{Sm}_k]) \to D^+_{(k,\mathbb{Q})}$$

between forget $\circ R_{\mathcal{MR}}$ *and* $R\Gamma$.

Proof. This is true by construction of the dR- and σ-components of $R_{\mathcal{MR}}$ in [Hu1]. In fact, the definition of $R\Gamma$ is a simplified version of the construction given there. (They are *not* identical though because $\mathcal{MR}$ takes the Hodge and weight filtration into account.) □

10.3 Comparison functors

We now have three candidates for categories of mixed motives: the triangulated categories of geometric motives and the abelian categories of absolute Hodge motives and of Nori motives (see Chapter 8).

Theorem 10.3.1. *The functor $R_{\mathcal{MR}}$ factors via a chain of functors*

$$C^b(\mathbb{Q}[Sm_k]) \to DM_{\mathrm{gm}} \to D^b(\mathcal{MM}_{\mathrm{Nori}}) \to D^b(\mathcal{MM}_{\mathrm{AH}}) \subset D_{\mathcal{MR}} \ .$$

The proof will be given at the end of the section. The argument is a bit involved.

Theorem 10.3.2 ([Hu2], [Hu3])**.** *There is a tensor triangulated functor*

$$R_{\mathcal{MR}} : DM_{\mathrm{gm}} \to D_{\mathcal{MR}}$$

such that for smooth X

$$H^i R_{\mathcal{MR}}(X) = H^*_{\mathcal{MR}}(X) \ .$$

For all $M \in DM_{\mathrm{gm}}$, the objects $H^i R_{\mathcal{MR}}(M)$ are absolute Hodge motives.

Proof. This is the main result of [Hu2]. Note that there is a Corrigendum [Hu3]. The second assertion is [Hu2] Theorem 2.3.6. □

Proposition 10.3.3. *Let $k \subset \mathbb{C}$.*

1. *There is a faithful tensor functor*

$$f : \mathcal{MM}_{\mathrm{Nori}} \to \mathcal{MM}_{\mathrm{AH}}$$

   such that the functor $R_{\mathcal{MR}} : C^b(\mathbb{Q}[\mathrm{Sm}_k]) \to D_{\mathcal{MR}}$ factors via $D^b(\mathcal{MM}_{\mathrm{Nori}}) \to D^b(\mathcal{MM}_{\mathrm{AH}})$.

2. *Every object in $\mathcal{MM}_{\mathrm{AH}}$ is a subquotient of an object in the image of $\mathcal{MM}_{\mathrm{Nori}}$.*

Proof. We want to use the universal property of Nori motives. Let $\iota : k \subset \mathbb{C}$ be the fixed embedding. The assignment $A \mapsto A_\iota$ (see Definition 10.2.1) is a fibre functor on the neutral Tannakian category $\mathcal{MM}_{\mathrm{AH}}$. We denote it H^*_{sing} because it agrees with singular cohomology of $X \otimes_k \mathbb{C}$ on $A = H^*_{\mathcal{MR}}(X)$.

We need to verify that the diagram $\mathrm{Pairs}^{\mathrm{eff}}$ of effective pairs from Chap. 8 can be represented in $\mathcal{MM}_{\mathrm{AH}}$ in a manner compatible with singular cohomology. More explicitly, let X be a variety and $Y \subset X$ a subvariety. Then $[Y \to X]$ is an object of DM_{gm}. Hence for every $i \geq 0$ there is

$$H^i_{\mathcal{MR}}(X, Y) = H^i R_{\mathcal{MR}}(X, Y) \in \mathcal{MM}_{\mathrm{AH}} \ .$$

By construction, we have

$$H^*_{\mathrm{sing}} H^i_{\mathcal{MR}}(X, Y) = H^i_{\mathrm{sing}}(X(\mathbb{C}), Y(\mathbb{C})) \ .$$

The edges in $\mathrm{Pairs}^{\mathrm{eff}}$ are also induced from morphisms in DM_{gm}. Moreover, the representation is compatible with the multiplicative structure on $\mathrm{Good}^{\mathrm{eff}}$.

By the universal property of Theorem 8.1.9, this yields a functor $\mathcal{MM}_{\mathrm{Nori}} \to \mathcal{MR}$. It is faithful, exact and a tensor functor. We claim that it factors via $\mathcal{MM}_{\mathrm{AH}}$. As $\mathcal{MM}_{\mathrm{AH}}$ is closed under subquotients in $\mathcal{MR}$, it is enough to check this on generators. By Corollary 8.2.21, the category $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$ is generated by objects of the form $H^i_{\mathrm{Nori}}(X, Y)$ for $X = W \setminus W_\infty$ with X smooth and Y a divisor with normal crossings. (In fact, it is generated by very good pairs; blow up the singularities without changing the motive by excision.) Let $Y_\bullet$ be the Čech nerve of the cover of Y by its normalization. This is the simplicial scheme described in detail in Section 3.3.6. Let

$$C_\bullet = \mathrm{Cone}(Y_\bullet \to X)[-1] \in C^-(\mathbb{Q}[\mathrm{Sm}_k]).$$

Then $H^i_{\mathcal{MR}}(X, Y) = H^i R_{\mathcal{MR}}(C_\bullet)$ is an absolute Hodge motive.

Consider $X_* \in C^b(\mathbb{Q}[\mathrm{Sm}_k])$. We apply Proposition 8.2.16 to $\mathcal{A} = \mathcal{MM}_{\mathrm{Nori}}$ and $\mathcal{A} = \mathcal{MM}_{\mathrm{AH}}$. Hence, there is $R_{\mathrm{Nori}}(X_*) \in D^b(\mathcal{MM}_{\mathrm{Nori}})$ such that the underlying vector space of $H^i R_{\mathrm{Nori}}(X_*)$ is singular cohomology. We claim that there is a natural morphism

$$f : R_{\mathrm{Nori}}(X_*) \to R_{\mathcal{MR}}(X_*).$$

It will automatically be a quasi-isomorphism because both compute singular cohomology of X_*.

We continue as in the proof of Proposition 8.2.16. We choose a rigidified affine cover $\tilde{U}_{X_*}$ of X_* and a very good filtration on the cover. This induces a very good filtration on $\mathrm{Tot} C_*(\tilde{U}_{X_*})$. This induces a double complex of very good pairs. Each very good pair may in turn be seen as complex with two entries. We apply $\tilde{R}_{\mathcal{MR}}$ to this triple complex and take the associated simple complex. On the one hand, the result is quasi-isomorphic to $R_{\mathcal{MR}}(X_*)$ because this is true in singular cohomology. On the other hand, it agrees with $f R_{\mathrm{Nori}}(X_*)$, also by construction.

Finally, we claim that every $M \in \mathcal{MM}_{\mathrm{AH}}$ it is subquotient of the image of a Nori motive. By definition of absolute Hodge motives it suffices to consider M of the form $H^i R_{\mathcal{MR}}(X_*)$ for $X_* \in C^b(\mathbb{Q}[\mathrm{Sm}_k])$. We have seen that $H^i R_{\mathcal{MR}}(X_*) = H^i f(R_{\mathrm{Nori}}(X_*))$, hence M is in the image of f. □

Remark 10.3.4. It is very far from clear whether the functor is also full or essentially surjective. The two properties are related because every object in $\mathcal{MM}_{\mathrm{AH}}$ is a subquotient of an object in the image of $\mathcal{MM}_{\mathrm{Nori}}$.

Theorem 10.3.5. *There is a functor*

$$DM_{\mathrm{gm}} \to D^b(\mathcal{MM}_{\mathrm{Nori}})$$

such the composition

$$C^b(\mathbb{Q}[\mathrm{Sm}_k]) \to DM_{\mathrm{gm}} \to D^b(\mathcal{MM}_{\mathrm{Nori}})$$

agrees with the functor R_{Nori} *of Proposition 8.2.16.*

Proof. This is a result of Harrer, see [Ha]. □

Proof of Theorem 10.3.1. We put together Theorem 10.3.5 and Theorem 10.3.3. □

10.4 Weights and Nori motives

Let $k \subset \mathbb{C}$ be a subfield. We are now going to explore the connection between Grothendieck motives and pure Nori motives and weights.

Definition 10.4.1. Let $n \in \mathbb{N}_0$. An object $M \in \mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$ is called *pure of weight n* if it is a subquotient of a motive of the form $H^n_{\mathrm{Nori}}(Y)$ with Y smooth and projective.

A motive is called *pure* if it is a direct sum of pure motives of some weights.

In particular, $H^*_{\mathrm{Nori}}(Y)$ is pure if Y is smooth and projective.

Definition 10.4.2.

1. The category of *effective Chow motives* $\mathrm{CHM}^{\mathrm{eff}}$ is given by the pseudo-abelian hull of the category with objects given by smooth, projective varieties and morphism form $[X]$ to $[Y]$ given by the Chow group $\mathrm{Ch}^{\dim X}(Y \times X)$ of algebraic cycles of codimension $\dim Y$ up to rational equivalence. The category of *Chow motives* CHM is given by the localization of the category of effective Chow motives with respect to the Lefschetz motive L which is the direct complement of $[\mathrm{Spec} k]$ in $\mathbb{P}^1$.

2. The category of *effective Grothendieck motives* $\mathrm{GRM}^{\mathrm{eff}}$ is given by the pseudo-abelian hull of the category with objects given by smooth, projective varieties and morphism form $[X]$ to $[Y]$ given by the group $A^{\dim X}(Y\times$

X) of algebraic cycles of codimension $\dim Y$ up to homological equivalence with respect to singular cohomology. The category of *Grothendieck motives* GRM is given by the localization of the category of effective Grothendieck motives with respect to the Lefschetz motive L.

In both cases, the composition is given by composition of correspondences.

Remark 10.4.3. There is a *contravariant* functor $X \mapsto [X]$ from the category of smooth, projective varieties over k to Chow or Grothendieck motives. It maps a morphism $f : Y \to X$ to the transpose of its graph Γ_f. The dimension of Γ_f is the same as the dimension of Y, hence it has codimension $\dim X$ in $X \times Y$. On the other hand, singular cohomology defines a well-defined *covariant* functor on Chow and Grothendieck motives. Note that it is not a tensor functor due to the signs in the Künneth formula.

This normalization is the original one, see e.g., [Man]. In recent years, it has also become common to use the covariant normalization instead, in particular in the case of Chow motives.

The category of Grothendieck motives is conjectured to be abelian and semi-simple. Jannsen has shown in [Ja2] that this is the case if and only if homological equivalence agrees with numerical equivalence.

Proposition 10.4.4. *Singular cohomology on* GRM *factors naturally via a faithful functor*

$$\mathrm{GRM} \to \mathcal{MM}_{\mathrm{Nori}}$$

whose image is contained in the category of pure Nori motives.

If the Hodge conjecture holds, then the inclusion is an equivalence of semi-simple abelian categories.

Proof. The opposite category of CHM is a full subcategory of the category of geometric motives DM_{gm} by [VSF, Chapter 5, Proposition 2.1.4]. Restricting the contravariant functor

$$DM_{\mathrm{gm}} \to D^b(\mathcal{MM}_{\mathrm{Nori}}) \xrightarrow{\bigoplus H^i} \mathcal{MM}_{\mathrm{Nori}}$$

to the subcategory yields a covariant functor

$$\mathrm{CHM} \to \mathcal{MM}_{\mathrm{Nori}} \; .$$

By definition, its image is contained in the category of pure Nori motives. Also by definition, a morphism in CHM is zero in GRM if it is zero in singular cohomology, and hence in $\mathcal{MM}_{\mathrm{Nori}}$. Therefore, the functor automatically factors via GRM. The induced functor then is faithful.

We now assume the Hodge conjecture. By [Ja1, Lemma 5.5], this implies that absolute Hodge cycles agree with cycles up to homological equivalence. Equivalently, the functor $\mathrm{GRM} \to \mathcal{MR}$ to mixed realizations is fully faithful. As it factors via $\mathcal{MM}_{\mathrm{Nori}}$, the inclusion $\mathrm{GRM} \to \mathcal{MM}_{\mathrm{Nori}}$ has to be full as well.

The endomorphisms of $[Y]$ for Y smooth and projective can be computed in $\mathcal{MR}$. Hence it is semi-simple because $H^*_{\mathcal{MR}}(Y)$ is polarizable, see [Hu1, Proposition 21.1.2 and 21.2.3]. This implies that its subquotients are the same as its direct summands. Hence, the functor from GRM to pure Nori motives is essentially surjective. □

Proposition 10.4.5. *Every Nori motive $M \in \mathcal{MM}_{\mathrm{Nori}}$ carries a unique bounded increasing filtration $(W_nM)_{n\in\mathbb{Z}}$ inducing the weight filtration in $\mathcal{MR}$. Every morphism of Nori motives is strictly compatible with the filtration.*

Proof. As the functor $\mathcal{MM}_{\mathrm{Nori}} \to \mathcal{MR}$ is faithful and exact, the filtration on $M \in \mathcal{MM}_{\mathrm{Nori}}$ is indeed uniquely determined by its image in M. Strictness of morphisms follows from the same property in $\mathcal{MR}$.

We turn to existence. Bondarko [Bo] constructed what he calls a weight structure on DM_{gm}. It induces a *weight filtration* on the values of any cohomological functor. We apply this to the functor to $\mathcal{MM}_{\mathrm{Nori}}$. In particular, the weight filtration on $H^n_{\mathrm{Nori}}(X,Y)$ is motivic for every vertex of $\mathrm{Pairs}^{\mathrm{eff}}$. The weight filtration on subquotients is the induced filtration, hence also motivic. As any object in $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$ is a subquotient of some $H^n_{\mathrm{Nori}}(X,Y)$, this finishes the proof in the effective case. The non-effective case follows immediately by localization. □

10.5 Periods of motives

Recall the chain of functors

$$DM_{\mathrm{gm}} \to D^b(\mathcal{MM}_{\mathrm{Nori}}) \to D^b(\mathcal{MM}_{\mathrm{AH}}) \to D^b((k,\mathbb{Q})-\mathrm{Vect})$$

constructed in the last section.

Definition 10.5.1.

1. Let $\mathcal{C}(\mathrm{gm})$ be the full subcategory of $(k,\mathbb{Q})-\mathrm{Vect}$ closed under subquotients which is generated by $\mathrm{H}(M)$ for $M \in DM_{\mathrm{gm}}$. Let $\mathbb{P}_{\mathrm{gm}} = \mathbb{P}(\mathcal{C}(\mathrm{gm})))$ be the *period algebra of geometric motives.*

2. Let $\mathcal{C}(\mathrm{Nori})$ be the full subcategory of $(k,\mathbb{Q})-\mathrm{Vect}$ closed under subquotients which is generated by $\mathrm{H}(M)$ for $M \in \mathcal{MM}_{\mathrm{Nori}}$. Let $\mathbb{P}_{\mathrm{Nori}}(k) = \mathbb{P}(\mathcal{C}(\mathrm{Nori}))$ be the *period algebra of Nori motives.*

3. Let $\mathcal{C}(\mathrm{AH})$ be the full subcategory of $(k,\mathbb{Q})-\mathrm{Vect}$ closed under subquotients which is generated by $\mathrm{H}(M)$ for $M \in \mathcal{MM}_{\mathrm{AH}}$. Let $\mathbb{P}_{\mathrm{AH}}(k) = \mathbb{P}(\mathcal{C}(\mathrm{AH}))$ be the *period algebra of absolute Hodge motives.*

Proposition 10.5.2. *We have*

$$\mathbb{P}(k) = \mathbb{P}_{\mathrm{gm}}(k) = \mathbb{P}_{\mathrm{Nori}}(k) = \mathbb{P}_{\mathrm{AH}}(k)\ .$$

Proof. From the functors between categories of motives, we have inclusions of subcategories of $(k, \mathbb{Q})-$Vect:

$$\mathcal{C}(\mathrm{gm}) \subset \mathcal{C}(\mathrm{Nori}) \subset \mathcal{C}(\mathrm{AH}) .$$

Moreover, the category $\mathcal{C}(\mathrm{Sm}_k)$ of Definition 9.4.1 is contained in $\mathcal{C}(\mathrm{gm})$. By definition, we also have $\mathcal{C}(\mathrm{AH}) = \mathcal{C}(\mathrm{Sm}_k)$. Hence, all categories are equal. Finally recall, that $\mathbb{P}(k) = \mathbb{P}(\mathrm{Sm}_k)$ by Theorem 9.4.2. $\square$

This allows easily to translate information on motives into information on periods. Here is an example:

Corollary 10.5.3. *Let $\mathcal{X}$ be an algebraic space, or, more generally, a Deligne-Mumford stack over k. Then the periods of $\mathcal{X}$ are contained in $\mathbb{P}(k)$.*

Proof. Every Deligne-Mumford stack defines a geometric motive by work of Choudhury [Ch]. Their periods are therefore contained in the periods of geometric motives. $\square$

Chapter 11

Kontsevich-Zagier Periods

This chapter follows closely the Diploma thesis of Benjamin Friedrich, see [Fr]. The results are due to him.

We work over $k = \mathbb{Q}$ or equivalently $\overline{\mathbb{Q}}$ throughout. Denote the integral closure of $\mathbb{Q}$ in $\mathbb{R}$ by $\widetilde{\mathbb{Q}}$. Note that $\widetilde{\mathbb{Q}}$ is a field.

In this section, we sometimes use X_0, ω_0 etc. to denote objects over $\widetilde{\mathbb{Q}}$ and X, ω etc. for objects over $\mathbb{C}$.

11.1 Definition

Recall the notion of a $\widetilde{\mathbb{Q}}$-semialgebraic set from Definition 2.6.1.

Definition 11.1.1. Let

- $G \subseteq \mathbb{R}^n$ be an oriented compact $\widetilde{\mathbb{Q}}$-semi-algebraic set which is equidimensional of dimension d, and
- ω a rational differential d-form on $\mathbb{R}^n$ having coefficients in $\overline{\mathbb{Q}}$, which does not have poles on G.

Then we call the complex number $\int_G \omega$ a *naive period* and denote the set of all naive periods for all G and ω by $\mathbb{P}_{\mathrm{nv}}$.

This set $\mathbb{P}_{\mathrm{nv}}$ enjoys additional structure.

Proposition 11.1.2. *The set $\mathbb{P}_{nv}$ is a unital $\overline{\mathbb{Q}}$-algebra.*

Proof. Multiplicative structure: In order to show that $\mathbb{P}_{\mathrm{nv}}$ is closed under multiplication, we write

$$p_i : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \longrightarrow \mathbb{R}^{n_i}, \quad i = 1, 2$$

for the natural projections and obtain

$$\left(\int_{G_1} \omega_1\right) \cdot \left(\int_{G_2} \omega_2\right) = \int_{G_1 \times G_2} p_1^* \omega_1 \wedge p_2^* \omega_2 \in \mathbb{P}_{\text{nv}}$$

by the Fubini formula.

Multiplication by $\overline{\mathbb{Q}}$: We find every $a \in \overline{\mathbb{Q}}$ as naive period with $G = [0,1] \subset \mathbb{R}$ with respect to the differential form adt. In particular, $1 \in \mathbb{P}_{\text{nv}}$.

Combining the last two steps, we can shift the dimension of the set G in the definition of a period number. Let $\alpha = \int_G \omega$. Represent $1 = \int_{[0,1]} dt$ and $1\alpha = \int_{G \times [0,1]} \omega \wedge dt$.

Additive structure: Let $\int_{G_1} \omega_1$ and $\int_{G_2} \omega_2 \in \mathbb{P}_{\text{nv}}$ be periods with domains of integration $G_1 \subseteq \mathbb{R}^{n_1}$ and $G_2 \subseteq \mathbb{R}^{n_2}$. Using the dimension shift described above, we may assume without loss of generality that $\dim G_1 = \dim G_2$. Using the inclusions

$$\begin{aligned} i_1 &: \mathbb{R}^{n_1} \cong \mathbb{R}^{n_1} \times \{1/2\} \times \{\underline{0}\} \subset \mathbb{R}^{n_1} \times \mathbb{R} \times \mathbb{R}^{n_2} \quad \text{and} \\ i_2 &: \mathbb{R}^{n_2} \cong \{\underline{0}\} \times \{-1/2\} \times \mathbb{R}^{n_2} \subset \mathbb{R}^{n_1} \times \mathbb{R} \times \mathbb{R}^{n_2}, \end{aligned}$$

we can write $i_1(G_1) \cup i_2(G_2)$ for the disjoint union of G_1 and G_2. With the projections $p_j : \mathbb{R}^{n_1} \times \mathbb{R} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_j}$ for $j = 1, 2$, we can lift ω_j on $\mathbb{R}^{n_j}$ to $p_j^* \omega_j$ on $\mathbb{R}^{n_1} \times \mathbb{R} \times \mathbb{R}^{n_2}$. For $q_1, q_2 \in \overline{\mathbb{Q}}$ we get

$$q_1 \int_{G_1} \omega_1 + q_2 \int_{G_2} \omega_2 = \int_{i_1(G_1) \cup i_2(G_2)} q_1 \cdot (1/2 + t) \cdot p_1^* \omega_1 + q_2 \cdot (1/2 - t) \cdot p_2^* \omega_2 \in \mathbb{P}_{\text{nv}},$$

where t is the coordinate of the "middle" factor $\mathbb{R}$ of $\mathbb{R}^{n_1} \times \mathbb{R} \times \mathbb{R}^{n_2}$. This shows that $\mathbb{P}_{\text{nv}}$ is a $\overline{\mathbb{Q}}$-vector space.

□

The Definition 11.1.1 was inspired by the one given in [KZ, p. 772]:

Definition 11.1.3 (Kontsevich-Zagier)**.** A *Kontsevich-Zagier period* is a complex number whose real and imaginary part are values of absolutely convergent integrals of rational functions with rational coefficients, over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients.

We will show at the end of this section, that Kontsevich-Zagier periods agree with naive periods in definition 11.1.1, see Theorem 11.2.4.

Examples of naive periods are

- $\displaystyle\int_1^2 \frac{dt}{t} = \log(2),$
- $\displaystyle\int_{x^2+y^2 \le 1} dx\, dy = \pi$ and

- $\int_G \frac{dt}{s} = \int_1^2 \frac{dt}{\sqrt{t^3+1}}$ = elliptic integrals, for $G := \{(t,s) \in \mathbb{R}^2 \,|\, 1 \leq t \leq 2,\, 0 \leq s,\, s^2 = t^3+1\}$.

As a problematic example, we consider the following identity.

Proposition 11.1.4 (cf. [K1, p. 62])**.** *We have*

$$\int_{0 \leq t_1 \leq t_2 \leq 1} \frac{dt_1 \wedge dt_2}{(1-t_1)\,t_2} = \zeta(2). \tag{11.1}$$

Proof. This equality follows by a simple power series manipulation: For $0 \leq t_2 < 1$, we have

$$\int_0^{t_2} \frac{dt_1}{1-t_1} = -\log(1-t_2) = \sum_{n=1}^{\infty} \frac{t_2^n}{n}.$$

Let $\epsilon > 0$. The power series $\sum_{n=1}^{\infty} \frac{t_2^{n-1}}{n}$ converges uniformly for $0 \leq t_2 \leq 1-\epsilon$ and we get

$$\int_{0 \leq t_1 \leq t_2 \leq 1-\epsilon} \frac{dt_1\, dt_2}{(1-t_1)\,t_2} = \int_0^{1-\epsilon} \sum_{n=1}^{\infty} \frac{t_2^{n-1}}{n}\, dt_2 = \sum_{n=1}^{\infty} \frac{(1-\epsilon)^n}{n^2}.$$

Applying Abel's Theorem [Fi, XII, 438, 6°, p. 411] at $(*)$, using $\sum_{n=1}^{\infty} \frac{1}{n^3} < \infty$ gives us

$$\int_{0 \leq t_1 \leq t_2 \leq 1} \frac{dt_1\, dt_2}{(1-t_1)\,t_2} = \lim_{\epsilon \to 0} \sum_{n=1}^{\infty} \frac{(1-\epsilon)^n}{n^2} \overset{(*)}{=} \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2).$$

□

Equation (11.1) is not a valid representation of $\zeta(2)$ as an integral for a naive period in our sense, because the pole locus $\{t_1 = 1\} \cup \{t_2 = 0\}$ of $\frac{dt_1 \wedge dt_2}{(1-t_1)\,t_2}$ is not disjoint with the domain of integration $\{0 \leq t_1 \leq t_2 \leq 1\}$. But (11.1) gives a valid period integral according to the original definition Kontsevich-Zagier — see Definition 11.1.3. We will show in Example 14.1 how to circumvent directly this difficulty by a blow-up. The general blow-up procedure which makes this possible is used in the proof of Theorem 11.2.4. This argument shows that Kontsevich-Zagier periods and naive periods are the same.

11.2 Comparison of Definitions of Periods

Theorem 11.2.1 (Friedrich [Fr])**.**

$$\mathbb{P}^{\text{eff}}(\mathbb{Q}) = \mathbb{P}^{\text{eff}}_{\text{nc}}(\mathbb{Q}) = \mathbb{P}^{\text{eff}}_{nv} \quad \textit{and} \quad \mathbb{P}(\mathbb{Q}) = \mathbb{P}_{\text{nc}}(\mathbb{Q}) = \mathbb{P}_{nv}\ .$$

The proof will take the rest of this section.

Lemma 11.2.2.

$$\mathbb{P}^{\text{eff}}_{\text{nc}}(\mathbb{Q}) \subseteq \mathbb{P}^{\text{eff}}_{nv} .$$

Proof. By definition, its elements of $\mathbb{P}^{\text{eff}}_{\text{nc}}(\mathbb{Q})$ are of the form $\int_\gamma \omega$ where $\gamma \in H_d^{\text{sing}}(X^{\text{an}}, D^{\text{an}}, \mathbb{Q})$ with X a smooth variety of dimension d and D a divisor with normal crossings and $\omega \in \Gamma(X, \Omega^d_X)$.

We choose an embedding

$$X \subseteq \underset{(x_0:\ldots:x_n)}{\mathbb{P}^n_{\mathbb{Q}}}$$

and equip $\mathbb{P}^n_{\mathbb{Q}}$ with coordinates as indicated. Lemma 2.6.5 provides us with a map

$$\psi : \mathbb{C}P^n \hookrightarrow \mathbb{R}^N$$

such that D^{an} and $\mathbb{C}P^n$ become $\widetilde{\mathbb{Q}}$-semi-algebraic subsets of $\mathbb{R}^N$. Then, by Proposition 2.6.8, the cohomology class $\psi_*\gamma$ has a representative which is a rational linear combination of singular simplices Γ_i, each of which is $\widetilde{\mathbb{Q}}$-semi-algebraic.

As $\mathbb{P}^{\text{eff}}_{\text{nv}}$ is a $\mathbb{Q}$-algebra by Proposition 11.1.2, it suffices to prove that

$$\int_{\psi^{-1}(\text{Im}\Gamma_i)} \omega \in \mathbb{P}^{\text{eff}}_{\text{nv}}.$$

We drop the index i from now. Set $G = \text{Im}\Gamma$. The claim will be clear as soon as we find a rational differential form ω' on $\mathbb{R}^N$ such that $\psi^*\omega' = \omega$, since then

$$\int_{\psi^{-1}(G)} \omega = \int_{\psi^{-1}(G)} \psi^*\omega' = \int_G \omega' \in \mathbb{P}^{\text{eff}}_{\text{nv}}.$$

After eventually applying a barycentric subdivision to Γ, we may assume w.l.o.g. that there exists a hyperplane in $\mathbb{C}P^n$, say $\{x_0 = 0\}$, which does not meet $\psi^{-1}(G)$. Furthermore, we may assume that $\psi^{-1}(G)$ lies entirely in U^{an} for U an open affine subset of $D \cap \{x_0 \neq 0\}$. (As usual, U^{an} denotes the complex analytic space associated to the base change to $\mathbb{C}$ of U.) The restriction of ω to the open affine subset can be represented in the form (cf. [Ha2, II.8.4A, II.8.2.1, II.8.2A])

$$\sum_{|J|=d} f_J(x_0, \ldots, x_n)\, d\left(\frac{x_{j_1}}{x_0}\right) \wedge \ldots \wedge d\left(\frac{x_{j_d}}{x_0}\right)$$

with $f_J(x_1, \cdots, x_n) \in \mathbb{Q}(x_0, \cdots, x_n)$ being homogenous of degree zero. This expression defines a rational differential form on all of $\mathbb{P}^n_{\mathbb{Q}}$ with coefficients in $\mathbb{Q}$ and it does not have poles on $\psi^{-1}(G)$.

We construct the rational differential form ω' on $\mathbb{R}^N$ with coefficients in $\mathbb{Q}(i)$ as follows

$$\omega'_I := \sum_{|J|=d} f_J\left(1, \frac{y_{10}+iz_{10}}{y_{00}+iz_{00}}, \cdots, \frac{y_{n0}+iz_{n0}}{y_{00}+iz_{00}}\right) d\left(\frac{y_{j_10}+iz_{j_10}}{y_{00}+iz_{00}}\right) \wedge \ldots \wedge d\left(\frac{y_{j_d0}+iz_{j_d0}}{y_{00}+iz_{00}}\right),$$

where we have used the notation from the proof of Lemma 2.6.5. Using the explicit form of ψ given in this proof, we obtain

$$\psi^* f_J\left(1, \frac{y_{10}+iz_{10}}{y_{00}+iz_{00}}, \cdots, \frac{y_{n0}+iz_{n0}}{y_{00}+iz_{00}}\right) = f_J\left(\frac{x_0\overline{x}_0}{|x_0|^2}, \frac{x_1\overline{x}_0}{|x_0|^2}, \ldots, \frac{x_n\overline{x}_0}{|x_0|^2}\right) = f_J(x_0, x_1, \ldots, x_n)$$

and

$$\psi^* d\left(\frac{y_{j0}+iz_{j0}}{y_{00}+iz_{00}}\right) = d\left(\frac{x_j\overline{x}_0}{|x_0|^2}\right) = d\left(\frac{x_j}{x_0}\right).$$

This shows that $\psi^*\omega' = \omega$ and we are done. □

Lemma 11.2.3.

$$\mathbb{P}^{\mathrm{eff}}_{nv} \subseteq \mathbb{P}^{\mathrm{eff}}_{\mathrm{nc}}(\overline{\mathbb{Q}}) \ .$$

Proof. We will use objects over various base fields. We will use subscripts to indicate which base field is used: A 0 for $\widetilde{\mathbb{Q}}$, a 1 for $\overline{\mathbb{Q}}$, a subscript $\mathbb{R}$ for $\mathbb{R}$ and none for $\mathbb{C}$. Furthermore, we fix an embedding $\overline{\mathbb{Q}} \subset \mathbb{C}$.

Let $\int_G \omega_{\mathbb{R}} \in \mathbb{P}_{\mathrm{nv}}$ be a naïve period with

- $G \subset \mathbb{R}^n$ an oriented $\widetilde{\mathbb{Q}}$-semi-algebraic set, equidimensional of dimension d, and
- $\omega_{\mathbb{R}}$ a rational differential d-form on $\mathbb{R}^n$ with coefficients in $\overline{\mathbb{Q}}$, which does not have poles on G.

The $\widetilde{\mathbb{Q}}$-semi-algebraic set $G \subset \mathbb{R}^n$ is given by polynomial inequalities and equalities. By omitting the inequalities but keeping the equalities in the definition of G, we see that G is supported on (the set of $\mathbb{R}$-valued points of) a variety $Y_{\mathbb{R}} \subseteq \mathbb{A}^n_{\mathbb{R}}$ of same dimension d. This variety $Y_{\mathbb{R}}$ is already defined over $\widetilde{\mathbb{Q}}$

$$Y_{\mathbb{R}} = Y_0 \times_{\widetilde{\mathbb{Q}}} \mathbb{R}$$

for a variety $Y_0 \subseteq \mathbb{A}^n_{\widetilde{\mathbb{Q}}}$ over $\widetilde{\mathbb{Q}}$. Similarly, the boundary ∂G of G is supported on a variety $E_{\mathbb{R}}$, likewise defined over $\widetilde{\mathbb{Q}}$

$$E_{\mathbb{R}} = E_0 \times_{\widetilde{\mathbb{Q}}} \mathbb{R}.$$

Note that E_0 is a divisor on Y_0. By eventually enlarging E_0, we may assume w.l.o.g. that E_0 contains the singular locus of Y_0. In order to obtain an abstract period, we need smooth varieties. The resolution of singularities according to Hironaka [Hi1] provides us with a Cartesian square

$$\begin{array}{ccc} \widetilde{E}_0 & \subseteq & \widetilde{Y}_0 \\ \downarrow & & \downarrow \pi_0 \\ E_0 & \subseteq & Y_0 \end{array} \tag{11.2}$$

where

- $\widetilde{Y}_0$ is smooth and quasi-projective,
- π_0 is proper, surjective and birational, and
- $\widetilde{E}_0$ is a divisor with normal crossings.

In fact, π_0 is an isomorphism away from $\widetilde{E}_0$ since the singular locus of Y_0 is contained in E_0

$$\pi_{0|\widetilde{U}_0} : \widetilde{U}_0 \xrightarrow{\sim} U_0 \tag{11.3}$$

with $\widetilde{U}_0 := \widetilde{Y}_0 \setminus \widetilde{E}_0$ and $U_0 := Y_0 \setminus E_0$.

We apply the analytification functor to the base change to $\mathbb{C}$ of the map $\pi_0 : \widetilde{Y}_0 \to Y_0$ and obtain a projection

$$\pi_{\text{an}} : \widetilde{Y}^{\text{an}} \to Y^{\text{an}}.$$

We want to show that the "strict transform" of G

$$\widetilde{G} := \overline{\pi_{\text{an}}^{-1}(G \setminus E^{\text{an}})} \subseteq \widetilde{Y}^{\text{an}}$$

can be triangulated. Since $\mathbb{C}P^n$ is the projective closure of $\mathbb{C}^n$, we have $\mathbb{C}^n \subset \mathbb{C}P^n$ and thus get an embedding

$$Y^{\text{an}} \subseteq \mathbb{C}^n \subset \mathbb{C}P^n.$$

We also choose an embedding

$$\widetilde{Y}^{\text{an}} \subseteq \mathbb{C}P^m$$

for some $m \in \mathbb{N}$. Using Lemma 2.6.5, we may consider both Y^{an} and $\widetilde{Y}^{\text{an}}$ as $\widetilde{\mathbb{Q}}$-semi-algebraic sets via some maps

$$\psi : Y^{\text{an}} \subset \mathbb{C}P^n \hookrightarrow \mathbb{R}^N, \quad \text{and}$$
$$\widetilde{\psi} : \widetilde{Y}^{\text{an}} \subseteq \mathbb{C}P^m \hookrightarrow \mathbb{R}^M.$$

In this setting, the induced projection

$$\pi_{\text{an}} : \widetilde{Y}^{\text{an}} \longrightarrow Y^{\text{an}}$$

becomes a $\widetilde{\mathbb{Q}}$-semi-algebraic map. The composition of ψ with the inclusion $G \subseteq Y^{\text{an}}$ is a $\widetilde{\mathbb{Q}}$-semi-algebraic map; hence $G \subset \mathbb{R}^N$ is $\widetilde{\mathbb{Q}}$-semi-algebraic by Fact 2.6.4. Since E^{an} is also $\widetilde{\mathbb{Q}}$-semi-algebraic via ψ, we find that $G \setminus E^{\text{an}}$ is $\widetilde{\mathbb{Q}}$-semi-algebraic. Again by Fact 2.6.4, $\pi_{\text{an}}^{-1}(G \setminus E^{\text{an}}) \subset \mathbb{R}^M$ is $\widetilde{\mathbb{Q}}$-semi-algebraic. Thus $\widetilde{G} \subset \mathbb{R}^M$, being the closure of a $\widetilde{\mathbb{Q}}$-semi-algebraic set, is $\widetilde{\mathbb{Q}}$-semi-algebraic. From Proposition 2.6.8, we see that $\widetilde{G}$ can be triangulated

$$\widetilde{G} = \cup_j \triangle_j, \tag{11.4}$$

where the $\triangle_j$ are (homeomorphic images of) d-dimensional simplices.

Our next aim is to define an algebraic differential form $\widetilde{\omega}_1$ replacing $\omega_{\mathbb{R}}$. We first make a base change in (11.2) from $\widetilde{\mathbb{Q}}$ to $\overline{\mathbb{Q}}$ and obtain

$$\begin{array}{ccc} \widetilde{E}_1 & \subseteq & \widetilde{Y}_1 \\ \downarrow & & \downarrow \pi_1 \\ E_1 & \subseteq & Y_1\,. \end{array}$$

The differential d-form $\omega_{\mathbb{R}}$ can be written as

$$\omega_{\mathbb{R}} = \sum_{|J|=d} f_J(x_1, \ldots, x_n)\, dx_{j_1} \wedge \ldots \wedge dx_{j_d}, \tag{11.5}$$

where $x_1, \ldots, x_n$ are coordinates of $\mathbb{R}^n$ and $f_J \in \overline{\mathbb{Q}}(x_1, \ldots, x_n)$. We can use equation (11.5) to define a differential form ω_1 on $\mathbb{A}^n_{\overline{\mathbb{Q}}}$

$$\omega_1 = \sum_{|J|=d} f_J(x_1, \ldots, x_n)\, dx_{j_1} \wedge \ldots \wedge dx_{j_d},$$

where now $x_1, \ldots, x_n$ denote coordinates of $\mathbb{A}^n_{\overline{\mathbb{Q}}}$. The pole locus of ω_1 gives us a variety $Z_1 \subset \mathbb{A}^n_{\overline{\mathbb{Q}}}$. We set

$$\begin{aligned} X_1 &:= Y_1 \setminus Z_1, \quad D_1 := E_1 \setminus Z_1, \quad \text{and} \\ \widetilde{X}_1 &:= \pi_1^{-1}(X_1), \quad \widetilde{D}_1 := \pi_1^{-1}(D_1). \end{aligned}$$

The restriction $\omega_{1|X_1}$ of ω_1 to X_1 is a (regular) algebraic differential form on X_1; the pullback

$$\widetilde{\omega}_1 := \pi_1^*(\omega_{1|X_1})$$

is an algebraic differential form on $\widetilde{X}_1$.

We consider the complex analytic spaces $\widetilde{X}^{\mathrm{an}}$, $\widetilde{D}^{\mathrm{an}}$, Z^{an} associated to the base change to $\mathbb{C}$ of $\widetilde{X}_1$, $\widetilde{D}_1$, Z_1. Since ω_1 has no poles on G, we have $G \cap Z^{\mathrm{an}} = \emptyset$; hence $\widetilde{G} \cap \pi_{\mathrm{an}}^{-1}(Z^{\mathrm{an}}) = \emptyset$. This shows $\widetilde{G} \subseteq \widetilde{X} = \widetilde{Y} \setminus \pi_{\mathrm{an}}^{-1}(Z^{\mathrm{an}})$.

Since G is oriented, so is $\pi_{\mathrm{an}}^{-1}(G \setminus E^{\mathrm{an}})$, because π_{an} is an isomorphism away from E^{an}. Every d-simplex $\triangle_j$ in (11.4) intersects $\pi_{\mathrm{an}}^{-1}(G \setminus E^{\mathrm{an}})$ in a dense open subset, hence inherits an orientation. As in the proof of Proposition 2.6.8, we choose orientation-preserving homeomorphisms from the standard d-simplex $\triangle_d^{\mathrm{std}}$ to $\triangle_j$

$$\sigma_j : \triangle_d^{\mathrm{std}} \longrightarrow \triangle_j.$$

These maps sum up to a singular chain

$$\widetilde{\Gamma} = \oplus_j\, \sigma_j \in \mathrm{C}_d^{\mathrm{sing}}(\widetilde{X}^{\mathrm{an}}; \mathbb{Q}).$$

It might happen that the boundary of the singular chain $\widetilde{\Gamma}$ is not supported on $\partial\widetilde{G}$. Nevertheless, it will always be supported on $\widetilde{D}^{\mathrm{an}}$: The set $\pi_{\mathrm{an}}^{-1}(G \setminus E^{\mathrm{an}})$ is oriented and therefore the boundary components of $\partial\triangle_j$ that do not belong to

$\partial\widetilde{G}$ cancel if they have non-zero intersection with $\pi_{\mathrm{an}}^{-1}(G \setminus E^{\mathrm{an}})$. Thus $\widetilde{\Gamma}$ gives rise to a singular homology class

$$\widetilde{\gamma} \in H_d^{\mathrm{sing}}(\widetilde{X}^{\mathrm{an}}, \widetilde{D}^{\mathrm{an}}; \mathbb{Q}).$$

We denote the base change to $\mathbb{C}$ of ω_1 and $\widetilde{\omega}_1$ by ω and $\widetilde{\omega}$, respectively. Now

$$\begin{aligned}\int_G \omega_1 = \int_G \omega = \int_{G\cap U^{\mathrm{an}}} \omega \\ \overset{(11.3)}{=} \int_{\pi^{-1}(G\cap U^{\mathrm{an}})} \pi^*\omega = \int_{\widetilde{G}\cap\widetilde{U}^{\mathrm{an}}} \widetilde{\omega} \\ = \int_{\widetilde{G}} \widetilde{\omega} = \int_{\widetilde{\Gamma}} \widetilde{\omega} = \int_{\widetilde{\gamma}} \widetilde{\omega} \in \mathbb{P}_{\mathrm{nc}}^{\mathrm{eff}}(\overline{\mathbb{Q}})\end{aligned}$$

is a period for the quadruple $(\widetilde{X}_1, \widetilde{D}_1, \widetilde{\omega}_1, \widetilde{\gamma})$. □

Proof of Theorem 11.2.1. It suffices to consider the effective case. By Theorem 9.4.2, we have $\mathbb{P}^{\mathrm{eff}}(\mathbb{Q}) = \mathbb{P}_{\mathrm{nc}}^{\mathrm{eff}}(\mathbb{Q})$. By Corollary 9.3.5, this is also the same as $\mathbb{P}^{\mathrm{eff}}(\overline{\mathbb{Q}})$. The result now follows by combining Lemma 11.2.2 and Lemma 11.2.3. □

Now, we show that naive periods and Kontsevich-Zagier periods coincide:

Theorem 11.2.4.

$$\mathbb{P}_{KZ}^{\mathrm{eff}} = \mathbb{P}_{nv}^{\mathrm{eff}} = \mathbb{P}^{\mathrm{eff}},\ \mathbb{P}_{KZ} = \mathbb{P}_{nv} = \mathbb{P}.$$

Proof. We will use that $\mathbb{P}_{\mathrm{nv}}^{\mathrm{eff}} = \mathbb{P}_{\mathrm{nc}}^{\mathrm{eff}} = \mathbb{P}^{\mathrm{eff}}$ (see Theorem 11.2.1) and work with effective periods only. We partially follow ideas of Belkale and Brosnan [BB].
First we show that $\mathbb{P}_{\mathrm{KZ}}^{\mathrm{eff}} \subseteq \mathbb{P}_{\mathrm{nc}}^{\mathrm{eff}}$: Assume we have given a period through an n-dimensional absolutely convergent integral $\int_\Delta \omega$, where $\omega = \frac{f(x_1,\ldots,x_n)}{g(x_1,\ldots,x_n)}$ is a rational function defined over $\mathbb{Q}$ and Δ a $\mathbb{Q}$-semialgebraic region defined by inequalities $h_i \geq 0$. This defines a rational differential form ω on $\mathbb{A}^n$. We can extend ω to a rational differential form on $\mathbb{P}^n$ (also denoted by ω) by adding a homogenous variable x_0. The closure $\bar{\Delta}$ of Δ in $\mathbb{P}^n(\mathbb{R})$ is a compact semialgebraic region, defined by $H_i \geq 0$ for some homogenous polynomials H_i. Let $H = \prod_i H_i$. Now we use resolution of singularities and obtain a blow-up

$$\sigma : X \to \mathbb{P}^n,$$

such that we have the following properties:
1. σ is an isomorphism outside the union of the pole locus of ω and the zero sets of all polynomials H_i.
2. The strict transform of the zero locus of H is a normal crossing divisor in X.

3. Near each point $P \in X$, there are local algebraic coordinates $x_1, ..., x_n$ and integers e_j, f_j for each $j = 1, ..., n$, such that

$$H \circ \sigma = \text{unit}_1 \times \prod_{j=1}^{n} x_j^{e_j}, \ \sigma^*\omega = \text{unit}_2 \times \prod_{j=1}^{n} x_j^{f_j} dx_1 \wedge \cdots \wedge dx_n.$$

Let $\tilde{\Delta}$ be the analytic closure of $\Delta \cap U$, where U is the set where σ is an isomorphism. Then $\tilde{\Delta}$ is compact, since it is a closed subset of the compact set $\sigma^{-1}(\bar{\Delta})$. The absolute convergence of $\int_\Delta \omega$ implies the local convergence of $\sigma^*\omega$ over regions $\{0 < x_i < \epsilon\}$ at point $P \in \tilde{\Delta}$. This is only possible, if all $f_j \geq 0$. Therefore, $\sigma^*\omega$ is regular (holomorphic) at the point P, and hence on the whole of $\tilde{\Delta}$.

Now we show that $\mathbb{P}^{\text{eff}}_{\text{nc}} \subseteq \mathbb{P}^{\text{eff}}_{\text{KZ}}$: This argument is indicated in Kontsevich-Zagier [KZ, pg. 773]. First, note that naive periods in $\mathbb{P}^{\text{eff}}_{\text{KZ}}$ can also be defined with $\bar{\mathbb{Q}}$-coefficients and the polynomials involved can be replaced by algebraic functions without changing the set $\mathbb{P}^{\text{eff}}_{\text{KZ}}$. A proof is not given in loc. cit., but this can be achieved by using auxiliary variables and minimal polynomials as in the proof that $\sqrt{2} \in \mathbb{P}^{\text{eff}}_{\text{KZ}}$. Assuming this, we now assume that we have given a smooth algebraic variety X of dimension n, a regular differential from ω of top degree (hence closed), a normal crossing divisor $D \subset X$, all this data defined over $\mathbb{Q}$, and a singular chain γ with boundary $\partial\gamma \subset D$. Now we can use the method of Lemma 11.2.2 and we can write

$$\int_\gamma \omega = \int_G \tilde{\omega},$$

where G is a $\tilde{\mathbb{Q}}$-semialgebraic subset of the required form, i.e., given by inequalities, and $\tilde{\omega}$ is a differential form with algebraic coefficients. □

Chapter 12

Formal periods and the period conjecture

Following Kontsevich (see [K1]), we now introduce another algebra $\tilde{\mathbb{P}}(k)$ of *formal periods* from the same data we have used in order to define the actual period algebra of a field in Chapter 9. It comes with an obvious surjective map to $\mathbb{P}(k)$.

The first aim of the chapter is to give a conceptual interpretation of $\tilde{\mathbb{P}}(k)$ as the ring of algebraic functions on the torsor between two fibre functors on Nori motives: singular cohomomology and algebraic de Rham cohomology.

We then discuss the period conjecture from this point of view.

12.1 Formal periods and Nori motives

Definition 12.1.1. Let $k \subset \mathbb{C}$ be a subfield. The space of *effective formal periods* $\tilde{\mathbb{P}}^{\mathrm{eff}}(k)$ is defined as the $\mathbb{Q}$-vector space generated by symbols (X, D, ω, γ), where X is an algebraic variety over k, $D \subset X$ a subvariety, $\omega \in H^d_{\mathrm{dR}}(X, D)$, $\gamma \in H_d(X(\mathbb{C}), D(\mathbb{C}), \mathbb{Q})$ with relations

1. linearity in ω and γ;
2. for every $f : X \to X'$ with $f(D) \subset D'$
$$(X, D, f^*\omega', \gamma) = (X', D', \omega', f_*\gamma)$$
3. for every triple $Z \subset Y \subset X$
$$(Y, Z, \omega, \partial\gamma) = (X, Y, \delta\omega, \gamma)$$
with ∂ the connecting morphism for relative singular homology and δ the connecting morphism for relative de Rham cohomology.

We write $[X, D, \omega, \gamma]$ for the image of the generator. The vector space $\tilde{\mathbb{P}}^{\mathrm{eff}}(k)$ is turned into an algebra via

$$(X, D, \omega, \gamma)(X', D', \omega', \gamma') = (X \times X', D \times X' \cup D' \times X, \omega \wedge \omega', \gamma \times \gamma') \ .$$

The space of *formal periods* is the localization $\tilde{\mathbb{P}}(k)$ of $\tilde{\mathbb{P}}^{\mathrm{eff}}(k)$ with respect to $[\mathbb{G}_m, \{1\}, \frac{dX}{X}, S^1]$, where S^1 is the unit circle in $\mathbb{C}^*$.

Remark 12.1.2. This is modeled after Kontsevich [K1] Definition 20, but does not agree with it. We will discuss this point in more detail in Remark 12.1.7.

Theorem 12.1.3. *(Nori) Let $k \subset \mathbb{C}$ be subfield. Let $G_{\mathrm{mot}}(k)$ be the Tannakian dual of the category of Nori motives with $\mathbb{Q}$-coefficents (sic!), see Definition 8.1.6. Let $X = \mathrm{Spec}\tilde{\mathbb{P}}(k)$. Then X is naturally isomorphic to the torsor of isomorphisms between singular cohomology and algebraic de Rham cohomology on Nori motives. It has a natural torsor structure under the base change of $G_{\mathrm{mot}}(k, \mathbb{Q})$ to k (in the fpqc-topology on the category of k-schemes):*

$$X \times_k G_{\mathrm{mot}}(k, \mathbb{Q})_k \to X.$$

Remark 12.1.4. This was first formulated in the case $k = \mathbb{Q}$ without proof by Kontsevich as [K1, Theorem 6]. He attributes it to Nori.

Proof. Consider the diagram $\mathrm{Pairs}^{\mathrm{eff}}$ of Definition 8.1.1 and the representations $T_1 = H^*_{\mathrm{dR}}(-)$ and $T_2 = H^*(-, k)$ (sic!). Note that $H_d(X(\mathbb{C}), D(\mathbb{C}), \mathbb{Q})$ is dual to $H^d(X(\mathbb{C}), D(\mathbb{C}), \mathbb{Q})$.

By the very definition, $\tilde{\mathbb{P}}^{\mathrm{eff}}(k)$ is the module $P_{1,2}(\mathrm{Pairs}^{\mathrm{eff}})$ of Definition 7.4.19. By Theorem 7.4.21, it agrees with the module $A_{1,2}(\mathrm{Pairs}^{\mathrm{eff}})$ of Definition 7.4.2. We are now in the situation of Section 7.4 and apply its main result, Theorem 7.4.10. In particular,

$$A_{1,2}(\mathrm{Pairs}^{\mathrm{eff}}) = A_{1,2}(\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}).$$

Recall that by Theorem 8.2.20, the diagram categories of $\mathrm{Pairs}^{\mathrm{eff}}$ and $\mathrm{Good}^{\mathrm{eff}}$ agree. This also shows that the modules

$$A_{1,2}(\mathrm{Pairs}^{\mathrm{eff}}) = A_{1,2}(\mathrm{Good}^{\mathrm{eff}})$$

agree. From now on, we may work with the diagram $\mathrm{Good}^{\mathrm{eff}}$ which has the advantage of admitting a commutative product structure. The algebra structures on $A_{1,2}(\mathrm{Good}^{\mathrm{eff}}) = P_{1,2}(\mathrm{Good}^{\mathrm{eff}}) = \tilde{\mathbb{P}}^{\mathrm{eff}}(k)$ agree.

We can apply the same considerations to the localized diagram Good. As in Proposition 7.2.5, localization on the level of diagrams or categories amounts to localization on the algebra. Hence,

$$A_{1,2}(\mathrm{Good}) = P_{1,2}(\mathrm{Good}) = \tilde{\mathbb{P}}(k)$$

and

$$X = \mathrm{Spec}A_{1,2}(\mathrm{Good}).$$

Also, by definition, $G_2(\text{Good})$ is the Tannakian dual of the category of Nori motives with k coefficients. By base change Lemma 6.5.6 it is the base change of the Tannaka dual of the category of Nori motives with $\mathbb{Q}$-coefficients. After these identifications, the operation

$$X \times_k G_{\text{mot}}(k, \mathbb{Q})_k \to X$$

is the one of Theorem 7.4.7.

By Theorem 7.4.10, it is a torsor because $\mathcal{MM}_{\text{Nori}}$ is rigid. □

Remark 12.1.5. There is a small subtlety here because our to fibre functors take values in different categories, $\mathbb{Q}-$Mod and $k-$Mod. As $H^*(X,Y,k) = H^*(X,Y,\mathbb{Q}) \otimes_{\mathbb{Q}} k$ and $\tilde{\mathbb{P}}(k)$ already is a k-algebra, the algebra of formal periods does not change when replacing $\mathbb{Q}$-coefficients with k-coefficients.

We can also view X as torsor in the sense of Definition 1.7.9. The description of the torsor structure was discussed extensively in Section 7.4, in particular Theorem 7.4.10. In terms of period matrices, it is given by the formula in [K1]:

$$P_{ij} \mapsto \sum_{k,\ell} P_{ik} \otimes P_{k\ell}^{-1} \otimes P_{\ell j}.$$

Corollary 12.1.6. *1. The algebra of effective formal periods $\tilde{\mathbb{P}}^{\text{eff}}(k)$ remains unchanged when we restrict in Definition 12.1.1 to (X, D, ω, γ) with X affine of dimension d, D of dimension $d-1$ and $X \smallsetminus D$ smooth, $\omega \in H^d_{\text{dR}}(X, D)$, $\gamma \in H_d(X(\mathbb{C}), D(\mathbb{C}), \mathbb{Q})$.*

2. $\tilde{\mathbb{P}}^{\text{eff}}(k)$ is generated as $\mathbb{Q}$-vector space by elements of the form $[X, D, \omega, \gamma]$ with X smooth of dimension d, D a divisor with normal crossings $\omega \in H^d_{\text{dR}}(X, D)$, $\gamma \in H_d(X(\mathbb{C}), D(\mathbb{C}), \mathbb{Q})$.

Proof. In the proof of Theorem 12.1.3, we have already argued that we can replace the diagram $\text{Pairs}^{\text{eff}}$ by the diagram Good^{eff}. The same argument also allows to replace it by $\text{VGood}^{\text{eff}}$.

By blowing up X, we get another good pair $(\tilde{X}, \tilde{D}, d)$. By excision, they have the same de Rham and singular cohomology as (X, D, d). Hence, we may identify the generators. □

Remark 12.1.7. We do not know whether it is enough to work only with formal periods of the form (X, D, ω, γ) with X smooth and D a divisor with normal crossings in Definition 12.1.1 as Kontsevich does in [K1, Definition 20]. By the Corollary, these symbols generate the algebra, but it is not clear to us if they also give all relations. Indeed, Kontsevich in loc. cit. only imposes the relation given by the connecting morphism of triples in an even more special case.

Moreover, Kontsevich considers differential forms of top degree rather than cohomology classes. They are automatically closed. He imposes Stokes' formula as an additional relation, hence this amounts to considering cohomology classes. Note, however, that not every de Rham class is of this form in general.

All formal effective periods (X, D, ω, γ) can be evaluated by "integrating" ω along γ. More precisely, recall (see Definition 5.4.1 the period pairing

$$H^d_{\mathrm{dR}}(X,D) \times H_d(X(\mathbb{C}), D(\mathbb{C})) \to \mathbb{C}$$

It maps $(\mathbb{G}_m, \{1\}, dX/X, S^1)$ to $2\pi i$.

Definition 12.1.8. Let

$$\mathrm{ev} : \tilde{\mathbb{P}}(k) \to \mathbb{C},$$

be the ring homomorphism induced by the period pairing. We denote by per the $\mathbb{C}$-valued point of $X = \operatorname{Spec} \tilde{\mathbb{P}}(k)$ defined by ev.

The elements in the image are precisely the element of the period algebra $\mathbb{P}(k)$ of Definition 9.3.1. By the results in Chapters 9, 10, and 11 (for $k = \mathbb{Q}$), it agrees with all other definitions of a period algebra. From this perspective, per is the $\mathbb{C}$-valued point of the torsor X of Theorem 12.1.3 comparing singular and algebraic de Rham cohomology. It is given by the period isomorphism per defined in Chapter 5.

The following statement of period number is a corollary from our previous results on formal periods.

Corollary 12.1.9. *The algebra $\mathbb{P}(k)$ is $\mathbb{Q}$-linearly generated by number of the form $(2\pi i)^j\alpha$ with $j \in \mathbb{Z}$, and α the period of (X, D, ω, γ) with X smooth affine, D a divisor with normal crossings, $\omega \in \Omega^d_X(X)$.*

This was also proved without mentioning motives as Theorem 9.4.2.

Proof. Recall that $2\pi i$ is itself a period of such a quadruple.

By Corollary 8.2.21, the category $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$ is generated by motives of good pairs (X, Y, d) of the form $X = W \setminus W_\infty$, $Y = W_0 \setminus (W_\infty \cap W_0)$ with W smooth projective of dimension d, $W_0 \cup W_\infty$ a divisor with normal crossings, $X' = \setminus W_0$ affine. Hence, their periods generated $\mathbb{P}^{\mathrm{eff}}(k)$ as a $\mathbb{Q}$-vector space.

Let $Y' = W_\infty \setminus (W_0 \cap W_\infty)$. By Lemma 8.3.8, the motive $H^d_{\mathrm{Nori}}(X,Y)$ is dual to $H^d_{\mathrm{Nori}}(X',Y')(d)$. By Lemma 9.2.9, this implies that the periods of the first agree with the periods of the latter up to a factor $(2\pi i)^d$.

As X' is affine and Y' a divisor with normal crossings, $H^d_{\mathrm{dR}}(X',Y')$ is generated by $\Omega^d_{X'}(X')$ by Proposition 3.3.19. □

Proposition 12.1.10. *Let K/k be algebraic. Then*

$$\tilde{\mathbb{P}}(K) = \tilde{\mathbb{P}}(k) ,$$

and hence also

$$\mathbb{P}(K) = \mathbb{P}(k) .$$

The second statement was already proved directly Corollary 9.3.5

Proof. It suffices to consider the case K/k finite. The general case follows by taking limits.

Generators of $\tilde{\mathbb{P}}(k)$ also define generators of $\tilde{\mathbb{P}}(K)$ by base change for the field extension K/k. The same is true for relations, hence we get a well-defined map $\tilde{\mathbb{P}}(k) \to \tilde{\mathbb{P}}(K)$.

We define a map in the opposite direction by viewing a K-variety as k-variety. More precisely, let (Y, E, m) be vertex of $\mathrm{Pairs}^{\mathrm{eff}}(K)$ and (Y_k, E_k, m) the same viewed as vertex of $\mathrm{Pairs}^{\mathrm{eff}}(k)$. As in the proof of Corollary 9.3.5, we have

$$\mathrm{H}(Y_k, E_k, m) = R_{K/k}\mathrm{H}(Y, E, m)$$

with $R_{K/k}$ as defined in Lemma 9.2.7. The same proof as in Lemma 9.2.7 (treating actual periods) also shows that the formal periods of (Y_k, E_k, m) agree with the formal periods (Y, E, m): □

12.2 The period conjecture

We exlore the relation to transcendence questions from the point of view of of Nori motives and their periods. We only treat the case where $k/\mathbb{Q}$ is algebraic. For more general fields, see Ayoub's remarks in [Ay].

Recall that $\tilde{\mathbb{P}}(\mathbb{Q}) = \tilde{\mathbb{P}}(k) = \tilde{\mathbb{P}}(\bar{\mathbb{Q}})$ under this assumption.

Conjecture 12.2.1 (Kontsevich-Zagier)**.** *Let $k/\mathbb{Q}$ be an algebraic field extension contained in $\mathbb{C}$. The evaluation map (see Definition 12.1.8)*

$$\mathrm{ev} : \tilde{\mathbb{P}}(k) \to \mathbb{P}(k)$$

is bijective.

Remark 12.2.2. We have already seen that the map is surjective. Hence injectivity is the true issue. Equivalently, we can conjecture that $\tilde{\mathbb{P}}(k)$ is an integral domain and ev a generic point.

In the literature [A1, A2, Ay, BC, Wu], there are sometimes alternative formulations of this conjecture, called "Grothendieck conjecture". We will explain this a little bit more.

Definition 12.2.3. Let $M \in \mathcal{MM}_{\mathrm{Nori}}$ be a Nori motive. Let

$$X(M)$$

be the torsor of isomorphisms between singular and algebraic de Rham cohomology on the Tannaka category $\langle M, M^\vee \rangle^\otimes$ generated by M and

$$\tilde{\mathbb{P}}(M) = \mathcal{O}(X(M))$$

the associated ring of formal periods. If $M = H^*_{\mathrm{Nori}}(Y)$ for a variety Y, we also write $\tilde{\mathbb{P}}(Y)$.

Let $G_{\mathrm{mot}}(M)$ and $G_{\mathrm{mot}}(Y)$ be the Tannaka duals of the category with respect to singular cohomology.

These are the finite dimensional building blocks of $\tilde{\mathbb{P}}(k)$ and $G_{\mathrm{mot}}(k)$, respectively.

Remark 12.2.4. By Theorem 7.4.10, the space $X(M)$ is a $G_{\mathrm{mot}}(M)$-torsor. Hence they share all properties that can be tested after a faithfully flat base change. In particular, they have the same dimension. Moreover, $X(M)$ is smooth because $G(M)$ is a group scheme over a field of characteristic zero.

Analogous to [Ay] and [A2, Prop. 7.5.2.2 and Prop. 23.1.4.1], we can ask:

Conjecture 12.2.5 (Grothendieck conjecture for Nori motives)**.** *Let $k/\mathbb{Q}$ be an algebraic extension contained in $\mathbb{C}$ and $M \in \mathcal{MM}_{\mathrm{Nori}}(k)$. The following equivalent assertions are true:*

1. *The evaluation map*
$$\mathrm{ev} : \tilde{\mathbb{P}}(M) \to \mathbb{C}$$
is injective.
2. *The point ev_M of $\mathrm{Spec}\,\tilde{\mathbb{P}}(M)$ is a generic point, and $X(M)$ connected.*
3. *The space $X(M)$ is connected, and the transcendence degree of the subfield of $\mathbb{C}$ generated by the image of ev_M is the same as the dimension of $G_{\mathrm{mot}}(M)$.*

Proof of equivalence. Assume that ev is injective. Then $\tilde{\mathbb{P}}(M)$ is contained in the field $\mathbb{C}$, hence integral. The map to $\mathbb{C}$ factors via the residue field of a point. It ev is injective, this has to be the generic point. The subfield generated by $\mathrm{ev}(M)$ is isomorphic to the function field. Its transcendence degree is the dimension of the integral domain.

Conversely, if $X(M)$ is connected, then it $\tilde{\mathbb{P}}(M)$ is integral because it is already smooth. If ev factors the generic point, its function field embeds into $\mathbb{C}$ and hence $\tilde{\mathbb{P}}(M)$ does. If the subfield generated by the image of ev in $\mathbb{C}$ has the maximal possible transcendence degree, then ev has to be generic. □

Lemma 12.2.6. *If Conjecture 12.2.5 is true for all M, then Conjecture 12.2.1 holds.*

Proof. By construction, we have
$$\tilde{\mathbb{P}}(k) = \mathrm{colim}_M \tilde{\mathbb{P}}(M).$$
Injectivity of the evaluation maps on the level of every M implies injectivity of the transition maps and injectivity of ev on the union. □

Remark 12.2.7. The converse is not obvious. It amounts to asking whether $\tilde{\mathbb{P}}(M)$ is contained in $\tilde{\mathbb{P}}(k)$. In our description with generators and relations, this means that all relations are given by relations within the category $\langle M, M^\vee\rangle^\otimes$. This is not clear a priori. We have a conditional result in the pure case.

Proposition 12.2.8. *Assume that the Hodge conjecture holds for all varieties. Let M be a pure Nori motive. Then $\tilde{\mathbb{P}}(M)$ injects into $\tilde{\mathbb{P}}(k)$.*

Proof. The algebra $\tilde{\mathbb{P}}(M)$ is generated by classes (ω,γ) with $\omega \in H^*_{\mathrm{dR}}(M) \oplus H^*_{\mathrm{dR}}(M)^\vee$ and $\gamma \in H_*(M,\mathbb{Q}) \oplus H_*(M,\mathbb{Q})^\vee$ of the same cohomological degree. The relations are given by chains of morphisms and morphisms in the opposite direction

$$M \to M_1 \leftarrow M_2 \to \cdots \leftarrow M$$

in the tensor category generated by the direct sum of these Nori motives.

In $\tilde{\mathbb{P}}(k)$, the relations between these same generators are given by chains in the category of *all* Nori motives. A priori, there are more of these.

By Proposition 10.4.5, we have a weight filtration on the category of Nori motives. Morphisms between pure motives of different weights vanish. We choose our generators pure and we apply the weight filtration to the whole chain defining a relation. This implies that there are no relations between pure generators of different weights. The relations between pure generators of the same weight are already induced from relations of this fixed weight. We now apply the Hodge conjecture again and in a semi-simple category. The only relations are the ones given by the simple objects in the subcategory. □

The third version of Conjecture 12.2.5 is very close to the point of view taken originally by Grothendieck in the pure case. In order to understand the precise relation, we have to establish some properties first.

We specialize to the case $\tilde{\mathbb{P}}(Y)$ for Y smooth and projective. In this case, singular cohomology $H^*(Y,\mathbb{Q})$ carries a pure $\mathbb{Q}$-Hodge structure, see Definition 10.2.2. Recall that the Mumford-Tate group $\mathrm{MT}(V)$ of a polarizable pure Hodge structure V is the smallest $\mathbb{Q}$-algebraic subgroup of $\mathrm{GL}(V)$ such that Hodge representation $h : \mathbb{S} \to \mathrm{G}L(V_\mathbb{R})$ factors via G as $h : \mathbb{S} \to G_\mathbb{R}$. Here, $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ is the Deligne torus. It is precisely the $\mathbb{Q}$-algebraic subgroup of $\mathrm{G}L(V_\mathbb{R})$ that fixes all Hodge tensors in all tensor powers $\bigoplus V^{\otimes m} \otimes V^{\vee\otimes n}$ [M]. Alternatively, it can be understood as the Tannaka dual of the subcategory of the category of Hodge structures generated by V. The group $\mathrm{MT}(V)$ is a reductive $\mathbb{Q}$-algebraic group by [GGK, Chap. I].

Proposition 12.2.9. *Let $k = \bar{\mathbb{Q}}$ and let Y be smooth and projective. Assume that the Hodge conjecture holds for all powers of Y. Then $G_{\mathrm{mot}}(Y)$ is the same as the Mumford-Tate group of Y.*

Proof. By Proposition 10.4.4 the Tannaka subcategory of $\mathcal{MM}_{\mathrm{Nori}}$ generated by $H^*_{\mathrm{Nori}}(Y)$ agrees with the Tannaka subcategory of the category of Grothendieck motives GRM. Note that the statement of Proposition 10.4.4 assumes the full Hodge conjecture. The same argument also gives the statement on the subcategories under the weaker assumption. For the rest of the argument we refer to Lemme 7.2.2.1 and Remarque 23.1.4.2 of [A2]. It amounts to saying that equivalent Tannaka categories have isomorphic Tannaka duals. □

Corollary 12.2.10 (Period Conjecture)**.** *Let Y be a smooth, projective variety over $\mathbb{Q}$. Assume Conjecture 12.2.5 for powers of Y and the Hodge conjecture. Then every polynomial relation among the periods of Y are of motivic nature, i.e., they are induced by algebraic cycles (correspondences) in powers of Y.*

In the case of elliptic curves this was stated as conjecture by Grothendieck [Gro1].

Proof. By Conjecture 12.2.5 all $\mathbb{Q}$-linear relations between periods are induced by morphisms of Nori motives. Under the Hodge conjecture, the category of pure Nori motives is equivalent to the category of Grothendieck motives by Proposition 10.4.4. By definition of Grothendieck motives (Definition 10.4.2) this means that morphisms are induced from algebraic cycles.

Polynomial relations are induced from the tensor structure, hence powers of Y. □

Arnold [Ar, pg. 93] remarked in a footnote that this is related to a conjecture of Leibniz which he made in a letter to Huygens from 1691. Leibniz essentially claims that all periods of *generic* meromorphic 1-forms are transcendental. Of course, precisely the meaning of "generic" is the essential question. The conjecture of Leibniz can be rephrased in modern form as in [Wu]:

Conjecture 12.2.11 (Integral Conjecture of Leibniz)**.** *Any period integral of a rational algebraic* 1*-form ω on a smooth projective variety X over a number field k over a path γ with $\partial\gamma \subset D$ (the polar divisor of ω) which does not come from a proper mixed k-Hodge substructure $H \subsetneq H_1(X \setminus D)$ is transcendental.*

This is only a statement about periods of type $i = 1$, i.e., for $H^1(X, D)$ (or, by duality $H_1(X \setminus D)$) on curves. The Leibniz conjecture follows essentially from the period conjecture in the case $i = 1$, since the Hodge conjecture holds on $H^1(X) \otimes H^1(X) \subset H^2(X)$. This conjecture is still open. See also [BC] for strongly related questions.

Wüstholz [Wu] has related this problem to many other transcendance results. One can give transcendance proofs assuming this conjecture:

Example 12.2.12. Let us show that $\log(\alpha)$ is transcendental for every algebraic $\alpha \neq 0, 1$ under the assumption of the Leibniz conjecture. One takes $X = \mathbb{P}^1$, and $\omega = d\log(z)$ and $\gamma = [1, \alpha]$. The polar divisor of ω is $D = \{0, \infty\}$, and the Hodge structure $H_1(X \setminus D) = H_1(\mathbb{C}^\times) = \mathbb{Z}(1)$ is irreducible as a Hodge structure. Hence, $\log(\alpha)$ is transcendental assuming Leibniz's conjecture.

There are also examples of elliptic curves in [Wu] related to Chudnovsky's theorem we mention below.

The third form of Conjecture 12.2.5 is also very useful in a computational sense. In this case, assuming the Hodge conjecture for all powers of Y, the motivic Galois group $G_{\mathrm{mot}}(Y)$ is the same as the *Mumford-Tate group* $\mathrm{MT}(Y)$ by Proposition 12.2.9.

André shows in [A2, Rem. 23.1.4.2]:

Corollary 12.2.13. *Let Y be a smooth, projective variety over $\mathbb{Q}$ and assume that the Hodge conjecture holds for all powers of Y. Then, assuming Grothendieck's conjecture,*

$$\mathrm{trdeg}_{\mathbb{Q}}\mathbb{P}(Y) = \dim_{\mathbb{Q}} \mathrm{MT}(Y).$$

Proof. We view the right hand side as $G_{\mathrm{mot}}(Y_{\bar{\mathbb{Q}}})$ by Proposition 12.2.9. By [A2, Paragraph 7.6.4], it is of finite index in $G_{\mathrm{mot}}(Y)$, hence has the same dimension. It has also the same dimension as the torsor $\tilde{\mathbb{P}}(Y)$. Under Grothendieck's conjecture, this is given by the transcendence degree of $\mathbb{P}(Y)$, see Conjecture 12.2.5. □

This corollary give a reasonable, completely unconditional testing conjecture for transcendence questions.

Example 12.2.14. (Tate motives) If the motive of Y is a Tate motives, e.g., $Y = \mathbb{P}^n$, then the conjecture is true, since $2\pi i$ is transcendant. The Mumford-Tate group is the 1-torus here. More generally, the conjecture holds for Artin-Tate motives, since the transcendance degree remains 1.

Example 12.2.15. (Elliptic curves) Let E be an elliptic curve over $\mathbb{Q}$. Then the Mumford-Tate group of E is either a 2-torus if E has complex multiplication, or $\mathrm{GL}_{2,\mathbb{Q}}$ otherwise (see [M]). Hence, the transcendence degree of $\mathbb{P}(E)$ is either 2 or 4. G. V. Chudnovsky [Ch] has proved that $\mathrm{trdeg}_{\mathbb{Q}}\mathbb{P}(E) = 2$ if E is an elliptic curve with complex multiplication, and it is ≥ 2 for all elliptic curves over $\mathbb{Q}$. Note that in this situation we have actually 5 period numbers ω_1, ω_2, η_1, η_2 and π around (see Section 13.4 for more details), but they are related by Legendre's relation $\omega_2\eta_1 - \omega_1\eta_2 = 2\pi i$, so that the transcendence degree cannot go beyond 4. Hence, it remains to show that the transcendence degree of the periods of an elliptic curve without complex multiplication is precisely 4, as predicted by the conjecture.

12.3 The case of 0-dimensional varieties

We go through all objects in the baby case of zero motives, i.e., the ones generated by 0-dimensional varieties.

Definition 12.3.1. Let $\mathrm{Pairs}^0 \subset \mathrm{Pairs}^{\mathrm{eff}}$ be the subdiagram of vertices (X, Y, n) with $\dim X = 0$. Let $\mathcal{MM}^0_{\mathrm{Nori}}$ be its diagram category with respect to the representation of $\mathrm{Pairs}^{\mathrm{eff}}$ given by singular cohomology with rational coefficients. Let $\mathrm{Var}^0 \subset \mathrm{Pairs}^0$ be the diagram defined by the opposite category of 0-dimensional k-varieties, or equivalently, the category of finite separable k-algebras.

If $\dim X = 0$, then $\dim Y = 0$ and X decomposes into a disjoint union of Y and $X \setminus Y$. Hence $H^*(X, Y, \mathbb{Q}) = H^*(X \setminus Y, \mathbb{Q})$ and it suffices to consider only vertices with $Y = \emptyset$. Moreover, all cohomology is concentrated in degree 0, and the pairs $(X, Y, 0)$ are all good and even very good. In particular, the multiplicative structure on Good restricts to the obvious multiplicative structure on Pairs^0 and Var^0.

We are always going to work with the multiplicative diagram Var^0 in the sequel.

Definition 12.3.2. Let $G^0_{\mathrm{mot}}(k)$ be the Tannaka dual of $\mathcal{MM}^0_{\mathrm{Nori}}$ and $\tilde{\mathbb{P}}^0(k)$ be the space of periods attached to $\mathcal{MM}^0_{\mathrm{Nori}}$.

The notation is a bit awkward because G^0 often denotes the connected component of unity of a group scheme G. Our $G^0_{\mathrm{mot}}(k)$ is very much *not* connected.

Our aim is to show that $G^0_{\mathrm{mot}}(k) = \mathrm{Gal}(\bar{k}/k)$ and $\tilde{\mathbb{P}}^0(k) \cong \bar{k}$ with the natural operation. In particular, the period conjecture (in any version) holds for 0-motives. This is essentially Grothendieck's treatment of Galois theory.

By construction of the coalgebra in Corollary 6.5.5, we have

$$A(\mathrm{Var}^0, H^0) = \mathrm{colim}_F \mathrm{End}(H^0|_F)^\vee \ ,$$

where F runs through a system of finite subdiagrams whose union is D.

We start with the case when F has a single vertex $\mathrm{Spec}K$, with K/k be a finite field extension, $Y = \mathrm{Spec}K$. The endomorphisms of the vertex are given by the elements of the Galois group $G = \mathrm{Gal}(K/k)$. We spell out $H^0(Y, \mathbb{Q})$. We have

$$Y(\mathbb{C}) = \mathrm{Mor}_k(\mathrm{Spec}\mathbb{C}, \mathrm{Spec}K) = \mathrm{Hom}_{k-\mathrm{alg}}(K, \mathbb{C})$$

the set of field embeddings of K into $\mathbb{C}$, viewed as a finite set with the discrete topology. Singular cohomology attaches a copy of $\mathbb{Q}$ to each point, hence

$$H^0(Y(\mathbb{C}), \mathbb{Q}) = \mathrm{Maps}(Y(\mathbb{C}), \mathbb{Q}) = \mathrm{Maps}(\mathrm{Hom}_{k-\mathrm{alg}}(K, \mathbb{C}), \mathbb{Q}).$$

As always, this is contravariant in Y, hence covariant in fields. The left operation of the Galois group G on K induces a left operation on $H^0(Y(\mathbb{C}), \mathbb{Q})$.

Let K/k be Galois of degree d. We compute the ring of endomorphisms of H^0 on the single vertex $\mathrm{Spec}K$ (see Definition 6.1.8)

$$E = \mathrm{End}(H^0|_{\mathrm{Spec}K}).$$

By definition, these are the endomorphisms of $H^0(\mathrm{Spec}K, \mathbb{Q})$ commuting with the operation of the Galois group. The set $Y(\mathbb{C})$ has a simply transitive action of

G. Hence, $\mathrm{Maps}(Y(\mathbb{C}),\mathbb{Q})$ is a free $\mathbb{Q}[G]^{op}$-module of rank 1. Its commutator E is then isomorphic to $\mathbb{Q}[G]$. This statement already makes the algebra structure on E explicit.

The diagram algebra does not change when we consider the diagram $\mathrm{Var}^0(K)$ containing all vertices of the form A with $A=\bigoplus_{i=1}^n K_i$, $K_i\subset K$.

There are two essential cases: If $K'\subset K$ is a subfield, we have a surjective map $Y(\mathbb{C})\to Y'(\mathbb{C})$. The compatibility condition with respect to this map implies that the value of the diagram endomorphism on K' is already determined by its value on K. If $A=K\bigoplus K$, then compatibility with the inclusion of the first and the second factor implies that the value of the diagram endomorphism on A is already determined by its value on K.

In more abstract language: The category $\mathrm{Var}^0(K)$ is equivalent to the category of finite G-sets. The algebra E is the group ring of the Galois group of this category under the representation $S\mapsto \mathrm{Maps}(S,\mathbb{Q})$.

Note that $K\otimes_k K=\bigoplus_\sigma K$, with σ running through the Galois group, is in $\mathrm{Var}^0(K)$. The category has fibre products. In the language of Definition 7.1.3, the diagram $\mathrm{Var}^0(K)$ has a commutative product structure (with trivial grading). By Proposition 7.1.5 and its proof, the diagram category is a tensor category, or equivalently, E carries a comultiplication.

We go through the construction in the proof of loc.cit. We start with an element of E and view it as an endomorphism of $H^0(Y\times Y(\mathbb{C}),\mathbb{Q})\cong H^0(Y(\mathbb{C}),\mathbb{Q})\otimes H^0(Y(\mathbb{C}),\mathbb{Q})$, hence as a tensor product of endomorphisms of $H^0(Y(\mathbb{C}),\mathbb{Q})$. The operation of $E=\mathbb{Q}[G]$ on $\mathrm{Maps}(Y(\mathbb{C})\times Y(\mathbb{C}),\mathbb{Q})$ is determined by the condition that it has to be compatible with the diagonal map $Y(\mathbb{C})\to Y(\mathbb{C})\times Y(\mathbb{C})$. This amounts to the diagonal embedding $\mathbb{Q}[G]\to\mathbb{Q}[G]\otimes\mathbb{Q}[G]$.

Thus we have shown that $E=\mathbb{Q}[G]$ as bialgebra. This means that

$$G_{\mathrm{mot}}(Y)=\mathrm{Spec}E^\vee=G$$

as a constant monoid (even group) scheme over $\mathbb{Q}$.

Passing to the limit over all K we get

$$G^0_{\mathrm{mot}}(k)=\mathrm{Gal}(\bar{k}/k)$$

as proalgebraic group schemes of dimension 0. As a byproduct, we see that the monoid attachted to $\mathcal{MM}^0_{\mathrm{Nori}}$ is a group, hence the category is rigid.

We now turn to periods, again in the case K/k finite and Galois. Note that $H^0_{\mathrm{dR}}(\mathrm{Spec}K)=K$ and the period isomorphism

$$\begin{aligned} K\otimes_k\mathbb{C}\to{}&\mathrm{Maps}(\mathrm{Hom}_{k-\mathrm{alg}}(K,\mathbb{C}),\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C},\\ v\mapsto{}&\quad(f\mapsto f(v))\end{aligned}$$

is the base change of the same map with values in K

$$K\otimes_k K\to\mathrm{Maps}(\mathrm{Hom}_{k-\mathrm{alg}}(K,K),\mathbb{Q})\otimes_{\mathbb{Q}}K.$$

In particular, all entries of the period matrix are in K. The space of formal periods of K is generated by symbols (ω,γ) where ω runs through a k-basis of K and γ through the set $\mathrm{Hom}_{k-\mathrm{alg}}(K,K)$ viewed as basis of a $\mathbb{Q}$-vector space. The relations coming from the operation of Galois group bring us down to a space of dimension $[K:k]$, hence the evaluation map is injective. Passing to the limit, we get

$$\tilde{\mathbb{P}}^0(k)=\bar{k}.$$

(We would get the same result by applying Proposition 12.1.10 and working only over $\bar{k}$.) The operation of $\mathrm{Gal}(\bar{k}/k)$ on $\tilde{\mathbb{P}}^0(k)$ is the natural one. More precisely, $g\in\mathrm{Gal}(\bar{k}/k)$ operates by applying g^{-1} because the operation is defined via γ, which is in the dual space. Note that the dimension of $\tilde{\mathbb{P}}^0(k)$ is also 0.

We have seen from general principles that the operation of $\mathrm{Gal}(\bar{k}/k)$ on $X^0(k)=\tilde{\mathbb{P}}^0(k)$ defines a torsor. In this case, we can trivialize it already over $\bar{k}$. We have

$$\mathrm{Mor}_k(\mathrm{Spec}\bar{k},X^0(k))=\mathrm{Hom}_{k-\mathrm{alg}}(\bar{k},\bar{k}).$$

By Galois theory, the operation of $\mathrm{Gal}(\bar{k}/k)$ on this set is simply transitive.

When we apply the same discussion to the ground field $\bar{k}$, we get $G^0_{\mathrm{mot}}(\bar{k})=\mathrm{Gal}(\bar{k}/\bar{k})$ and $\tilde{\mathbb{P}}^0(\bar{k})=\bar{k}$. We see that the (formal) period algebra has not changed, but the motivic Galois group has. It is still true that $\mathrm{Spec}\bar{k}$ is a torsor under the motivic Galois group, but now viewed as $\bar{k}$-schemes, where both consist of a single point!

Part IV

Examples

Chapter 13

Elementary examples

13.1 Logarithms

In this section, we give one of the most simple examples for a cohomological period in the sense of Chap. 9. Let

$$X := \mathbb{A}^1_{\mathbb{Q}} \setminus \{0\} = \operatorname{Spec} \mathbb{Q}[t, t^{-1}]$$

be the affine line with the point 0 deleted and

$$D := \{1, \alpha\} \quad \text{with} \quad \alpha \neq 0, 1$$

a divisor on X. The singular homology of the pair $(X(\mathbb{C}), D(\mathbb{C})) = (\mathbb{C}^\times, \{1, \alpha\})$ is generated by a small loop σ turning counter-clockwise around 0 once and the interval $[1, \alpha]$. In order to compute the algebraic de Rham cohomology of (X, D), we first note that by Section 3.2, $H^\bullet_{\mathrm{dR}}(X, D)$ is the cohomology of the complex of global sections of the cone complex $\widetilde{\Omega}^\bullet_{X,D}$, since X is affine and the sheaves $\widetilde{\Omega}^p_{X,D}$ are quasi-coherent, hence acyclic for the global section functor. We spell out the complex $\Gamma(X, \widetilde{\Omega}^\bullet_{X,D})$ in detail

$$\begin{array}{l} 0 \\ \uparrow \\ \Gamma(X, \widetilde{\Omega}^1_{X,D}) = \Gamma\big(X, \Omega^1_X \oplus \bigoplus_j i_* \mathcal{O}_{D_j}\big) = \mathbb{Q}[t, t^{-1}]dt \oplus \underset{1}{\mathbb{Q}} \oplus \underset{\alpha}{\mathbb{Q}} \\ \uparrow d \\ \Gamma(X, \mathcal{O}_X) = \mathbb{Q}[t, t^{-1}] \end{array}$$

and observe that the evaluation map

$$\mathbb{Q}[t,t^{-1}] \twoheadrightarrow \underset{1}{\mathbb{Q}} \oplus \underset{\alpha}{\mathbb{Q}}$$
$$f(t) \mapsto \big(f(1), f(\alpha)\big)$$

is surjective with kernel

$$(t-1)(t-\alpha)\mathbb{Q}[t,t^{-1}] = \mathrm{span}_{\mathbb{Q}}\{t^{n+2} - (\alpha+1)t^{n+1} + \alpha t^n \mid n \in \mathbb{Z}\}.$$

Differentiation maps this kernel to

$$\mathrm{span}_{\mathbb{Q}}\{(n+2)t^{n+1} - (n+1)(\alpha+1)t^n - n\alpha t^{n-1} \mid n \in \mathbb{Z}\}dt.$$

Therefore we get

$$\begin{aligned} H^1_{\mathrm{dR}}(X,D) &= \Gamma(X_0, \widetilde{\Omega}_{X,D}) \,/\, \Gamma(X, \mathcal{O}_X) \\ &= \mathbb{Q}[t,t^{-1}]dt \oplus \underset{1}{\mathbb{Q}} \oplus \underset{\alpha}{\mathbb{Q}} \,/\, d(\mathbb{Q}[t,t^{-1}]) \\ &= \mathbb{Q}[t,t^{-1}]dt / \mathrm{span}_{\mathbb{Q}}\{(n+2)t^{n+1} - (n+1)(\alpha+1)t^n - n\alpha t^{n-1}\}dt. \end{aligned}$$

By the last line, we see that the class of $t^n dt$ in $H^1_{\mathrm{dR}}(X,D)$ for $n \neq -1$ is linearly dependent of

- $t^{n-1}dt$ and $t^{n-2}dt$, and
- $t^{n+1}dt$ and $t^{n+2}dt$,

hence we see by induction that $\frac{dt}{t}$ and dt generate $H^1_{\mathrm{dR}}(X,D)$. Therefore, $H^1_{\mathrm{dR}}(X,D)$ is spanned by

$$\frac{dt}{t} \quad \text{and} \quad \frac{1}{\alpha-1}dt.$$

We obtain the following period matrix P for $H^1(X,D)$:

	$\frac{1}{\alpha-1}dt$	$\frac{dt}{t}$
$[1,\alpha]$	1	$\log\alpha$
σ	0	$2\pi i$

(13.1)

In Section 7.4.3 we have seen how the torsor structure on the periods of (X,D) is given by a triple coproduct Δ in terms of the matrix P:

$$P_{ij} \mapsto \sum_{k,\ell} P_{ik} \otimes P^{-1}_{k\ell} \otimes P_{\ell j}\ .$$

The inverse period matrix in this example is given by:

$$P^{-1} = \begin{pmatrix} 1 & \frac{-\log\alpha}{2\pi i} \\ 0 & \frac{1}{2\pi i} \end{pmatrix}$$

and thus we get for the triple coproduct of the most important entry $\log(\alpha)$

$$\Delta(\log\alpha) = \log\alpha \otimes \tfrac{1}{2\pi i} \otimes 2\pi i - 1 \otimes \tfrac{\log\alpha}{2\pi i} \otimes 2\pi i + 1 \otimes 1 \otimes \log\alpha\ . \tag{13.2}$$

13.2 More Logarithms

In this section, we describe a variant of the cohomological period in the previous section. We define

$$D_0 := \{1, \alpha, \beta\} \quad \text{with} \quad \alpha \neq 0, 1 \quad \text{and} \quad \beta \neq 0, 1, \alpha,$$

but keep $X := \mathbb{A}^1_{\mathbb{Q}} \setminus \{0\} = \operatorname{Spec} \mathbb{Q}[t, t^{-1}]$.

Then, $H_1^{\text{sing}}(X, D; \mathbb{Q})$ is generated by the loop σ from the first example and the intervals $[1, \alpha]$ and $[\alpha, \beta]$. Hence, the differential forms $\frac{dt}{t}$, dt and $2t\,dt$ give a basis of $H^1_{\text{dR}}(X, D)$: If they were linearly dependent, the period matrix P would not be of full rank

	$\frac{dt}{t}$	dt	$2t\,dt$
σ	$2\pi i$	0	0
$[1, \alpha]$	$\log \alpha$	$\alpha - 1$	$\alpha^2 - 1$
$[\alpha, \beta]$	$\log\left(\frac{\beta}{\alpha}\right)$	$\beta - \alpha$	$\beta^2 - \alpha^2$.

We observe that $\det P = 2\pi i(\alpha - 1)(\beta - \alpha)(\beta - 1) \neq 0$.

We have

$$P^{-1} = \begin{pmatrix} \frac{1}{2\pi i} & 0 & 0 \\ \frac{\log\beta(\alpha^2-1)-\log\alpha(\beta^2-1)}{2\pi i(\beta-\alpha)(\alpha-1)(\beta-1)} & \frac{\alpha+\beta}{(\alpha-1)(\beta-1)} & \frac{\alpha+1}{(\alpha-\beta)(\beta-1)} \\ \frac{-\log\beta(\alpha-1)+\log\alpha(\beta-1)}{2\pi i(\beta-\alpha)(\alpha-1)(\beta-1)} & \frac{-1}{(\alpha-1)(\beta-1)} & \frac{-1}{(\alpha-\beta)(\beta-1)} \end{pmatrix},$$

and therefore we get for the triple coproduct for the entry $\log(\alpha)$:

$$\begin{aligned}
\Delta(\log\alpha) = &\log\alpha \otimes \frac{1}{2\pi i} \otimes 2\pi i \\
&+ (\alpha - 1) \otimes \frac{-\log\beta(\alpha^2 - 1) + \log\alpha(\beta^2 - 1)}{2\pi i(\beta - \alpha)(\alpha - 1)(\beta - 1)} \otimes 2\pi i \\
&+ (\alpha - 1) \otimes \frac{\alpha + \beta}{(\alpha - 1)(\beta - 1)} \otimes \log\alpha \\
&+ (\alpha - 1) \otimes \frac{\alpha + 1}{(\alpha - \beta)(\beta - 1)} \otimes \log\left(\frac{\beta}{\alpha}\right) \\
&+ (\alpha^2 - 1) \otimes \frac{\log\beta(\alpha - 1) - \log\alpha(\beta - 1)}{2\pi i(\beta - \alpha)(\alpha - 1)(\beta - 1)} \otimes 2\pi i \\
&+ (\alpha^2 - 1) \otimes \frac{-1}{(\alpha - 1)(\beta - 1)} \otimes \log\alpha \\
&+ (\alpha^2 - 1) \otimes \frac{-1}{(\alpha - \beta)(\beta - 1)} \otimes \log\left(\frac{\beta}{\alpha}\right) \\
= &\log\alpha \otimes \frac{1}{2\pi i} \otimes 2\pi i - 1 \otimes \frac{\log\alpha}{2\pi i} \otimes 2\pi i + 1 \otimes 1 \otimes \log\alpha \ .
\end{aligned}$$

Compare this with Equation 13.2 !

13.3 Quadratic Forms

Let

$$\begin{array}{rccc} Q(\underline{x}): & \mathbb{Q}^3 & \longrightarrow & \mathbb{Q} \\ & \underline{x} = (x_0, x_1, x_2) & \mapsto & \underline{x}\, A\, \underline{x}^T \end{array}$$

be a quadratic form with $A \in \mathbb{Q}^{3\times 3}$ being a regular, symmetric matrix.

The zero-locus of $Q(\underline{x})$

$$\overline{X} := \{\underline{x} \in \mathbb{P}^2(\mathbb{Q}) \,|\, Q(\underline{x}) = 0\}$$

is a *quadric* or non-degenerate *conic*. We are interested in its affine piece

$$X := \overline{X} \cap \{x_0 \neq 0\} \subset \mathbb{Q}^2 \subset \mathbb{P}^2(\mathbb{Q}).$$

We show that we can assume $Q(\underline{x})$ to be of a particular nice form. A non-zero vector $v \in \mathbb{Q}^3$ is called *Q-anisotropic*, if $Q(v) \neq 0$. Since $\operatorname{char} \mathbb{Q} \neq 2$, there exist such vectors, just suppose the contrary:

$$\begin{array}{lcr} Q(1,0,0) = 0 & \text{gives} & A_{11} = 0, \\ Q(0,1,0) = 0 & \text{gives} & A_{22} = 0, \\ Q(1,1,0) = 0 & \text{gives} & 2 \cdot A_{12} = 0 \end{array}$$

and A would be degenerate. In particular

$$Q(1,\lambda,0) = Q(1,0,0) + 2\lambda Q(1,1,0) + \lambda^2 Q(0,1,0)$$

will be different form zero for almost all $\lambda \in \mathbb{Q}$. Hence, we can assume that $(1,0,0)$ is anisotropic after applying a coordinate transformation of the form

$$x_0' := x_0, \quad x_1' := -\lambda x_0 + x_1, \quad x_2' := x_2.$$

After another affine change of coordinates, we can also assume that A is a diagonal matrix. An inspection reveals that we can choose this coordinate transformation such that the x_0-coordinate is left unaltered. (Just take for e_1 the anisotropic vector $(1,0,0)$ in the proof.) Such a transformation does not change the isomorphism type of X, and we can take X to be cut out by an equation of the form

$$ax^2 + by^2 = 1 \quad \text{for} \quad a, b \in \mathbb{Q}^\times$$

with affine coordinates $x := \frac{x_1}{x_0}$ and $y := \frac{x_2}{x_0}$. Since X is affine, the sheaves Ω_X^p are acyclic, hence we can compute its algebraic de Rham cohomology by

$$H^\bullet_{\mathrm{dR}}(X) = \mathrm{h}^\bullet \Gamma(X, \Omega_X^\bullet),$$

so we write down the complex $\Gamma(X, \Omega_X^\bullet)$ in detail

$$\begin{array}{l} 0 \\ \uparrow \\ \Gamma(X, \Omega_X^1) = \mathbb{Q}[x,y]/(ax^2+by^2-1)\{dx, dy\} \,/\, (axdx + bydy) \\ {\scriptstyle d}\uparrow \\ \Gamma(X, \mathcal{O}_X) = \mathbb{Q}[x,y]/(ax^2+by^2-1). \end{array}$$

Obviously, $H^1_{\mathrm{dR}}(X)$ can be presented with generators $x^n y^m dx$ and $x^n y^m dy$ for $m, n \in \mathbb{N}_0$ modulo numerous relations. Using $axdx + bydy = 0$, we get

- $y^m\, dy = d\, \frac{y^{m+1}}{m+1} \sim 0$
- $x^n\, dx = d\, \frac{x^{n+1}}{n+1} \sim 0$

$n \geq 1$
- $x^n y^m\, dy = \frac{-n}{m+1} x^{n-1} y^{m+1}\, dx + d\, \frac{x^n y^{m+1}}{m+1}$
  $\sim \frac{-n}{m+1} x^{n-1} y^{m+1}\, dx \quad \text{for} \quad n \geq 1, m \geq 0$
- $x^n y^{2m}\, dx = x^n \left(\frac{1-ax^2}{b}\right)^m dx \sim 0$
- $x^n y^{2m+1}\, dx = x^n \left(\frac{1-ax^2}{b}\right)^m y\, dx$
- $xy\, dx = \frac{-x^2}{2}\, dy + d\, \frac{x^2 y}{2}$
  $\sim \frac{by^2-1}{2a}\, dy$
  $= \frac{b}{2a} y^2\, dy - \frac{1}{2a}\, dy \sim 0$

$n \geq 2$
- $x^n y\, dx = \frac{-b}{a} x^{n-1} y^2\, dy + x^n y\, dx + \frac{b}{a} x^{n-1} y^2\, dy$
  $= \frac{-b}{a} x^{n-1} y^2\, dy + \frac{x^{n-1} y}{2a}\, d(ax^2 + by^2 - 1)$
  $= \frac{-b}{a} x^{n-1} y^2\, dy + d\left(\frac{(x^{n-1}y)(ax^2+by^2-1)}{2a}\right)$
  $\sim \frac{-b}{a} x^{n-1} y^2\, dy$
  $= \left(x^{n+1} - \frac{x^{n-1}}{a}\right) dy$
  $= \left(-(n+1)x^n y + \frac{n-1}{a} x^{n-2} y\right) dx + d\left(x^{n+1} y - \frac{x^{n-1}}{a} y\right)$

$\Rightarrow\ x^n y\, dx \sim \frac{n-1}{(n+2)a} x^{n-2} y\, dx \quad \text{for} \quad n \geq 2.$

Thus we see that all generators are linearly dependent of $y\, dx$

$$H^1_{\mathrm{dR}}(X) = \mathrm{h}^1 \Gamma(X, \Omega_X^\bullet) = \mathbb{Q}\ y\, dx.$$

What about the base change to $\mathbb{C}$ of X? We use the symbol $\sqrt{}$ for the principal branch of the square root. Over $\mathbb{C}$, the change of coordinates

$$u := \sqrt{a}x - i\sqrt{b}y, \quad v := \sqrt{a}x + i\sqrt{b}y$$

gives

$$\begin{aligned} X &= \mathrm{Spec}\mathbb{C}[x,y]/(ax^2+by^2-1) \\ &= \mathrm{Spec}\mathbb{C}[u,v]/(uv-1) \\ &= \mathrm{Spec}\mathbb{C}[u,u^{-1}] \\ &= \mathbb{A}^1_{\mathbb{C}} \setminus \{0\}. \end{aligned}$$

Hence, the first singular homology group $H_\bullet^{\mathrm{sing}}(X,\mathbb{Q})$ of X is generated by

$$\sigma : [0,1] \to X(\mathbb{C}), s \mapsto u = e^{2\pi i s},$$

i.e., a circle with radius 1 turning counter-clockwise around $u=0$ once.

The period matrix consists of a single entry

$$\begin{aligned} \int_\sigma y\,dx &= \int_\sigma \frac{v-u}{2i\sqrt{b}}\, d\,\frac{u+v}{2\sqrt{a}} \\ &\overset{\text{Stokes}}{=} \int_\sigma \frac{v\,du - u\,dv}{4i\sqrt{ab}} \\ &= \frac{1}{2i\sqrt{ab}} \int_\sigma \frac{du}{u} \\ &= \frac{\pi}{\sqrt{ab}}. \end{aligned}$$

The denominator squared is nothing but the discriminant of the quadratic form Q

$$\operatorname{disc} Q := \det A \in \mathbb{Q}^\times/{\mathbb{Q}^\times}^2.$$

This is an important invariant, that distinguishes some, but not all isomorphism classes of quadratic forms. Since $\operatorname{disc} Q$ is well-defined modulo $(\mathbb{Q}^\times)^2$, it makes sense to write

$$H^1_{\mathrm{dR}}(X) = \mathbb{Q}\,\frac{\pi}{\sqrt{\operatorname{disc} Q}} \subset H^1_{\mathrm{sing}}(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}.$$

13.4 Elliptic Curves

In this section, we give another well-known example for a cohomological period in the sense of Chap. 9.

An *elliptic curve* E is a one-dimensional non-singular complete and connected group variety over a field k, together with the origin 0, a k-rational point. An elliptic curve has genus $g=1$, where the genus g of a smooth projective curve is defined as

$$g := \dim_k \Gamma(E, \Omega^1_E)\,.$$

We refer to the book [Sil] of Silverman for the theory of elliptic curves, but try to be self-contained in the following. For simplicity, we assume $k=\mathbb{Q}$. It can be shown, using the Riemann-Roch theorem that such an elliptic curve E can be given as the zero locus in $\mathbb{P}^2(\mathbb{Q})$ of a Weierstraß equation

$$Y^2 Z = 4X^3 - g_2 X Z^2 - g_3 Z^3 \tag{13.3}$$

with Eisenstein series coefficients $g_2 = 60G_4, g_3 = 140G_6$ and projective coordinates X, Y and Z.

By the classification of compact, oriented real surfaces, the base change of E to $\mathbb{C}$ gives us a complex torus E^{an}, i.e., an isomorphism

$$E^{\mathrm{an}} \cong \mathbb{C}/\Lambda_{\omega_1,\,\omega_2} \tag{13.4}$$

in the complex analytic category with

$$\Lambda_{\omega_1,\,\omega_2} := \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$$
$$\text{for } \omega_1, \omega_2 \in \mathbb{C} \text{ linearly independent over } \mathbb{R},$$

being a lattice of full rank. Thus, all elliptic curves over $\mathbb{C}$ are diffeomorphic to the standard torus $S^1 \times S^1$, but carry different complex structures as the parameter $\tau := \omega_2/\omega_1$ varies. We can describe the isomorphism (13.4) quite explicitly using periods. Let α and β be a basis of

$$H_1^{\mathrm{sing}}(E, \mathbb{Z}) = H_1^{\mathrm{sing}}(S^1 \times S^1, \mathbb{Z}) = \mathbb{Z}\,\alpha \,\oplus\, \mathbb{Z}\,\beta.$$

The $\mathbb{Q}$-vector space $\Gamma(E, \Omega^1_E)$ is spanned by the holomorphic differential

$$\omega = \frac{dX}{Y}.$$

The map

$$\begin{aligned} E^{\mathrm{an}} &\to \quad \mathbb{C}/\Lambda_{\omega_1,\,\omega_2} \\ P &\mapsto \int_O^P \omega \ \text{ modulo } \Lambda_{\omega_1,\,\omega_2} \end{aligned} \tag{13.5}$$

then gives the isomorphism of Equation 13.4. Here $O = [0:1:0]$ denotes the group theoretic origin in E. The integrals

$$\omega_1 := \int_\alpha \omega \quad \text{and} \quad \omega_2 := \int_\beta \omega$$

are called the periods of E. Up to a $\mathbb{Z}$-linear change of basis, they are precisely the above generators of the lattice $\Lambda_{\omega_1,\,\omega_2}$.

The inverse map $\mathbb{C}/\Lambda_{\omega_1,\,\omega_2} \to E^{\mathrm{an}}$ for the isomorphism (13.5) can be described in terms of the Weierstraß $\wp$-function of the lattice $\Lambda := \Lambda_{\omega_1,\,\omega_2}$

$$\wp(z) = \wp(z, \Lambda) := \frac{1}{z^2} + \sum_{\substack{\omega \in \Lambda \\ \omega \neq 0}} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$$

and takes the form

$$\begin{aligned}\mathbb{C}/\Lambda_{\omega_1,\,\omega_2} &\to E^{\mathrm{an}} \subset \mathbb{C}P_{\mathrm{an}}{}^2\\ z &\mapsto [\wp(z) : \wp'(z) : 1], \Lambda_{\omega_1,\omega_2} \mapsto (0:1:0).\end{aligned}$$

Note that under the natural projection $\pi : \mathbb{C} \to \mathbb{C}/\Lambda_{\omega_1,\,\omega_2}$ any meromorphic function f on the torus $\mathbb{C}/\Lambda_{\omega_1,\,\omega_2}$ lifts to a doubly-periodic function $\pi^* f$ on the complex plane $\mathbb{C}$ with periods ω_1 and ω_2

$$f(x + n\omega_1 + m\omega_2) = f(x) \quad \text{for all} \quad n, m \in \mathbb{Z} \quad \text{and} \quad x \in \mathbb{C}.$$

This example is possibly the origin of the "period" terminology.

The defining coefficients G_4, G_6 of E can be recovered from $\Lambda_{\omega_1,\,\omega_2}$ by the Eisenstein series

$$G_{2k} := \sum_{\substack{\omega\in\Lambda\\ \omega\neq 0}} \omega^{-2k} \quad \text{for} \quad k = 2, 3.$$

Therefore, the periods ω_1 and ω_2 determine the elliptic curve E uniquely. However, they are not invariants of E, since they depend on the chosen Weierstraß equation of E. A change of coordinates which preserves the shape of (13.3), must be of the form

$$X' = u^2 X, \quad Y' = u^3 Y, \quad Z' = Z \quad \text{for} \quad u \in \mathbb{Q}^\times.$$

In the new parametrization X', Y', Z', we have

$$\begin{aligned}&G_4' = u^4 G_4, \quad G_6' = u^6 G_6,\\ &\omega' = u^{-1}\omega\\ &\omega_1' = u^{-1}\omega_1 \quad \text{and} \quad \omega_2' = u^{-1}\omega_2.\end{aligned}$$

Hence, $\tau = \omega_2/\omega_1$ is a better invariant of the isomorphism class of E. The value of the j-function (a modular function)

$$j(\tau) = 1728\frac{g_2^3}{g_2^3 - 27g_3^2} = q^{-1} + 744 + 196884q + \cdots \ (q = \exp(2\pi i\tau)$$

on τ indeed distinguishes non-isomorphic elliptic curves E over $\mathbb{C}$:

$$E \cong E' \text{ if and only if } j(E) = j(E')\,.$$

Hence, the moduli space of elliptic curves over $\mathbb{C}$ is the affine line.

A similar result holds over any algebraically closed field K of characteristic different from $2, 3$. For fields K that are not algebraically closed, the set of K-isomorphism classes of elliptic curves isomorphic over $\bar{K}$ to a fixed curve E/K is the Weil-Châtelet group of E [Sil], an infinite group for K a number field.

However, E has two more cohomological periods which are also called *quasi-periods*. In section 13.5, we will prove that the meromorphic differential form

$$\eta := X\frac{dX}{Y}$$

spans $H^1_{\mathrm{dR}}(E)$ together with $\omega = \frac{dX}{Y}$, i.e., modulo exact forms this form is a generator of $H^1(E, \mathcal{O}_E)$ in the Hodge decomposition. Like ω corresponds to dz under (13.5), η corresponds to $\wp(z)dz$. The quasi-periods then are

$$\eta_1 := \int_\alpha \eta, \quad \eta_2 := \int_\beta \eta.$$

We obtain the following period matrix for E:

$$\begin{array}{c|cc} & \frac{dX}{Y} & X\frac{dX}{Y} \\ \hline \alpha & \omega_1 & \eta_1 \\ \beta & \omega_2 & \eta_2 \end{array} \tag{13.6}$$

Lemma 13.4.1. *One has the Legendre relation (negative determinant of period matrix)*

$$\omega_2\eta_1 - \omega_1\eta_2 = \pm 2\pi i.$$

Proof. Consider the Weierstraß ζ-function [Sil]

$$\zeta(z) := \frac{1}{z} + \sum_{\substack{\omega\in\Lambda \\ \omega\neq 0}} \left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right).$$

It satisfies $\zeta'(z) = -\wp(z)$. Since $\zeta'(z) = -\wp(z)$ and $\wp$ is periodic, we have that $\eta(w) = \zeta(z+w) - \zeta(z)$ is independent of z. Hence, the complex path integral counter-clockwise around the fundamental domain centered at some point $a \notin \Lambda_{\omega_1,\omega_2}$ yields

$$\begin{aligned} 2\pi i &= \int_a^{a+\omega_1} \zeta(z)dz + \int_{a+\omega_1}^{a+\omega_1+\omega_2} \zeta(z)dz - \int_{a+\omega_2}^{a+\omega_1+\omega_2} \zeta(z)dz - \int_a^{a+\omega_2} \zeta(z)dz \\ &= \int_a^{a+\omega_2} \left(\zeta(z+\omega_1) - \zeta(z)\right) dz - \int_a^{a+\omega_1} \left(\zeta(z+\omega_2) - \zeta(z)\right) dz \\ &= \omega_2\eta_1 - \omega_1\eta_2, \end{aligned}$$

where $\eta_i = \eta(\omega_i)$. □

In the following two examples, all four periods are calculated and yield Γ-values besides $\sqrt{\pi}$, π and algebraic numbers. Such period expressions for elliptic curves with complex multiplication are nowadays called the Chowla-Lerch-Selberg formula, after Lerch [L] and Chowla-Selberg [CS]. See also the thesis of B. Gross [Gr].

Example 13.4.2. Let E be the elliptic curve with $G_6 = 0$ and affine equation $Y^2 = 4X^3 - 4X$.Then one has [Wa]

$$\omega_1 = \int_1^\infty \frac{dx}{\sqrt{x^3 - x}} = \frac{1}{2} B\left(\frac{1}{4}, \frac{1}{2}\right) = \frac{\Gamma(1/4)^2}{2^{3/2}\pi^{1/2}}, \quad \omega_2 = i\omega_1,$$

and

$$\eta_1 = \frac{\pi}{\omega_1} = \frac{(2\pi)^{3/2}}{\Gamma(1/4)^2}, \quad \eta_2 = -i\eta_1.$$

E has complex multiplication with ring $\mathbb{Z}[i]$ (Gaußian integers).

Example 13.4.3. Look at the elliptic curve E with $G_4 = 0$ and affine equation $Y^2 = 4X^3 - 4$.Then one has [Wa]

$$\omega_1 = \int_1^\infty \frac{dx}{\sqrt{x^3 - 1}} = \frac{1}{3} B\left(\frac{1}{6}, \frac{1}{2}\right) = \frac{\Gamma(1/3)^3}{2^{4/3}\pi}, \quad \omega_2 = \rho\omega_1,$$

($\rho = \frac{-1+\sqrt{-3}}{2}$) and

$$\eta_1 = \frac{2\pi}{\sqrt{3}\omega_1} = \frac{2^{7/3}\pi^2}{3^{1/2}\Gamma(1/3)^3}, \quad \eta_2 = \rho^2\eta_1.$$

E has complex multiplication with ring $\mathbb{Z}[\rho]$ (Eisenstein numbers).

Both of these example have complex multiplication. As we have explained in Example 12.2.15, G. V. Chudnovsky [Ch] has proved that $\mathrm{trdeg}_{\mathbb{Q}}\mathbb{P}(E) = 2$ if E is an elliptic curve with complex multiplication. This means that ω_1 and π are both transcendant and algebraically independent, and ω_2, η_1 and η_2 are algebraically dependent. The transcendance of ω_1 for all elliptic curves is a theorem of Th. Schneider [S]. Of course, the transcendance of π is Lindemann's theorem.

For elliptic without complex multiplication it is conjectured that the Legendre relation is the only algebraic relation among the 5 period numbers ω_1, ω_2, η_1, η_2 and π. But this is still open.

13.5 Periods of 1-forms on arbitrary curves

Let X be a smooth, projective curve of geometric genus g over k, where $k \subset \mathbb{C}$. We denote the associated analytic space by X^{an}.

In the classical literature, different types of meromorphic differential forms on X^{an} and their periods were considered. The survey of Messing [Me] gives a historical account, see also [GH, pg. 459]. In this section, we mention these notions, translate them into a modern language, and relate them to cohomological periods in the sense of Chap. 9, since the terminology is still used in many areas of mathematics, e.g., in transcendence theory.

A *meromorphic* 1-form ω on X^{an} is locally given by $f(z)dz$, where f is meromorphic. Any meromorphic function has poles in a discrete and finite set D in X^{an}. Using a local coordinate z at a point $P \in X^{\mathrm{an}}$, we can write $f(z) = z^{-\nu(P)} \cdot h(z)$, where h is holomorphic and $h(P) \neq 0$. In particular, a meromorphic 1-form is a section of the holomorphic line bundle $\Omega^1_{X^{\mathrm{an}}}(kD)$ for some integer $k \geq 0$. We say that ω has *logarithmic poles*, if $\nu(P) \leq 1$ at all points of D. A *rational* 1-form is a section of the line bundle $\Omega^1_X(kD)$ on X. In particular, we can speak of rational 1-forms defined over k, if X is defined over k.

Proposition 13.5.1. *Meromorphic* 1*-forms on* X^{an} *are the same as rational* 1*-forms on* X*.*

Proof. Since X is projective, and meromorphic 1-forms are section of the line bundle $\Omega^1_X(kD)$ for some integer $k \geq 0$, this follows from Serre's GAGA principle [Se1]. □

In the following, we will mostly use the analytic language of meromorphic forms.

Definition 13.5.2. A *differential of the first kind* on X^{an} is a holomorphic 1-form (hence closed). A *differential of the second kind* is a closed meromorphic 1-form with vanishing residues. A *differential of the third kind* is a closed meromorphic 1-form with at most logarithmic poles along some divisor $D^{\mathrm{an}} \subset X^{\mathrm{an}}$.

Note that forms of the second and third kind include forms of the first kind.

Theorem 13.5.3. *Any meromorphic* 1*-form* ω *on* X^{an} *can be written as*

$$\omega = df + \omega_1 + \omega_2 + \omega_3,$$

where df *is an exact form,* ω_1 *is of the first kind,* ω_2 *is of the second kind, and* ω_3 *is of the third kind. This decomposition is unique up to exact forms, if* ω_3 *is chosen not to be of second kind, and* ω_2 *not to be of the first kind.*

The first de Rham cohomology of X^{an} *is given by*

$$H^1_{\mathrm{dR}}(X^{\mathrm{an}}, \mathbb{C}) \cong \frac{1-\textit{forms of the second kind}}{\textit{exact forms}}$$

The inclusion of differentials of the first kind into differentials of the second kind is given by the Hodge filtration

$$H^0(X^{\mathrm{an}}, \Omega^1_{X^{\mathrm{an}}}) \subset H^1_{\mathrm{dR}}(X^{\mathrm{an}}, \mathbb{C}).$$

For differentials of the third kind, we note that

$$\begin{aligned} F^1H^1(X^{\mathrm{an}} \smallsetminus D^{\mathrm{an}}, \mathbb{C}) &= H^0(X^{\mathrm{an}}, \Omega^1_{X^{\mathrm{an}}}\langle D^{\mathrm{an}} \rangle) \\ &\cong \frac{1-\textit{forms of the third kind with poles in } D^{\mathrm{an}}}{\textit{exact forms} + \textit{forms of the first kind}}. \end{aligned}$$

Proof. Let ω be a meromorphic 1-form on X^{an}. The residue theorem states that the sum of the residues of ω is zero. Suppose that ω has poles in the finite subset $D \subset X^{\mathrm{an}}$. Then look at the exact sequence

$$0 \to H^0(X^{\mathrm{an}}, \Omega^1_{X^{\mathrm{an}}}) \to H^0(X^{\mathrm{an}}, \Omega^1_{X^{\mathrm{an}}}\langle D\rangle) \xrightarrow{\mathrm{Res}} \bigoplus_{P \in D} \mathbb{C} \xrightarrow{\Sigma} H^1(X^{\mathrm{an}}, \Omega^1_{X^{\mathrm{an}}}).$$

It shows that there exists a 1-form $\omega_3 \in H^0(X^{\mathrm{an}}, \Omega^1_{X^{\mathrm{an}}}(\log D))$ of the third kind which has the same residues as ω. In addition, the form $\omega - \omega_3$ is of the second kind, i.e., it has perhaps poles but no residues. Now, look at the meromorphic de Rham complex

$$\Omega^0_{X^{\mathrm{an}}}(*) \xrightarrow{d} \Omega^1_{X^{\mathrm{an}}}(*)$$

of all meromorphic differential forms on X^{an} (with arbitrary poles along arbitrary divisors). The cohomology sheaves of it are given by [GH, pg. 457]

$$\mathcal{H}^0\Omega^\bullet_{X^{\mathrm{an}}}(*) = \mathbb{C}, \quad \mathcal{H}^1\Omega^\bullet_{X^{\mathrm{an}}}(*) = \bigoplus_{P \in X^{\mathrm{an}}} \mathbb{C}\,.$$

These isomorphisms are induced by the inclusion of constant functions and the residue map respectively. With the help of the spectral sequence abutting to $H^*(X^{\mathrm{an}}, \Omega^*_{X^{\mathrm{an}}}(*))$ [GH, pg. 458], one obtains an exact sequence

$$0 \to H^1_{\mathrm{dR}}(X^{\mathrm{an}}, \mathbb{C}) \to \frac{H^0(X^{\mathrm{an}}, \Omega^1_{X^{\mathrm{an}}}(*))}{\text{exact forms}} \xrightarrow{\mathrm{Res}} \bigoplus_{P \in X^{\mathrm{an}}} \mathbb{C},$$

and the claim follows. The identification with $F^1H^1(X^{\mathrm{an}} \smallsetminus D^{\mathrm{an}}, \mathbb{C})$ is by definition of the Hodge filtration. □

Corollary 13.5.4. *In the algebraic category, if X is defined over $k \subset \mathbb{C}$, we have that*

$$H^1_{\mathrm{dR}}(X) \cong \frac{1-\textit{rational forms of the second kind over } k}{\textit{exact forms}}$$

We can now define periods of differentials of the first, second, and third kind.

Definition 13.5.5. Periods of the n-th kind (n=1,2,3) in the sense of Definition 9.1.1 are periods of differentials ω of the n-th kind, i.e., integrals

$$\int_\gamma \omega\,,$$

where γ is a closed path avoiding the poles of D for $n = 2$ and which is contained in $X \setminus D$ for $n = 3$.

Usually, in the literature periods of 1-forms of the first kind are called periods, and periods of 1-forms of the second kind and not of the first kind are called quasi-periods.

Theorem 13.5.6. *Let X be a smooth, projective curve over k as above.*

Periods of the second kind (and hence also periods of the first kind) are cohomological periods in the sense of 9.3.1 of the first cohomology group $H^1(X)$. Periods of the third kind with poles along D are periods of the cohomology group $H^1(U)$, where $U = X \setminus D$.

Every period of any smooth, quasi-projective curve U over k is of the first, second or third kind on a smooth compacification X of U.

Proof. The first assertion follows from the definition of periods of the n-th kind, since differentials of the n-th kind represent cohomology classes in $H^1(X)$ for $n = 1, 2$ and in $H^1(X \setminus D)$ for $n = 3$. If U is a smooth, quasiprojective curve over k, then we choose a smooth compactification X and the assertion follows from the exact sequence

$$0 \to H^0(X^{\mathrm{an}}, \Omega^1_{X^{\mathrm{an}}}) \to H^0(X^{\mathrm{an}}, \Omega^1_{X^{\mathrm{an}}}\langle D\rangle) \stackrel{\mathrm{Res}}{\to} \bigoplus_{P \in D} \mathbb{C} \stackrel{\Sigma}{\to} H^1(X^{\mathrm{an}}, \Omega^1_{X^{\mathrm{an}}}).$$

by Theorem 13.5.3. □

Examples 13.5.7. In the elliptic curve case of section 13.4, $\omega = \frac{dX}{Y}$ is 1-form of the first kind, and $\eta = X\frac{dX}{Y}$ a 1-form of the second kind, but not of the first kind. Some periods (and quasi-periods) of this sort were computed in the two Examples 13.4.2,13.4.3. For an example of the third kind, look at $X = \mathbb{P}^1$ and $D = \{0, \infty\}$ where $\omega = \frac{dz}{z}$ is a generator with period $2\pi i$. Compare this with section 13.1 where also logarithms occur as periods. For periods of differentials of the third kind on modular and elliptic curves see [Br].

Finally, let X be a smooth, projective curve of genus g defined over $\mathbb{Q}$. Then there is a $\mathbb{Q}$-basis $\omega_1, \ldots, \omega_g, \eta_1, \ldots, \eta_g$ of $H^1_{\mathrm{dR}}(X)$, where the ω_i are of the first kind and the η_j of the second kind. One may choose a basis $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ for $H_1^{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{Z})$, such that after a change of basis over $\mathbb{Q}$, we have $\int_{\alpha_j} \omega_i = \delta_{ij}$ and $\int_{\beta_j} \eta_i = \delta_{ij}$.

The period matrix is then given by a block matrix:

	$\omega_\bullet$	$\eta_\bullet$
$\alpha_\bullet$	$\mathbb{I}$	τ'
$\beta_\bullet$	τ	$\mathbb{I}$

(13.7)

where, by Riemann's bilinear relations [GH, pg. 123], τ is a matrix in the *Siegel upper half space* $\mathbb{H}_g$ of symmetric complex matrices with positive definite imaginary part. In the example of elliptic curves, section 13.4 the matrix τ is the (1×1)-matrix given by $\tau = \omega_2/\omega_1 \in \mathbb{H}$.

For transcendence results for periods of curves and abelian varieties we refer to the survey of Wüstholz [Wu], and our discussion in Section 12.2 of Part III.

Chapter 14

Multiple zeta values

This chapter follows partly the Diploma thesis of Benjamin Friedrich, see [Fr]. We study in some detail the very important class of periods called multiple zeta values (MZV). These are periods of mixed Tate motives.

14.1 A ζ-value

In Prop. 11.1.4, we saw how to write $\zeta(2)$ as a Kontsevich-Zagier period:

$$\zeta(2) = \int_{0 \le x \le y \le 1} \frac{dx \wedge dy}{(1-x)\, y}.$$

The problem was that this identity did not give us a valid representation of $\zeta(2)$ as a naïve period, since the pole locus of the integrand and the domain of integration are not disjoint. We show how to circumvent this difficulty, as an example of Theorem 11.2.1.

First we define (often ignoring the difference between X and X^{an}),

$$\begin{aligned}
&Y := \mathbb{A}^2 \quad \text{with coordinates } x \text{ and } y,\\
&Z := \{x = 1\} \cup \{y = 0\},\\
&X := Y \setminus Z,\\
&D := (\{x = 0\} \cup \{y = 1\} \cup \{x = y\}) \setminus Z,\\
&\triangle := \{(x,y) \in Y \,|\, x, y \in \mathbb{R},\, 0 \le x \le y \le 1\} \quad \text{a triangle in } Y, \quad \text{and}\\
&\omega := \frac{dx \wedge dy}{(1-x)\, y},
\end{aligned}$$

thus getting

$$\zeta(2) = \int_{\triangle} \omega,$$

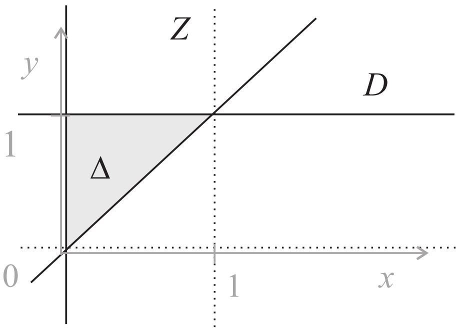


Figure 14.1: The configuration $Z, D, \triangle$

with $\omega \in \Gamma(X, \Omega^2_X)$ and $\partial\triangle \subset D \cup \{(0,0),(1,1)\}$, see Figure 14.1.

Now we blow up Y in the points $(0,0)$ and $(1,1)$ obtaining $\pi : \widetilde{Y} \to Y$. We denote the strict transform of Z by $\widetilde{Z}$, $\pi^*\omega_0$ by $\widetilde{\omega}$ and $\widetilde{Y} \setminus \widetilde{Z}$ by $\widetilde{X}$. The "strict transform" $\overline{\pi^{-1}(\triangle \setminus \{(0,0),(1,1)\})}$ will be called $\widetilde{\triangle}$ and (being $\widetilde{\mathbb{Q}}$-semi-algebraic hence triangulable — cf. Proposition 2.6.9) gives rise to a singular chain

$$\widetilde{\gamma} \in H_2^{\mathrm{sing}}(\widetilde{X}, \widetilde{D}; \mathbb{Q}).$$

Since π is an isomorphism away from the exceptional locus, this exhibits

$$\zeta(2) = \int_\triangle \omega = \int_{\widetilde{\gamma}} \widetilde{\omega} \in \mathbb{P}_a = \mathbb{P}$$

as a naïve period, see Figure 14.2.

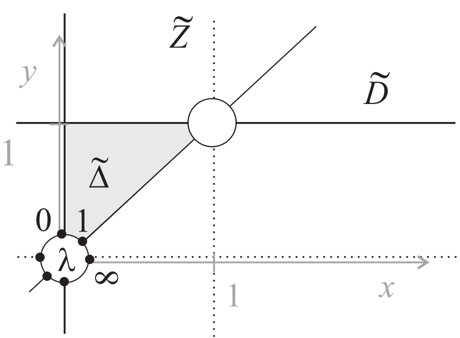


Figure 14.2: The configuration $\widetilde{Z}, \widetilde{D}, \widetilde{\triangle}$

We will conclude this example by writing out $\widetilde{\omega}$ and $\widetilde{\triangle}$ more explicitly. Note that $\widetilde{Y}$ can be described as the subvariety

$$\mathbb{A}^2_{\mathbb{Q}} \times \mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q}) \quad \text{with coordinates} \quad (\widetilde{x}, \widetilde{y}, \lambda_0 : \lambda_1, \mu_0 : \mu_1)$$

cut out by

$$\widetilde{x}\lambda_0 = \widetilde{y}\lambda_1 \quad \text{and} \quad (\widetilde{x}-1)\mu_0 = (\widetilde{y}-1)\mu_1.$$

With this choice of coordinates π takes the form

$$\begin{array}{cccc} \pi : & \widetilde{Y} & \to & Y \\ & (\widetilde{x},\widetilde{y},\lambda_0:\lambda_1,\mu_0:\mu_1) & \mapsto & (\widetilde{x},\widetilde{y}) \end{array}$$

and we have $\widetilde{X} := \widetilde{Y} \setminus (\{\lambda_0 = 0\} \cup \{\mu_1 = 0\})$. We can embed $\widetilde{X}$ into affine space

$$\begin{gathered} \widetilde{X} \to \mathbb{A}^4_{\mathbb{Q}} \\ (\widetilde{x},\widetilde{y},\lambda_0:\lambda_1,\mu_0:\mu_1) \mapsto (\widetilde{x},\widetilde{y},\frac{\lambda_1}{\lambda_0},\frac{\mu_0}{\mu_1}) \end{gathered}$$

and so have affine coordinates $\widetilde{x}$, $\widetilde{y}$, $\lambda := \frac{\lambda_1}{\lambda_0}$ and $\mu := \frac{\mu_0}{\mu_1}$ on $\widetilde{X}$.

Now, near $\pi^{-1}(0,0)$, the form $\widetilde{\omega}$ is given by

$$\widetilde{\omega} = \frac{d\widetilde{x}\wedge d\widetilde{y}}{(1-\widetilde{x})\,\widetilde{y}} = \frac{d(\lambda\widetilde{y})\wedge d\widetilde{y}}{(1-\widetilde{x})\,\widetilde{y}} = \frac{d\lambda\wedge d\widetilde{y}}{1-\widetilde{x}},$$

while near $\pi^{-1}(1,1)$ we have

$$\widetilde{\omega} = \frac{d\widetilde{x}\wedge d\widetilde{y}}{(1-\widetilde{x})\,\widetilde{y}} = \frac{d\widetilde{x}\wedge d(\widetilde{y}-1)}{(1-\widetilde{x})\,\widetilde{y}} = \frac{d\widetilde{x}\wedge d(\mu(\widetilde{x}-1))}{(1-\widetilde{x})\,\widetilde{y}} = \frac{-d\widetilde{x}\wedge d\mu}{\widetilde{y}}.$$

The region $\widetilde{\triangle}$ is given by

$$\widetilde{\triangle} = \{(\widetilde{x},\widetilde{y},\lambda,\mu) \in \widetilde{X}(\mathbb{C}) \,|\, \widetilde{x},\widetilde{y},\lambda,\mu \in \mathbb{R},\ \ 0 \le \widetilde{x} \le \widetilde{y} \le 1,\ \ 0 \le \lambda \le 1,\ \ 0 \le \mu \le 1\}.$$

14.2 Definition of multiple zeta values

Recall that the Riemann ζ-function is defined as

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s}, \quad \mathrm{Re}(s) > 1.$$

It has an analytic continuation to the whole complex plane with a simple pole at $s = 1$.

Definition 14.2.1. For integers $s_1, ..., s_r \ge 1$ with $s_1 \ge 2$ one defines the *multiple zeta values* (MZV)

$$\zeta(s_1, ..., s_r) := \sum_{n_1 > n_2 > ... > n_r \ge 1} n_1^{-s_1} \cdots n_r^{-s_r}.$$

The number $n = s_1 + \cdots + s_r$ is the *weight* of $\zeta(s_1, ..., s_r)$. The *length* is r.

Lemma 14.2.2. *$\zeta(s_1, ..., s_r)$ is convergent.*

Proof. Clearly, $\zeta(s_1, ..., s_r) \le \zeta(2, 1, ..., 1)$. We use the formula

$$\sum_{n=1}^{m-1} n^{-1} \le 1 + \log(m-1),$$

which is proved by comparing with the Riemann integral of $1/x$. Using induction, this implies that

$$\zeta(2,1,...,1) \le \sum_{n_1=1}^{\infty} n_1^{-2} \sum_{1\le n_r<\cdots<n_2\le n_1-1} n_2^{-1}\cdots n_r^{-1} \le \sum_{n_1=1}^{\infty} \frac{(1+\log(n_1-1))^r}{n_1^2},$$

which is convergent. □

Lemma 14.2.3. *The positive even ζ-values are given by*

$$\zeta(2m) = (-1)^{m+1}\frac{(2\pi)^{2m}}{2(2m)!}B_{2m},$$

where B_{2m} is a Bernoulli number*, defined via*

$$\frac{t}{e^t-1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}.$$

The first Bernoulli numbers are $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$. All odd Bernoulli B_m numbers vanish for odd $m \ge 3$.

Proof. One uses the power series

$$x\cot(x) = 1 - 2\sum_{n=1}^{\infty} \frac{x^2}{n^2\pi^2 - x^2}.$$

The geometric series expansion gives

$$x\cot(x) = 1 - 2\sum_{n=1}^{\infty} \frac{\left(\frac{x}{n\pi}\right)^2}{1-\left(\frac{x}{n\pi}\right)^2} = 1 - 2\sum_{m=1}^{\infty}\frac{x^{2m}}{\pi^{2m}}\zeta(2m).$$

On the other hand,

$$x\cot(x) = ix\frac{e^{ix}+e^{-ix}}{e^{ix}-e^{-ix}} = ix\frac{e^{2ix}+1}{e^{2ix}-1} = ix + \frac{2ix}{e^{2ix}-1} = ix + \sum_{m=0}^{\infty} B_m\frac{(2ix)^m}{m!}.$$

The claim then follows by comparing coefficients. □

Corollary 14.2.4. *$\zeta(2) = \frac{\pi^2}{6}$ and $\zeta(4) = \frac{\pi^4}{90}$.*

$\zeta(s)$ satisfies a functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s).$$

Using it, one can show:

Corollary 14.2.5. *$\zeta(-m) = -\frac{B_{m+1}}{m+1}$ for $m \geq 1$. In particular, $\zeta(-2m) = 0$ for $m \geq 1$. These are called the trivial zeroes of $\zeta(s)$.*

Remark 14.2.6. J. Zhao has generalized the analytic continuation and the functional equation for multiple zeta values [Z2].

In the following, we want to study MZV as periods. They satisfy many relations. Already Euler knew that $\zeta(2,1) = \zeta(3)$. This can be shown as follows:

$$\begin{aligned}
\zeta(3) + \zeta(2,1) &= \sum_{n=1}^{\infty} \frac{1}{n^3} + \sum_{1 \leq k < n} \frac{1}{n^2 k} = \sum_{1 \leq k \leq n} \frac{1}{n^2 k} = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{n} \frac{1}{k} \\
&= \sum_{k,n \geq 1} \frac{1}{n^2} \left(\frac{1}{k} - \frac{1}{n+k} \right) = \sum_{k,n \geq 1} \frac{1}{nk(n+k)} \\
&= \sum_{k,n \geq 1} \left(\frac{1}{n} + \frac{1}{k} \right) \frac{1}{(n+k)^2} = \sum_{k,n \geq 1} \frac{1}{n(n+k)^2} + \sum_{k,n \geq 1} \frac{1}{k(n+k)^2} \\
&= 2\zeta(2,1).
\end{aligned}$$

Other relations of this type are

$$\begin{gathered}
\zeta(2,1,1) = \zeta(4), \\
\zeta(2,2) = \frac{3}{4}\zeta(4), \\
\zeta(3,1) = \frac{1}{4}\zeta(4), \\
\zeta(2)^2 = \frac{5}{2}\zeta(4), \\
\zeta(5) = \zeta(3,1,1) + \zeta(2,1,2) + \zeta(2,2,1) \\
\zeta(5) = \zeta(4,1) + \zeta(3,2) + \zeta(2,3).
\end{gathered}$$

The last two relations are special cases of the sum relation:

$$\zeta(n) = \sum_{s_1 + \cdots + s_r = n} \zeta(s_1, ..., s_r).$$

It was conjectured by Zagier [Z] that the $\mathbb{Q}$-vector space Z_n of MZV of weight n has dimension d_n, where d_n is the coefficient of t^n in the power series

$$\sum_{n=0}^{\infty} d_n t^n = \frac{1}{1 - t^2 - t^3},$$

so that one has a recursion $d_n = d_{n-2} + d_{n-3}$. For example $d_4 = 1$, which can be checked using the above relations. By convention, $d_0 = 1$. This conjecture is still open, however one knows that d_n is an upper bound for $\dim_{\mathbb{Q}}(Z_n)$ [B1, Te]. It is also conjectured that the MZV of different weights are independent over $\mathbb{Q}$, so that the space of all MZV should be a direct sum

$$Z = \bigoplus_{n \geq 0} Z_n.$$

Hoffman [Hof] conjectured that all MZV containing only $s_i \in \{2, 3\}$ form a basis of Z. Brown [B1] showed in 2010 that this set forms a generating set. Broadhurst et. al. [BBV] conjecture that the $\zeta(s_1, ..., s_r)$ with $s_i \in \{2, 3\}$ a *Lyndon word* form a transcendence basis. A Lyndon word in two letters with an order, e.g. $2 < 3$, is a string in these two letters that is strictly smaller in lexicographic order than all of its circular shifts.

14.3 Kontsevich's integral representation

Define one-forms $\omega_0 := \frac{dt}{t}$ and $\omega_1 := \frac{dt}{1-t}$. We have seen that

$$\zeta(2) = \int_{0 \leq t_1 \leq t_2 \leq 1} \omega_0(t_2)\omega_1(t_1).$$

In a similar way, we get that

$$\zeta(n) = \int_{0 \leq t_1 \leq \cdots \leq t_n \leq 1} \omega_0(t_n)\omega_0(t_{n-1}) \cdots \omega_1(t_1).$$

We will now write this as

$$\zeta(n) = I(\underbrace{0 \ldots 01}_{n}).$$

Definition 14.3.1. For $\epsilon_1, ..., \epsilon_n \in \{0, 1\}$, we define the Kontsevich-Zagier periods

$$I(\epsilon_n \ldots \epsilon_1) := \int_{0 \leq t_1 \leq \cdots \leq t_n \leq 1} \omega_{\epsilon_n}(t_n)\omega_{\epsilon_{n-1}}(t_{n-1}) \cdots \omega_{\epsilon_1}(t_1).$$

Note that this definition differs from parts of the literature in terms of the order, but it has the advantage that there is no sign in the following formula:

Theorem 14.3.2 (Attributed to Kontsevich by Zagier [Z])**.**

$$\zeta(s_1, ..., s_r) = I(\underbrace{0 \ldots 01}_{s_1}\underbrace{0 \ldots 01}_{s_2} \ldots \underbrace{0 \ldots 01}_{s_r}).$$

In particular, the MZV are Kontsevich-Zagier periods.

Proof. We will define more generally

$$I(0;\epsilon_n\dots\epsilon_1;z) := \int_{0\le t_1\le\dots\le t_n\le z} \omega_{\epsilon_n}(t_n)\omega_{\epsilon_{n-1}}(t_{n-1})\cdots\omega_{\epsilon_1}(t_1)$$

for $0 \le z \le 1$. Then we show that

$$I(0;\underbrace{0\dots01}_{s_1}\underbrace{0\dots01}_{s_2}\cdots\underbrace{0\dots01}_{s_r};z) = \sum_{n_1>n_2>\dots>n_r\ge1} \frac{z^{n_1}}{n_1^{s_1}\cdots n_r^{s_r}}.$$

Convergence is always ok for $z < 1$, but at the end we will have it for $z = 1$ be Abel's theorem. We proceed by induction on $n = \sum_{i=1}^r s_i$. We start with $n = 1$:

$$I(0;1;z) = \int_0^z \omega_1(t) = \int_0^z \sum_{n\ge0} t^n dt = \sum_{n\ge0} \frac{z^{n+1}}{n+1} = \sum_{n\ge1} \frac{z^n}{n}.$$

The induction step has two cases:

$$I(0;0\underbrace{0\dots01}_{s_1}\underbrace{0\dots01}_{s_2}\cdots\underbrace{0\dots01}_{s_r};z) = \int_0^z \frac{dt_n}{t_n} I(0;\underbrace{0\dots01}_{s_1}\underbrace{0\dots01}_{s_2}\cdots\underbrace{0\dots01}_{s_r};t_n)$$
$$= \int_0^z \frac{dt_n}{t_n} \sum_{n_1>n_2>\dots>n_r\ge1} \frac{t_n^{n_1}}{n_1^{s_1}\cdots n_r^{s_r}} = \sum_{n_1>n_2>\dots>n_r\ge1} \frac{z^{n_1}}{n_1^{s_1+1}\cdots n_r^{s_r}}.$$

$$I(0;1\underbrace{0\dots01}_{s_1}\underbrace{0\dots01}_{s_2}\cdots\underbrace{0\dots01}_{s_r};z) = \int_0^z \frac{dt_n}{1-t_n} I(0;\underbrace{0\dots01}_{s_1}\underbrace{0\dots01}_{s_2}\cdots\underbrace{0\dots01}_{s_r};t_n)$$
$$= \int_0^z dt_n \sum_{m=0}^{\infty} t_n^m \sum_{n_1>n_2>\dots>n_r\ge1} \frac{t_n^{n_1}}{n_1^{s_1}\cdots n_r^{s_r}} = \sum_{m=0}^{\infty} \sum_{n_1>n_2>\dots>n_r\ge1} \int_0^z dt_n \frac{t_n^{n_1+m}}{n_1^{s_1}\cdots n_r^{s_r}}$$
$$= \sum_{n_0>n_1>n_2>\dots>n_r\ge1} \frac{z^{n_0}}{n_1^{s_1}\cdots n_r^{s_r}}.$$

In the latter step we strictly use $z < 1$ to have convergence. It does not occur at the end of the induction, since the string starts with a 0. Convergence is proven by Abel's theorem at the end. □

14.4 Shuffle and Stuffle relations for MZV

In this section, we present a slightly more abstract viewpoint on multiple zeta values and their relations by looking only at the strings representing a MZV integral. It turns out that there are two types of multiplications on those strings, called the shuffle and stuffle products, which induce the usual multiplication on the integrals, but which have a different definition. Comparing both leads to all kind of relations between multiple zeta values. The reader may also consult [IKZ, Hof, HO, He] for more information.

A MZV can be represented via a tuple $(s_1, ..., s_r)$ of integers or a string

$$s = \underbrace{0\dots01}_{s_1}\underbrace{0\dots01}_{s_2}\cdots\underbrace{0\dots01}_{s_r}$$

of 0's and 1's. There is a one-to-one correspondence between strings with a 0 on the left and a 1 on the right and all tuples $(s_1, ..., s_r)$ with all $s_i \geq 1$ and $s_1 \geq 2$. For any tuple $s = (s_1, ..., s_r)$, we denote the associated string by $\tilde{s}$. We will formalize the algebras arising from this set-up.

Definition 14.4.1 (Hoffman Algebra)**.** Let

$$\mathfrak{h} := \mathbb{Q}\langle x, y\rangle = \mathbb{Q} \oplus \mathbb{Q}x \oplus \mathbb{Q}y \oplus \mathbb{Q}xy \oplus \mathbb{Q}yx \oplus \cdots$$

be the free non-commutative graded algebra in two variables x, y (both of degree 1). There are subalgebras

$$\mathfrak{h}_1 := \mathbb{Q} \oplus \mathfrak{h}y, \quad \mathfrak{h}_0 := \mathbb{Q} \oplus x\mathfrak{h}y.$$

The generator in degree 0 is denoted by $\mathbb{I}$.

We will now identify x and y with 0 and 1, if it is convenient. For example any generator, i.e., a noncommutative word in x and y of length n can be viewed as a string $\epsilon_n \cdots \epsilon_1$ in the letters 0 and 1. With this identification, there is obviously an evaluation map such that

$$\zeta : \mathfrak{h} \longrightarrow \mathbb{R}, \quad \epsilon_n \cdots \epsilon_1 \mapsto I(\epsilon_n, ..., \epsilon_1)$$

holds on the generators of $\mathfrak{h}$. In addition, if s is the string

$$s = \epsilon_n \cdots \epsilon_1 = \underbrace{0\dots01}_{s_1}\underbrace{0\dots01}_{s_2}\cdots\underbrace{0\dots01}_{s_r},$$

then we have $\zeta(s_1, ..., s_n) = \zeta(s)$ by Theorem 14.3.2.

We will now define two different multiplications

$$\text{Ш}, * : \mathfrak{h} \times \mathfrak{h} \longrightarrow \mathfrak{h},$$

called shuffle and stuffle, such that ζ becomes a ring homomorphism in both cases.

Definition 14.4.2. Define the *shuffle permutations* for $r + s = n$ as

$$\Sigma_{r,s} := \{\sigma \in \Sigma_n \mid \sigma(1) < \sigma(2) < \cdots < \sigma(r), \sigma(r+1) < \sigma(r+2) < \cdots < \sigma(r+s)\}.$$

Define the action of $\sigma \in \Sigma_{r,s}$ on the set $\{1, 2, ..., n\}$ as

$$\sigma(x_1...x_n) := x_{\sigma^{-1}(1)}...x_{\sigma^{-1}(n)}.$$

The *shuffle product* is then defined as

$$x_1...x_r \text{Ш} x_{r+1}...x_n := \sum_{\sigma\in\Sigma_{r,s}} \sigma(x_1...x_n).$$

Theorem 14.4.3. *The shuffle product* Ш *defines an associative, bilinear operation with unit* $\mathbb{I}$ *and hence an algebra structure on* $\mathfrak{h}$ *such that* ζ *is a ring homomorphism. It satisfies the recursive formula*

$$u \text{Ш} v = a(u' \text{Ш} v) + b(u \text{Ш} v'),$$

if $u = au'$ *and* $v = bv'$ *as strings.*

Proof. We only give a proof for the product formula $\zeta(\tilde{a} \text{Ш} \tilde{b}) = \zeta(a)\zeta(b)$; the rest is straightforward. Assume $a = (a_1, ..., a_r)$ is of weight m and $b = (b_1, ..., b_s)$ is of weight n. Then, by Fubini, the product $\zeta(a)\zeta(b)$ is an integral over the product domain

$$\Delta = \{0 \le t_1 \le \cdots \le t_m \le 1\} \times \{0 \le t_{m+1} \le \cdots \le t_{m+n} \le 1\}.$$

Ignoring subsets of measure zero,

$$\Delta = \coprod_\sigma \Delta_\sigma$$

indexed by all shuffles $\sigma \in \Sigma_{r,s}$, and where

$$\Delta_\sigma = \{(t_1, ..., t_{m+s} \mid 0 \le t_{\sigma^{-1}(1)} \le \cdots \le t_{\sigma^{-1}(n)} \le 1\}.$$

The proof follows then from the additivity of the integral. □

This induces binary relations as in the following examples.

Example 14.4.4. One has

$$(01) \text{Ш} (01) = 2(0101) + 4(0011)$$

and hence we have

$$\zeta(2)^2 = 2\zeta(2,2) + 4\zeta(3,1).$$

In a similar way,

$$(01) \text{Ш} (001) = (010011) + 3(001011) + 9(000111) + (001101),$$

which implies that

$$\zeta(2)\zeta(3,1) = \zeta(2,3,1) + 3\zeta(3,2,1) + 9\zeta(4,1,1) + \zeta(3,1,2),$$

and

$$(01) \text{Ш} (011) = 3(01011) + 6(00111) + (01101)$$

implies that

$$\zeta(2)\zeta(2,1) = 3\zeta(2,2,1) + 6\zeta(3,1,1) + \zeta(2,1,2).$$

Definition 14.4.5. The *stuffle product*

$$* : \mathfrak{h} \times \mathfrak{h} \longrightarrow \mathfrak{h}$$

is defined on tuples $a = (a_1, ..., a_r)$ and $b = (b_1, ..., b_s)$ as

$$\begin{aligned} a * b := &(a_1, ..., a_r, b_1, ..., b_s) + (a_1, ..., a_r + b_1, ..., b_s) \\ &+ (a_1, ..., a_{r-1}, b_1, a_r, b_2, ..., b_s) + (a_1, ..., a_{r-1} + b_1, a_r, b_2, ..., b_s) + \cdots \end{aligned}$$

The definition is made such that one has the formula $\zeta(a)\zeta(b) = \zeta(a * b)$ in the formula defining multiple zeta values.

Theorem 14.4.6. *The stuffle product $*$ defines an associative, bilinear multiplication on $\mathfrak{h}$ inducing an algebra $(\mathfrak{h}, *)$ with unit $\mathbb{I}$. One has $\zeta(a)\zeta(b) = \zeta(a*b)$ on tuples a and b. Furthermore, there is a recursion formula*

$$u * v = (a, u' * v) + (b, u * v') + (a, b, u' * v')$$

for tuples $u = (a, u')$ and $v = (b, v')$ with first entry a and b.

Proof. Again, we only give a proof for the product formula $\zeta(a)\zeta(b) = \zeta(a * b)$. Assume $a = (a_1, ..., a_r)$ is of weight m and $b = (a_{r+1}, ..., a_{r+s})$ is of weight n. The claim follows from a decomposition of the summation range:

$$\begin{aligned} &\zeta(a_1, ..., a_r)\zeta(a_{r+1}, ..., a_{r+s}) \\ &= \sum_{n_1 > n_2 > ... > n_r \geq 1} n_1^{-a_1} \cdots n_r^{-a_r} \cdot \sum_{n_{r+1} > n_{r+2} > ... > n_{r+s} \geq 1} n_{r+1}^{-a_{r+1}} \cdots n_{r+s}^{-a_{r+s}} = \\ &= \sum_{n_1 > n_2 > ... > n_r > n_{r+1} > n_{r+2} > ... > n_{r+s} \geq 1} n_1^{-a_1} \cdots n_r^{-a_r} n_{r+1}^{-a_{r+1}} \cdots n_{r+s}^{-a_{r+s}} \\ &+ \sum_{n_1 > n_2 > ... > n_r = n_{r+1} > n_{r+2} > ... > n_{r+s} \geq 1} n_1^{-a_1} \cdots n_r^{-(a_r + a_{r+1})} \cdots n_{r+s}^{-a_{r+s}} \\ &+ \text{ etc.} \end{aligned}$$

where all terms in the stuffle set occur once. □

This induces again binary relations as in the following examples.

Example 14.4.7.

$$\begin{aligned} \zeta(2)\zeta(3,1) &= \zeta(2,3,1) + \zeta(5,1) + \zeta(3,2,1) + \zeta(3,3) + \zeta(3,1,2) \\ \zeta(2)^2 &= 2\zeta(2,2) + \zeta(4). \end{aligned}$$

More generally,

$$\zeta(a)\zeta(b) = \zeta(a,b) + \zeta(a+b) + \zeta(b,a), \text{ for } a, b \geq 2..$$

Since we have $\zeta(\tilde{a}Ш\tilde{b}) = \zeta(a * b)$ we can define the unary double-shuffle relation as

$$\zeta(\tilde{a}Ш\tilde{b} - a * b) = 0.$$

Example 14.4.8. We have $\zeta(2)^2 = 2\zeta(2,2) + 4\zeta(3,1)$ using shuffle and $\zeta(2)^2 = 2\zeta(2,2) + \zeta(4)$ using the stuffle. Therefore one has

$$4\zeta(3,1) = \zeta(4).$$

In the literature [Hof, HO, IKZ, He] more relations were found, e.g., a modified version of this relation, called the *regularized double-shuffle relation*:

$$\zeta\left(\sum_{b\in(1)*a} b - \sum_{\tilde{c}\in(1)Ш\tilde{a}} c\right) = 0.$$

Example 14.4.9. Let $a = \widetilde{(2)} = (01)$. Then $(1)Ш(01) = (101) + 2(011)$ and $(1) * (2) = (1,2) + (3) + (2,1)$. Therefore, the corresponding relation is

$$\zeta(1,2) + 2\zeta(2,1) = \zeta(1,2) + \zeta(3) + \zeta(2,1), \text{ hence}$$

$$\zeta(2,1) = \zeta(3).$$

Like in this example, all non-convergent contributions cancel in the relation. It is conjectured that the regularized double-shuffle relation generates all relations among MZV. There are more relations: the sum theorem (mentioned above), the duality theorem, the derivation theorem and Ohno's theorem, which implies the first three [HO, He].

We will finish this subsection with some formulas mentioned by Brown [B1], mainly due to Broadhurst and Zagier:

$$\zeta(\underbrace{3,1,\ldots,3,1}_{2n}) = \frac{1}{2n+1}\zeta(\underbrace{2,2,\ldots,2}_{2n}) = \frac{2\pi^{4n}}{(4n+2)!}.$$

$$\zeta(\underbrace{2,...,2}_{b},3,\underbrace{2,...,2}_{a}) = \sum_{m+r=a+b+1} c_{m,r,a,b}\frac{\pi^{2m}}{(4m+1)!}\zeta(2r+1),$$

where $c_{m,r,a,b} = 2(-1)^r\left(\binom{2r}{2a+2} - \left(1-2^{-2r}\right)\binom{2r}{2b+1}\right) \in \mathbb{Q}$ $(m \geq 0, r \geq 1)$.

In the next section, we relate multiple zeta values to Nori motives and also to mixed Tate motives. This give a more conceptual embedding of such periods in the sense of Chapter 10, see in particular Section 10.5.

14.5 Multiple zeta values and moduli space of marked curves

In this short section, we indicate how one can relate multiple zeta values to Nori motives and to mixed Tate motives.

Multiple zeta values can also be seen as periods of certain cohomology groups of moduli spaces in such a way that they appear naturally as Nori motives. Recall that the moduli space $M_{0,n}$ of smooth rational curves with n marked points can be compactified to the space $\overline{M}_{0,n}$ of stable curves with n markings [K2]. Manin and Goncharov [GM] have observed the following.

Theorem 14.5.1. *For each convergent multiple zeta value $p = \zeta(s_1, ..., s_r)$ of weight $n = s_1 + ... + s_r$, one can construct divisors A, B in $\overline{M}_{0,n+3}$ such that p is a period of the cohomology group $H^n(\overline{M}_{0,n+3} \setminus A, B \setminus (A \cap B))$.*

The group $H^n(\overline{M}_{0,n+3} \setminus A, B \setminus (A \cap B))$ defines of course immediately a motive in Nori's sense.

Example 14.5.2. The fundamental example is $\zeta(2)$, which we already described in section 14.1. Here $\overline{M}_{0,5}$ is a compactification of

$$M_{0,5} = (\mathbb{P} \setminus \{0, 1, \infty\})^2 \setminus \text{ diagonal},$$

since $\overline{M}_{0,5}$ is the blow up $(0, 0)$, $(1, 1)$ and (∞, ∞) in $\mathbb{P}^1 \times \mathbb{P}^1$. This realizes $\zeta(2)$ as the integral

$$\zeta(2) = \int_{0 \le t_1 \le t_2 \le 1} \frac{dt_1}{1 - t_2} \frac{dt_2}{t_2} .$$

We leave it to the reader to make the divisors A and B explicit.

This viewpoint was very much refined in Brown's thesis [B3]. Recent related research for higher polylogarithms and elliptic polylogarithms can be found in [B4].

Levine [L2] has defined an abelian category as a full subcategory of the triangulated category of geometrical motives, see Chapter 10 for the notion of geometric motives. It is a full subcategory generated by the Tate objects $\mathbb{Q}(n)$. There is also a variant, called mixed Tate motives over $\mathbb{Z}$, see [Te, DG, B1]. The Theorem above implies:

Theorem 14.5.3 (Brown)**.** *Multiple zeta values together with $(2\pi i)^n$ are precisely all the periods of all mixed Tate motives over $\mathbb{Z}$.*

Proof. This is a result of Brown, see [B1, D3]. □

14.6 Multiple Polylogarithms

In this section, we study a variation of cohomology groups in a 2-parameter family of varieties over $\mathbb{Q}$, the so-called *double logarithm variation*, for which multiple polylogarithms appear as coefficients. This viewpoint gives more examples of Kontsevich-Zagier periods occuring as cohomological periods of canonical cohomology groups at particular values of the parameters. The degeneration of the parameters specializes such periods to simpler ones.

First define the *hyperlogarithm* as the iterated integral

$$\mathrm{I}_n(a_1,\ldots,a_n) := \int_{0\le t_1\le\cdots\le t_n\le 1} \frac{dt_1}{t_1-a_1}\wedge\cdots\wedge\frac{dt_n}{t_n-a_n}$$

with $a_1,\ldots,a_n\in\mathbb{C}$ (cf. [Z1, p. 168]). Note that, the order of terms here is different from the previous order, also in the infinite sum below.

These integrals specialize to the *multiple polylogarithm* (cf. [*loc. cit.*])

$$\mathrm{Li}_{m_1,\ldots,m_n}\left(\frac{a_2}{a_1},\cdots,\frac{a_n}{a_{n-1}},\frac{1}{a_n}\right) := (-1)^n\,\mathrm{I}_{\sum m_n}(a_1,\underbrace{0,\ldots,0}_{m_1-1},\ldots,a_n,\underbrace{0,\ldots,0}_{m_n-1}),$$

which is convergent if $1<|a_1|<\cdots<|a_n|$ (cf. [G3, 2.3, p. 9]). Alternatively, we can describe the multiple polylogarithm as a power series (cf. [G3, Thm. 2.2, p. 9])

$$\mathrm{Li}_{m_1,\ldots,m_n}(x_1,\ldots,x_n) = \sum_{0<k_1<\cdots<k_n} \frac{x_1^{k_1}\cdots x_n^{k_n}}{k_1^{m_1}\cdots k_n^{m_n}} \quad \text{for} \quad |x_i|<1. \tag{14.1}$$

Of special interest to us will be the *dilogarithm* $\mathrm{Li}_2(x)=\sum_{k>0}\frac{x^k}{k^2}$ and the *double logarithm* $\mathrm{Li}_{1,1}(x,y)=\sum_{0<k<l}\frac{x^k y^l}{kl}$.

Remark 14.6.1. At first, the functions $\mathrm{Li}_{m_1,\ldots,m_n}(x_1,\ldots,x_n)$ only make sense for $|x_i|<1$, but they can be analytically continued to multivalued meromorphic functions on $\mathbb{C}^n$ (cf. [Z1, p. 2]), for example $\mathrm{Li}_1(x)=-\log(1-x)$. One has $\mathrm{Li}_2(1)=\frac{\pi^2}{6}$.

14.6.1 The Configuration

Let us consider the configuration

$$\begin{aligned}
&Y := \mathbb{A}^2 \quad \text{with coordinates } x \text{ and } y,\\
&Z := \{x=a\}\cup\{y=b\} \quad \text{with} \quad a\neq 0,1 \quad \text{and} \quad b\neq 0,1\\
&X := Y\setminus Z\\
&D := (\{x=0\}\cup\{y=1\}\cup\{x=y\})\setminus Z,
\end{aligned}$$

see Figure 14.3.

We denote the irreducible components of the divisor D as follows:

$$\begin{aligned} D_1 &:= \{x = 0\} \setminus \{(0, b)\}, \\ D_2 &:= \{y = 1\} \setminus \{(a, 1)\}, \quad \text{and} \\ D_3 &:= \{x = y\} \setminus \{(a, a), (b, b)\}. \end{aligned}$$

By projecting from Y onto the y- or x-axis, we get isomorphisms for the associated complex analytic spaces

$$D_1^{\text{an}} \cong \mathbb{C} \setminus \{b\}, \quad D_2^{\text{an}} \cong \mathbb{C} \setminus \{a\}, \quad \text{and} \quad D_3^{\text{an}} \cong \mathbb{C} \setminus \{a, b\}.$$

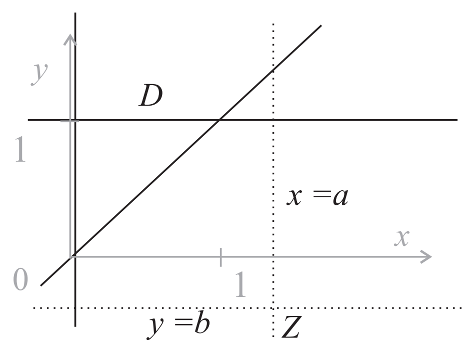


Figure 14.3: The algebraic pair (X, D)

14.6.2 Singular Homology

We can easily give generators for the second singular homology of the pair (X, D), see Figure 14.4.

- Let $\alpha : [0, 1] \to \mathbb{C}$ be a smooth path, which does not meet a or b. We define a "triangle"

$$\triangle := \{\big(\alpha(s), \alpha(t)\big) \mid 0 \leq s \leq t \leq 1\}.$$

- Consider the closed curve in $\mathbb{C}$

$$C_b := \left\{ \frac{a}{b + \epsilon e^{2\pi i s}} \mid s \in [0, 1] \right\},$$

which divides $\mathbb{C}$ into two regions: an inner one containing $\frac{a}{b}$ and an outer one. We can choose $\epsilon > 0$ small enough such that C_b separates $\frac{a}{b}$ from 0 to 1, i.e., such that 0 and 1 are contained in the outer region. This allows us to find a smooth path $\beta : [0,1] \to \mathbb{C}$ from 0 to 1 not meeting C_b. We define a "slanted tube"

$$S_b := \left\{ \left(\beta(t) \cdot (b + \epsilon e^{2\pi i s}), b + \epsilon e^{2\pi i s}\right) \mid s, t \in [0,1] \right\}$$

which winds around $\{y = b\}$ and whose boundary components are supported on D_1 (corresponding to $t = 0$) and D_3 (corresponding to $t = 1$). The special choice of β guarantees $S_b \cap Z(\mathbb{C}) = \emptyset$.

- Similarly, we choose $\epsilon > 0$ such that the closed curve

$$C_a := \left\{ \frac{b-1}{a - 1 - \epsilon e^{2\pi i s}} \mid s \in [0,1] \right\}$$

separates $\frac{b-1}{a-1}$ form 0 and 1. Let $\gamma : [0,1] \to \mathbb{C}$ be a smooth path from 0 to 1 which does not meet C_a. We have a "slanted tube"

$$S_a := \left\{ \left(a + \epsilon e^{2\pi i s}, 1 + \gamma(t) \cdot (a + \epsilon e^{2\pi i s} - 1)\right) \mid s, t \in [0,1] \right\}$$

winding around $\{x = a\}$ with boundary supported on D_2 and D_3.

- Finally, we have a torus

$$T := \{(a + \epsilon e^{2\pi i s}, b + \epsilon e^{2\pi i t}) \mid s, t \in [0,1]\}.$$

The 2-form $ds \wedge dt$ defines an orientation on the unit square $[0,1]^2 = \{(s,t) \mid s, t \in [0,1]\}$. Hence the manifolds with boundary $\triangle$, S_b, S_a, T inherit an orientation, and since they can be triangulated, they give rise to smooth singular chains. By abuse of notation we will also write $\triangle$, S_b, S_a, T for these smooth singular chains. The homology classes of $\triangle$, S_b, S_a and T will be denoted by γ_0, γ_1, γ_2 and γ_3, respectively.

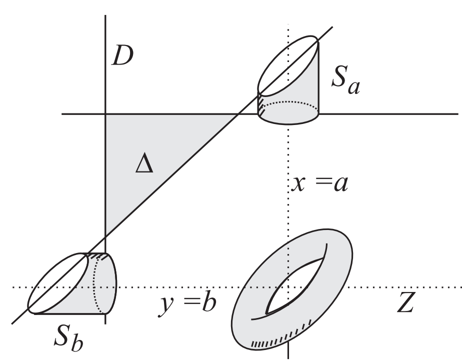


Figure 14.4: Generators of $H_2^{\text{sing}}(X, D; \mathbb{Q})$

An inspection of the long exact sequence in singular homology will reveal that $\gamma_0, \ldots, \gamma_3$ form a system generators (see the following proof)

$$H_2^{\text{sing}}(D, \mathbb{Q}) \longrightarrow H_2^{\text{sing}}(X, \mathbb{Q}) \longrightarrow H_2^{\text{sing}}(X, D, \mathbb{Q}) \longrightarrow H_1^{\text{sing}}(D, \mathbb{Q}) \xrightarrow{i_1} H_1^{\text{sing}}(X, \mathbb{Q}).$$

Proposition 14.6.2. *With notation as above, we have for the second singular homology of the pair (X, D)*

$$H_2^{\text{sing}}(X, D; \mathbb{Q}) = \mathbb{Q}\,\gamma_0 \oplus \mathbb{Q}\,\gamma_1 \oplus \mathbb{Q}\,\gamma_2 \oplus \mathbb{Q}\,\gamma_3.$$

Proof. For $c := a$ and $c := b$, the inclusion of the circle $\{c + \epsilon e^{2\pi i s} \mid s \in [0, 1]\}$ into $\mathbb{C} \setminus \{c\}$ is a homotopy equivalence, hence the product map $T \hookrightarrow X(\mathbb{C})$ is also a homotopy equivalence. This shows

$$H_2^{\text{sing}}(X, \mathbb{Q}) = \mathbb{Q}\,T,$$

while $H_1^{\text{sing}}(X, \mathbb{Q})$ has rank two with generators

- one loop winding counterclockwise around $\{x = a\}$ once, but not around $\{y = b\}$, thus being homologous to both $\partial S_a \cap D_2(\mathbb{C})$ and $-\partial S_a \cap D_3(\mathbb{C})$, and
- another loop winding counterclockwise around $\{y = b\}$ once, but not around $\{x = a\}$, thus being homologous to $\partial S_b \cap D_1(\mathbb{C})$ and $-\partial S_b \cap D_3(\mathbb{C})$.

In order to compute the Betti-numbers b_i of D, we use the spectral sequence for the closed covering $\{D_i\}$

$$E_2^{p,q} := \begin{array}{cccccc} \cdots & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & \bigoplus_{i=1}^3 H^1_{\text{dR}}(D_i, \mathbb{C}) & 0 & 0 & \cdots \\ \cdots & 0 & \text{Ker}\delta & \text{Coker}\delta & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & \cdots \end{array} \quad \Rightarrow \quad E_\infty^n := H^n_{\text{dR}}(D, \mathbb{C}),$$

where

$$\delta : \bigoplus_{i=1}^{3} H^0_{\text{dR}}(D_i, \mathbb{C}) \longrightarrow \bigoplus_{i<j} H^0_{\text{dR}}(D_{ij}, \mathbb{C}).$$

Note that this spectral sequence degenerates. Since D is connected, we have $b_0 = 1$, i.e.,

$$1 = b_0 = \text{rank}_{\mathbb{C}} E^0_\infty = \text{rank}_{\mathbb{C}} E_2^{0,0} = \text{rank}_{\mathbb{C}} \text{Ker}\delta.$$

Hence

$$\begin{aligned} \text{rank}_{\mathbb{C}}\text{Coker}\delta &= \text{rank}_{\mathbb{C}}\text{codomain}\,\delta - \text{rank}_{\mathbb{C}}\text{domain}\,\delta + \text{rank}_{\mathbb{C}}\text{Ker}\delta \\ &= (1+1+1) - (1+1+1) + 1 = 1, \end{aligned}$$

and so

$$\begin{aligned} b_1 &= \mathrm{rank}_{\mathbb{C}} E^1_\infty = \mathrm{rank}_{\mathbb{C}} E^{1,0}_2 + \mathrm{rank}_{\mathbb{C}} E^{0,1}_2 \\ &= \sum_{i=1}^{3} \mathrm{rank}_{\mathbb{C}} H^1_{\mathrm{dR}}(D_i, \mathbb{C}) + \mathrm{rank}_{\mathbb{C}} \mathrm{Coker}\delta \\ &= \mathrm{rank}_{\mathbb{C}} H^1(\mathbb{C} \setminus \{b\}, \mathbb{C}) + \mathrm{rank}_{\mathbb{C}} H^1(\mathbb{C} \setminus \{a\}, \mathbb{C}) + \mathrm{rank}_{\mathbb{C}} H^1(\mathbb{C} \setminus \{a, b\}, \mathbb{C}) + 1 \\ &= (1 + 1 + 2) + 1 = 5. \end{aligned}$$

We can easily specify generators of $H^{\mathrm{sing}}_1(D, \mathbb{Q})$ as follows

$$H^{\mathrm{sing}}_1(D, \mathbb{Q}) = \mathbb{Q}\cdot(\partial S_b \cap D_1) \oplus \mathbb{Q}\cdot(\partial S_a \cap D_2) \oplus \mathbb{Q}\cdot(\partial S_b \cap D_3) \oplus \mathbb{Q}\cdot(\partial S_a \cap D_3) \oplus \mathbb{Q}\cdot\partial\triangle.$$

Clearly $b_2 = \mathrm{rank}_{\mathbb{C}} H^{\mathrm{sing}}_2(D, \mathbb{Q}) = 0$. Now we can compute $\mathrm{Ker} i_1$ and obtain

$$\mathrm{Ker} i_1 = \mathbb{Q}\cdot\partial\triangle \oplus \mathbb{Q}\cdot(\partial S_b \cap D_1(\mathbb{C}) + \partial S_b \cap D_3(\mathbb{C})) \oplus \mathbb{Q}\cdot(\partial S_a \cap D_2(\mathbb{C}) + \partial S_a \cap D_3(\mathbb{C})).$$

This shows finally

$$\mathrm{rank}_{\mathbb{Q}} H^{\mathrm{sing}}_2(X, D; \mathbb{Q}) = \mathrm{rank}_{\mathbb{Q}} H^{\mathrm{sing}}_2(X, \mathbb{Q}) + \mathrm{rank}_{\mathbb{Q}} \mathrm{Ker} i_1 = 1 + 3 = 4.$$

From these explicit calculations we also derive the linear independence of $\gamma_0 = [\triangle]$, $\gamma_1 = [S_b]$, $\gamma_2 = [S_a]$, $\gamma_3 = [T]$ and Proposition 14.6.2 is proved. □

14.6.3 Smooth Singular Homology

Recall the definition of smooth singular cohomology (cf. Theorem 2.2.5). With the various sign conventions made so far, the boundary map $\delta : \mathrm{C}^\infty_2(X, D; \mathbb{Q}) \to \mathrm{C}^\infty_1(X, D; \mathbb{Q})$ is given by

$$\delta : \mathrm{C}^\infty_2(X, \mathbb{Q}) \oplus \bigoplus_{i=1}^{3} \mathrm{C}^\infty_1(D_i, \mathbb{Q}) \oplus \bigoplus_{i<j} \mathrm{C}^\infty_0(D_{ij}, \mathbb{Q}) \to \mathrm{C}^\infty_1(X, \mathbb{Q}) \oplus \bigoplus_{i=1}^{3}(D_i, \mathbb{Q})$$

$$(\underset{\emptyset}{\sigma}, \underset{1}{\sigma_1}, \underset{2}{\sigma_2}, \underset{3}{\sigma_3}, \underset{12}{\sigma_{12}}, \underset{13}{\sigma_{13}}, \underset{23}{\sigma_{23}}) \mapsto$$

$$(\underset{\emptyset}{\partial\sigma + \sigma_1 + \sigma_2 + \sigma_3}, \underset{1}{-\partial\sigma_1 + \sigma_{12} + \sigma_{13}}, \underset{2}{-\partial\sigma_2 - \sigma_{12} + \sigma_{23}}, \underset{3}{-\partial\sigma_3 - \sigma_{13} - \sigma_{23}}).$$

Thus the following elements of $\mathrm{C}^\infty_2(X, D; \mathbb{Q})$ are cycles

- $\Gamma_0 := (\underset{\emptyset}{\triangle}, \underset{1}{-\partial\triangle \cap D_1(\mathbb{C})}, \underset{2}{-\partial\triangle \cap D_2(\mathbb{C})}, \underset{3}{-\partial\triangle \cap D_3(\mathbb{C})}, \underset{12}{D_{12}(\mathbb{C})}, \underset{13}{-D_{13}(\mathbb{C})}, \underset{23}{D_{23}(\mathbb{C})})$,
- $\Gamma_1 := (\underset{\emptyset}{S_b}, \underset{1}{-\partial S_b \cap D_1(\mathbb{C})}, \underset{2}{0}, \underset{3}{-\partial S_b \cap D_3(\mathbb{C})}, \underset{12}{0}, \underset{13}{0}, \underset{23}{0})$,
- $\Gamma_2 := (\underset{\emptyset}{S_a}, \underset{1}{0}, \underset{2}{-\partial S_a \cap D_2(\mathbb{C})}, \underset{3}{0}, \underset{12}{-\partial S_a \cap D_3(\mathbb{C})}, \underset{13}{0}, \underset{23}{0})$ and
- $\Gamma_3 := (\underset{\emptyset}{T}, \underset{1}{0}, \underset{2}{0}, \underset{3}{0}, \underset{12}{0}, \underset{13}{0}, \underset{23}{0})$.

Under the isomorphism $H^\infty_2(X, D; \mathbb{Q}) \xrightarrow{\sim} H^{\mathrm{sing}}_2(X, D; \mathbb{Q})$ the classes of these cycles $[\Gamma_0]$, $[\Gamma_1]$, $[\Gamma_2]$, $[\Gamma_3]$ are mapped to γ_0, γ_1, γ_2, γ_3, respectively.

14.6.4 Algebraic de Rham cohomology and period matrix of (X, D)

Recall the definition of the complex $\widetilde{\Omega}^{\bullet}_{X,D}$. We consider

$$\Gamma(X, \widetilde{\Omega}^2_{X,D}) = \Gamma(X, \Omega^2_X) \oplus \bigoplus_{i=1}^{3} \Gamma(D_i, \Omega^1_{D_i}) \oplus \bigoplus_{i<j} \Gamma(D_{ij}, \mathcal{O}_{D_{ij}})$$

together with the following cycles of $\Gamma(X, \widetilde{\Omega}^2_{X,D})$

- $\omega_0 := (\underset{\emptyset}{\frac{dx\wedge dy}{(x-a)(y-b)}}, \underset{1}{0}, \underset{2}{0}, \underset{3}{0}, \underset{12}{0}, \underset{13}{0}, \underset{23}{0})$,
- $\omega_1 := (\underset{\emptyset}{0}, \underset{1}{\frac{-dy}{y-b}}, \underset{2}{0}, \underset{3}{0}, \underset{12}{0}, \underset{13}{0}, \underset{23}{0})$,
- $\omega_2 := (\underset{\emptyset}{0}, \underset{1}{0}, \underset{2}{\frac{-dx}{x-a}}, \underset{3}{0}, \underset{12}{0}, \underset{13}{0}, \underset{23}{0})$, and
- $\omega_3 := (\underset{\emptyset}{0}, \underset{1}{0}, \underset{2}{0}, \underset{3}{0}, \underset{12}{0}, \underset{13}{0}, \underset{23}{1})$.

By computing the (transposed) period matrix $P_{ij} := \langle \Gamma_j, \omega_i \rangle$ and checking its non-degeneracy, we will show that $\omega_0, \ldots, \omega_3$ span $H^2_{\mathrm{dR}}(X, D)$.

Proposition 14.6.3. *Let X and D be as above. Then the second algebraic de Rham cohomology group $H^2_{\mathrm{dR}}(X, D)$ of the pair (X, D) is generated by the cycles $\omega_0, \ldots, \omega_3$ considered above.*

Proof. Easy calculations give us the (transposed) period matrix P:

	Γ_0	Γ_1	Γ_2	Γ_3
ω_0	1	0	0	0
ω_1	$\mathrm{Li}_1(\frac{1}{b})$	$2\pi i$	0	0
ω_2	$\mathrm{Li}_1(\frac{1}{a})$	0	$2\pi i$	0
ω_3	?	$2\pi i \mathrm{Li}_1(\frac{b}{a})$	$2\pi i \log\left(\frac{a-b}{1-b}\right)$	$(2\pi i)^2$.

For example,

- $$\begin{aligned} P_{1,1} = \langle \Gamma_1, \omega_1 \rangle &= \int_{-\partial S_b \cap D_1(\mathbb{C})} \frac{-dy}{y-b} \\ &= \int_{|y-b|=\epsilon} \frac{dy}{y-b} \\ &= 2\pi i, \end{aligned}$$
- $$\begin{aligned} P_{3,3} = \langle \Gamma_3, \omega_3 \rangle &= \int_T \frac{dx}{x-a} \wedge \frac{dy}{y-b} \\ &= \left(\int_{|x-a|=\epsilon} \frac{dx}{x-a}\right) \cdot \left(\int_{|y-b|=\epsilon} \frac{dy}{y-b}\right) \quad \text{by Fubini} \\ &= (2\pi i)^2, \end{aligned}$$

- $P_{1,0} = \langle \Gamma_0, \omega_1 \rangle = \int_{-\partial\triangle\cap D_1(\mathbb{C})} \frac{-dy}{y-b}$
$= \int_0^1 \frac{-\alpha(t)}{\alpha(t)-b}$
$= -[\log(\alpha(t)) - b]_0^1$
$= -\log\left(\frac{1-b}{-b}\right)$
$= -\log\left(1 - \frac{1}{b}\right)$
$= \mathrm{Li}_1\left(\frac{1}{b}\right)$, and

- $P_{3,1} = \langle \Gamma_1, \omega_3 \rangle = \int_{S_b} \frac{dx}{x-a} \wedge \frac{dy}{y-b}$
$= \int_{[0,1]^2} \frac{d\left(\beta(t)\cdot\left(b+\epsilon e^{2\pi i s}\right)\right)}{\beta(t)\cdot(b+\epsilon e^{2\pi i s})-a} \wedge \frac{d\left(b+\epsilon e^{2\pi i s}\right)}{\epsilon e^{2\pi i s}}$
$= \int_{[0,1]^2} \frac{b+\epsilon e^{2\pi i s}}{\beta(t)\cdot(b+\epsilon e^{2\pi i s})-a} d\beta(t) \wedge 2\pi i ds$
$= -\int_0^1 \left[\frac{a\log\left(\beta(t)\cdot\left(b+\epsilon e^{2\pi i s}\right)-a\right)-2\pi i\beta(t)bs}{\beta(t)\cdot(-\beta(t)b+a)}\right]_0^1 d\beta(t)$
$= -2\pi i \int_0^1 \frac{d\beta(t)}{\beta(t)-\frac{a}{b}}$
$= -2\pi i \left[\log\left(\beta(t) - \frac{a}{b}\right)\right]_0^1$
$= -2\pi i \log\left(\frac{1-\frac{a}{b}}{-\frac{a}{b}}\right)$
$= -2\pi i \log\left(1 - \frac{a}{b}\right)$
$= 2\pi i \, \mathrm{Li}_1\left(\frac{b}{a}\right)$.

Obviously the period matrix P is non-degenerate and so Proposition 14.6.3 is proved. □

What about the entry $P_{3,0}$?

Proposition 14.6.4. $P_{3,0} = \mathrm{Li}_{1,1}\left(\frac{b}{a}, \frac{1}{b}\right)$.

For the proof we need to show that $\langle \Gamma_0, \omega_3 \rangle = \mathrm{Li}_{1,1}\left(\frac{b}{a}, \frac{1}{b}\right)$, where $\mathrm{Li}_{1,1}(x, y)$ is an analytic continuation of the double logarithm defined for $|x|, |y| < 1$ in Subsection 14.6.

Lemma 14.6.5. *The integrals*

$$\mathrm{I}_2^{\alpha}\left(\frac{1}{xy}, \frac{1}{y}\right) = \int_{0\le s\le t\le 1} \frac{d\alpha(s)}{\alpha(s) - \frac{1}{xy}} \wedge \frac{d\alpha(t)}{\alpha(t) - \frac{1}{y}}$$

with $\alpha : [0,1] \to \mathbb{C}$ *a smooth path from* 0 *to* 1*, and* $\frac{1}{xy}, \frac{1}{b} \in \mathbb{C} \setminus \mathrm{Im}\alpha$*, defined above on page 288, provide a genuine analytic continuation of* $\mathrm{Li}_{1,1}(x, y)$ *to a multivalued function which is defined on* $\{(x, y) \in \mathbb{C}^2 \mid x, y \neq 0, xy \neq 1, y \neq 1\}$.

Proof. We describe this analytic continuation in detail. Our approach is similar to the one taken in [G3, 2.3, p. 9], but differs from that in [Z2a, p. 7].

Let $B^{\mathrm{an}} := (\mathbb{C} \setminus \{0,1\})^2$ be the parameter space and choose a point $(a,b) \in B^{\mathrm{an}}$. For $\epsilon > 0$ we denote by $D_\epsilon(a,b)$ the polycylinder

$$D_\epsilon(a,b) := \{(a,b) \in B^{\mathrm{an}} \,|\, |a'-a| < \epsilon, |b'-b| < \epsilon\}.$$

If $\alpha : [0,1] \to \mathbb{C}$ is a smooth path from 0 to 1 passing through neither a nor b, then there exists an $\epsilon > 0$ such that $\mathrm{Im}\alpha$ does not meet any of the discs

$$\begin{aligned} D_{2\epsilon}(a) &:= \{a' \in \mathbb{C} \,|\, |a'-a| < 2\epsilon\}, \quad \text{and} \\ D_{2\epsilon}(b) &:= \{b' \in \mathbb{C} \,|\, |b'-b| < 2\epsilon\}. \end{aligned}$$

Hence the power series (14.2) below

$$\begin{aligned} \frac{1}{\alpha(s)-a'}\frac{1}{\alpha(t)-b'} &= \frac{1}{\alpha(s)-a}\frac{1}{1-\frac{a'-a}{\alpha(s)-a}}\frac{1}{\alpha(t)-b}\frac{1}{1-\frac{b'-b}{\alpha(t)-b}} \\ &= \sum_{k,l=0}^{\infty} \underbrace{\frac{1}{(\alpha(s)-a)^{k+1}(\alpha(t)-b)^{l+1}}}_{c_{k.l}} (a'-a)^k(b'-b)^l \quad (14.2) \end{aligned}$$

has coefficients $c_{k,l}$ satisfying

$$|c_{k,l}| < \left(\frac{1}{2\epsilon}\right)^{k+l+2}.$$

In particular, (14.2) converges uniformly for $(a',b') \in D_\epsilon(a,b)$ and we see that the integral

$$\begin{aligned} \mathrm{I}_2^\alpha(a',b') &:= \int_{0\le s\le t\le 1} \frac{d\alpha(s)}{\alpha(s)-a'} \wedge \frac{d\alpha(t)}{\alpha(t)-b'} \\ &= \sum_{k,l=0} \left(\int_{0\le s\le t\le 1} \frac{d\alpha(s)}{(\alpha(s)-a)^{k+1}} \wedge \frac{d\alpha(t)}{(\alpha(t)-b)^{l+1}}\right)(a'-a)^k(b'-b)^l \end{aligned}$$

defines an analytic function of $D_\epsilon(a,b)$. In fact, by the same argument we get an analytic function I_2^α on all of $(\mathbb{C} \setminus \mathrm{Im}\alpha)^2$.

Now let $\alpha_r : [0,1] \to \mathbb{C} \setminus (D_{2\epsilon}(a) \cup D_{2\epsilon}(b))$ with $r \in [0,1]$ be a smooth homotopy of paths from 0 to 1, i.e. $\alpha_r(0) = 0$ and $\alpha_r(1) = 1$ for all $r \in [0,1]$. We show

$$\mathrm{I}_2^{\alpha_0}(a',b') = \mathrm{I}_2^{\alpha_1}(a',b') \quad \text{for all} \quad (a',b') \in D_\epsilon(a,b).$$

Define a subset $\Gamma \subset \mathbb{C}^2$

$$\Gamma := \{(\alpha_r(s), \alpha_r(t)) \,|\, 0 \le s \le t \le 1, r \in [0,1]\}.$$

The boundary of Γ is built out of five components (each being a manifold with boundary)

- $\Gamma_{s=0} := \{(0, \alpha_r(t)) \,|\, r, t \in [0,1]\}$,

- $\Gamma_{s=t} := \{(\alpha_r(s), \alpha_r(s)) \,|\, r, s \in [0,1]\}$,
- $\Gamma_{t=1} := \{(\alpha_r(s), 1) \,|\, r, s \in [0,1]\}$,
- $\Gamma_{r=0} := \{(\alpha_0(s), \alpha_0(t)) \,|\, 0 \le s \le t \le 1\}$,
- $\Gamma_{r=1} := \{(\alpha_1(s), \alpha_1(t)) \,|\, 0 \le s \le t \le 1\}$.

Let $(a', b') \in D_\epsilon(a, b)$. Since the restriction of $\frac{dx}{x-a'} \wedge \frac{dy}{y-b'}$ to $\Gamma_{s=0}$, $\Gamma_{s=t}$ and $\Gamma_{t=1}$ is zero, we get by Stokes' theorem

$$\begin{aligned} 0 = \int_\Gamma 0 &= \int_\Gamma d\frac{dx}{x-a'} \wedge \frac{dy}{y-b'} \\ &= \int_{\partial\Gamma} \frac{dx}{x-a'} \wedge \frac{dy}{y-b'} \\ &= \int_{\Gamma_{r=1} - \Gamma_{r=0}} \frac{dx}{x-a'} \frac{dy}{y-b'} \\ &= \mathrm{I}_2^{\alpha_1}(a', b') - \mathrm{I}_2^{\alpha_0}(a', b'). \end{aligned}$$

For each pair of smooth paths $\alpha_0, \alpha_1 : [0,1] \to \mathbb{C}$ from 0 to 1, we can find a homotopy α_r relative to $\{0,1\}$ between both paths. Since $\mathrm{Im}\alpha_r$ is compact, we also find a point $(a,b) \in B^{\mathrm{an}} = (\mathbb{C} \setminus \{0,1\})^2$ and an $\epsilon > 0$ such that $\mathrm{Im}\alpha_r$ does not meet $D_{2\epsilon}(a,b)$ or $D_{2\epsilon}(a,b)$. Then $\mathrm{I}_2^{\alpha_0}$ and $\mathrm{I}_2^{\alpha_1}$ must agree on $D_\epsilon(a,b)$. By the identity principle for analytic functions of several complex variables [Gun, A, 3, p. 5], the functions $\mathrm{I}_2^{\alpha}(a', b')$, each defined on $(\mathbb{C} \setminus \mathrm{Im}\alpha)^2$, patch together to give a multivalued analytic function on $B^{\mathrm{an}} = (\mathbb{C} \setminus \{0,1\})^2$.

Now assume $1 < |b| < |a|$, then we can take $\alpha = \mathrm{id} : [0,1] \to \mathbb{C}$, $s \mapsto s$, and obtain

$$\mathrm{I}_2^{\mathrm{id}}(a,b) = \mathrm{I}_2(a,b) = \mathrm{Li}_{1,1}\left(\frac{b}{a}, \frac{1}{y}\right),$$

where $\mathrm{Li}_{1,1}(x,y)$ is the double logarithm defined for $|x|, |y| < 1$ in Subsection 14.6. Thus we have proved the lemma. □

Definition 14.6.6 (Double logarithm)**.** We call the analytic continuation from Lemma 14.6.5 the *double logarithm* as well and continue to use the notation $\mathrm{Li}_{1,1}(x,y)$.

The period matrix P is thus given by:

	Γ_0	Γ_1	Γ_2	Γ_3
ω_0	1	0	0	0
ω_1	$\mathrm{Li}_1(\frac{1}{b})$	$2\pi i$	0	0
ω_2	$\mathrm{Li}_1(\frac{1}{a})$	0	$2\pi i$	0
ω_3	$\mathrm{Li}_{1,1}\left(\frac{b}{a}, \frac{1}{b}\right)$	$2\pi i \mathrm{Li}_1(\frac{b}{a})$	$2\pi i \log\left(\frac{a-b}{1-b}\right)$	$(2\pi i)^2$.

14.6.5 Varying parameters a and b

The homology group $H_2^{\text{sing}}(X, D; \mathbb{Q})$ of the pair (X, D) carries a $\mathbb{Q}$-MHS $(W_\bullet, F^\bullet)$. The weight filtration is given in terms of the $\{\gamma_j\}$:

$$W_p H_2^{\text{sing}}(X, D; \mathbb{Q}) = \begin{cases} 0 & \text{for} \quad p \leq -5 \\ \mathbb{Q}\gamma_3 & \text{for} \quad p = -4, -3 \\ \mathbb{Q}\gamma_1 \oplus \mathbb{Q}\gamma_2 \oplus \mathbb{Q}\gamma_3 & \text{for} \quad p = -2, -1 \\ \mathbb{Q}\gamma_0 \oplus \mathbb{Q}\gamma_1 \oplus \mathbb{Q}\gamma_2 \oplus \mathbb{Q}\gamma_3 & \text{for} \quad p \geq 0, \end{cases}$$

The Hodge filtration is given in terms of the $\{\omega_i^*\}$:

$$F^p H_2^{\text{sing}}(X, D; \mathbb{Q}) = \begin{cases} \mathbb{C}\omega_0^* \oplus \mathbb{C}\omega_1^* \oplus \mathbb{C}\omega_2^* \oplus \mathbb{C}\omega_3^* & \text{for} \quad p \leq -2 \\ \mathbb{C}\omega_0^* \oplus \mathbb{C}\omega_1^* \oplus \mathbb{C}\omega_2^* & \text{for} \quad p = -1 \\ \mathbb{C}\omega_0^* & \text{for} \quad p = 0 \\ 0 & \text{for} \quad p \geq 1. \end{cases}$$

This $\mathbb{Q}$-MHS resembles very much the $\mathbb{Q}$-MHS considered in [G1, 2.2, p. 620] and [Z2a, 3.2, p. 6]. Nevertheless a few differences are note-worthy:

- Goncharov defines the weight filtration slightly different, for example his lowest weight is -6.
- The entry $P_{3,2} = 2\pi i \log\left(\frac{a-b}{1-b}\right)$ of the period matrix P differs by $(2\pi i)^2$, or put differently, the basis $\{\gamma_0, \gamma_1, \gamma_2 - \gamma_3, \gamma_3\}$ is used.

Up to now, the parameters a and b of the configuration (X, D) have been fixed. By varying a and b, we obtain a family of configurations: Equip $\mathbb{A}^2_{\mathbb{C}}$ with coordinates a and b and let

$$B := \mathbb{A}^2_{\mathbb{C}} \setminus (\{a = 0\} \cup \{a = 1\} \cup \{b = 0\} \cup \{b = 1\})$$

be the parameter space. Take another copy of $\mathbb{A}^2_{\mathbb{C}}$ with coordinates x and y and define total spaces

$$\underline{X} := (\underset{(a,b,x,y)}{B \times \mathbb{A}^2_{\mathbb{C}}}) \setminus (\{x = a\} \cup \{y = b\}), \quad \text{and}$$
$$\underline{D} := \text{“}B \times D\text{”} = \underline{X} \cap (\{x = 0\} \cup \{y = 1\} \cup \{x = y\}).$$

We now have a projection

$$\begin{array}{ccccc} \underline{D} & \hookrightarrow & \underline{X} & & (a, b, x, y) \\ & \searrow & \downarrow \pi & & \downarrow \\ & & B & & (a, b) \end{array},$$

whose fiber over a closed point $(a, b) \in B$ is precisely the configuration (X, D) for the parameter choice a, b. π is a flat morphism. The assignment

$$(a, b) \mapsto (V_{\mathbb{Q}}, W_\bullet, F^\bullet),$$

where

$$V_{\mathbb{Q}} := \operatorname{span}_{\mathbb{Q}}\{s_0, \ldots, s_3\},$$

$$V_{\mathbb{C}} := \mathbb{C}^4 \quad \text{with standard basis } e_0, \ldots, e_3,$$

$$s_0 := \begin{pmatrix} 1 \\ \mathrm{Li}\left(\frac{1}{b}\right) \\ \mathrm{Li}_1\left(\frac{1}{a}\right) \\ \mathrm{Li}_{1,1}\left(\frac{b}{a}, \frac{1}{b}\right) \end{pmatrix}, \; s_1 := \begin{pmatrix} 0 \\ 2\pi i \\ 0 \\ 2\pi i \mathrm{Li}_1\left(\frac{b}{a}\right) \end{pmatrix}, \; s_2 := \begin{pmatrix} 0 \\ 0 \\ 2\pi i \\ 2\pi i \log\left(\frac{a-b}{1-b}\right) \end{pmatrix}, \; s_3 := \begin{pmatrix} 0 \\ 0 \\ 0 \\ (2\pi i)^2 \end{pmatrix},$$

$$W_p V_{\mathbb{Q}} = \begin{cases} 0 & \text{for} \quad p \leq -5 \\ \mathbb{Q}s_3 & \text{for} \quad p = -4, -3 \\ \mathbb{Q}s_1 \oplus \mathbb{Q}s_2 \oplus \mathbb{Q}s_3 & \text{for} \quad p = -2, -1 \\ V_{\mathbb{Q}} & \text{for} \quad p \geq 0, \quad \text{and} \end{cases}$$

$$F^p V_{\mathbb{C}} = \begin{cases} V_{\mathbb{C}} & \text{for} \quad p \leq -2 \\ \mathbb{C}e_0 \oplus \mathbb{C}e_1 \oplus \mathbb{C}e_2 & \text{for} \quad p = -1 \\ \mathbb{C}e_0 & \text{for} \quad p = 0 \\ 0 & \text{for} \quad p \geq 1 \end{cases}$$

defines a good unipotent variation of $\mathbb{Q}$-MHS on B^{an}. Note that the Hodge filtration $F^\bullet$ does not depend on $(a, b) \in B^{\mathrm{an}}$.

One of the main characteristics of good unipotent variations of $\mathbb{Q}$-MHS is that they can be extended to a compactification of the base space (if the complement is a divisor with normal crossings).

The algorithm for computing these extensions, so called *limit mixed $\mathbb{Q}$-Hodge structures*, can be found for example in [H, 7, p. 24f] and [Z2b, 4, p. 12].

In a first step, we extend the variation to the divisor $\{a = 1\}$ minus the point $(1, 0)$ and then in a second step we extend it to the point $(1, 0)$. In particular, we assume that a branch has been picked for each entry P_{ij} of P. We will follow [Z2b, 4.1, p. 14f] very closely.

First step: Let σ be the loop winding counterclockwise around $\{a = 1\}$ once, but not around $\{a = 0\}$, $\{b = 0\}$ or $\{b = 1\}$. If we analytically continue an entry P_{ij} of P along σ we possibly get a second branch of the same multivalued function. In fact, the matrix resulting from analytic continuation of every entry along σ will be of the form

$$P \cdot T_{\{a=1\}},$$

where

$$T_{\{a=1\}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is the *monodromy matrix* corresponding to σ. The *local monodromy logarithm* is defined as

$$N_{\{a=1\}} = \frac{\log T_{\{a=1\}}}{2\pi i} = \frac{1}{2\pi i}\sum_{n=1}^{\infty}\frac{-1}{n}\left(\begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix} - T_{\{a=1\}}\right)^n$$
$$= \begin{pmatrix}0&0&0&0\\0&0&0&0\\\frac{-1}{2\pi i}&0&0&0\\0&0&0&0\end{pmatrix}.$$

We want to extend our $\mathbb{Q}$-MHS along the tangent vector $\frac{\partial}{\partial a}$, i.e. we introduce a local coordinate $t := a-1$ and compute the *limit period matrix*

$$P_{\{a=1\}} := \lim_{t\to 0} P\cdot e^{-\log(t)\cdot N_{\{a=1\}}}$$
$$= \lim_{t\to 0}\begin{pmatrix}1&0&0&0\\\mathrm{Li}_1\left(\frac{1}{b}\right)&2\pi i&0&0\\\mathrm{Li}_1\left(\frac{1}{1+t}\right)&0&2\pi i&0\\\mathrm{Li}_{1,1}\left(\frac{b}{1+t},\frac{1}{b}\right)&2\pi i\mathrm{Li}_1\left(\frac{b}{1+t}\right)&2\pi i\log\left(\frac{1-b+t}{1-b}\right)&(2\pi i)^2\end{pmatrix}\cdot\begin{pmatrix}1&0&0&0\\0&1&0&0\\\frac{\log(t)}{2\pi i}&0&1&0\\0&0&0&1\end{pmatrix}$$
$$= \lim_{t\to 0}\begin{pmatrix}1&0&0&0\\\mathrm{Li}_1\left(\frac{1}{b}\right)&2\pi i&0&0\\\mathrm{Li}_1\left(\frac{1}{1+t}\right)+\log(t)&0&2\pi i&0\\\mathrm{Li}_{1,1}\left(\frac{b}{1+t},\frac{1}{b}\right)+\log\left(\frac{1-b+t}{1-b}\right)\cdot\log(t)&2\pi i\mathrm{Li}_1\left(\frac{b}{1+t}\right)&2\pi i\log\left(\frac{1-b+t}{1-b}\right)&(2\pi i)^2\end{pmatrix}$$
$$\overset{(*)}{=}\begin{pmatrix}1&0&0&0\\\mathrm{Li}_1\left(\frac{1}{b}\right)&2\pi i&0&0\\0&0&2\pi i&0\\-\mathrm{Li}_2\left(\frac{1}{1-b}\right)&2\pi i\mathrm{Li}_1(b)&0&(2\pi i)^2\end{pmatrix}.$$

Here we used at $(*)$

- $P_{\{a=1\}_{2,0}} = \lim_{t\to 0}\mathrm{Li}_1\left(\frac{1}{1+t}\right)+\log(t)$
  $= \lim_{t\to 0} -\log\left(1-\frac{1}{1+t}\right)+\log(t)$
  $= \lim_{t\to 0} -\log(t)+\log(1+t)+\log(t)$
  $= 0,$ and

- $P_{\{a=1\}_{3,0}} = \lim_{t\to 0}\mathrm{Li}_{1,1}\left(\frac{b}{1+t},\frac{1}{b}\right)+\log\left(\frac{1-b+t}{1-b}\right)\cdot\log(t)$
  $= \mathrm{Li}_{1,1}\left(b,\frac{1}{b}\right)$ by L'Hospital
  $= -\mathrm{Li}_2\left(\dfrac{1}{1-b}\right).$

The vectors s_0, s_1, s_2, s_3 spanning the $\mathbb{Q}$-lattice of the limit $\mathbb{Q}$-MHS on $\{a = 1\} \setminus \{(1,0)\}$ are now given by the columns of the limit period matrix

$$s_0 = \begin{pmatrix} 1 \\ \mathrm{Li}_1\left(\frac{1}{b}\right) \\ 0 \\ -\mathrm{Li}_2\left(\frac{1}{1-b}\right) \end{pmatrix}, \quad s_1 = \begin{pmatrix} 0 \\ 2\pi i \\ 0 \\ 2\pi i \mathrm{Li}_1(b) \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 \\ 0 \\ 2\pi i \\ 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ (2\pi i)^2 \end{pmatrix}.$$

The weight and Hodge filtration of the limit $\mathbb{Q}$-MHS can be expressed in terms of the s_j and the standard basis vectors e_i of $\mathbb{C}^4$. This gives us a variation of $\mathbb{Q}$-MHS on the divisor $\{a = 1\} \setminus \{(1,0)\}$. This variation is actually (up to signs) an extension of Deligne's famous *dilogarithm variation* considered for example in [Kj, 4.2, p. 38f]. In loc. cit. the geometric origin of this variation is explained in detail.

Second step: We now extend this variation along the tangent vector $\frac{-\partial}{\partial b}$ to the point $(1,0)$, i.e. we write $b = -t$ with a local coordinate t. Let σ be the loop in $\{a = 1\} \setminus \{(1,0)\}$ winding counterclockwise around $(1,0)$ once, but not around $(1,1)$. Then the monodromy matrix corresponding to σ is given by

$$T_{(1,0)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

hence the local monodromy logarithm is given by

$$N_{(1,0)} = \frac{\log T_{(1,0)}}{2\pi i} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2\pi i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus we get for the limit period matrix

$$\begin{aligned} P_{(1,0)} &:= \lim_{t \to 0} P_{\{a=1\}} \cdot e^{-\log(t) \cdot N_{(1,0)}} \\ &= \lim_{t \to 0} \begin{pmatrix} 1 & 0 & 0 & 0 \\ \mathrm{Li}_1\left(\frac{-1}{t}\right) & 2\pi i & 0 & 0 \\ 0 & 0 & 2\pi i & 0 \\ -\mathrm{Li}_2\left(\frac{1}{1+t}\right) & 2\pi i \,\mathrm{Li}_1(-t) & 0 & (2\pi i)^2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{-\log(t)}{2\pi i} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \lim_{t \to 0} \begin{pmatrix} 1 & 0 & 0 & 0 \\ \mathrm{Li}_1\left(\frac{-1}{t}\right) - \log(t) & 2\pi i & 0 & 0 \\ 0 & 0 & 2\pi i & 0 \\ -\mathrm{Li}_2\left(\frac{1}{1+t}\right) - \mathrm{Li}_1(-t) \cdot \log(t) & 0 & 0 & (2\pi i)^2 \end{pmatrix} \\ &\overset{(*)}{=} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2\pi i & 0 & 0 \\ 0 & 0 & 2\pi i & 0 \\ -\zeta(2) & 0 & 0 & (2\pi i)^2 \end{pmatrix}. \end{aligned}$$

We remark that in the last matrix we see a decomposition into two (2×2)-blocks, one consisting of a Tate motive, the other involving $\zeta(2)$.

Here we used at $(*)$

- $$\begin{aligned} {P_{(1,0)}}_{1,0} &= \lim_{t\to 0} \mathrm{Li}_1\bigl(\tfrac{-1}{t}\bigr) - \log(t) \\ &= \lim_{t\to 0} -\log\bigl(1+\tfrac{1}{t}\bigr) - \log(t) \\ &= \lim_{t\to 0} -\log(1+t) + \log(t) - \log(t) \\ &= 0, \quad \text{and} \end{aligned}$$

- $$\begin{aligned} {P_{(1,0)}}_{3,0} &= \lim_{t\to 0} -\mathrm{Li}_2\Bigl(\frac{1}{1+t}\Bigr) - \mathrm{Li}_1(-t)\cdot\log(t) \\ &= \lim_{t\to 0} \mathrm{Li}_2\Bigl(\frac{1}{1+t}\Bigr) + \log(1+t)\cdot\log(t) \\ &= -\mathrm{Li}_2(1) \quad \text{by L'Hospital} \\ &= -\zeta(2). \end{aligned}$$

As in the previous step, the vectors s_0, s_1, s_2, s_3 spanning the $\mathbb{Q}$-lattice of the limit $\mathbb{Q}$-MHS are given by the columns of the limit period matrix $P_{(1,0)}$ and weight and Hodge filtrations by the formulae in subsection 14.6.5.

So we obtained $-\zeta(2)$ as a "period" of a limiting $\mathbb{Q}$-MHS.

Chapter 15

Miscellaneous periods: an outlook

In this chapter, we collect several other important examples of periods in the literature for the convenience of the reader.

15.1 Special values of L-functions

The Beilinson conjectures give a formula for the values (more precisely, the leading coefficients) of L-functions of motives at integral points. We sketch the formulation in order to explain that these numbers are periods.

In this section, fix the base field $k = \mathbb{Q}$. Let $\Gamma = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ be the absolute Galois group. For any prime p, let $I_p \subset \Gamma$ be the inertia group. Let $\mathrm{Fr}_p \in \Gamma/I_p$ be the Frobenius.

Let M be a mixed motive, i.e., an object in the conjectural $\mathbb{Q}$-linear abelian category of mixed motives over $\mathbb{Q}$. For any prime l, it has an l-adic realization M_l which is a finite dimensional $\mathbb{Q}_l$-vector space with a continuous operation of the absolute Galois group $G_\mathbb{Q} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

Definition 15.1.1. Let M as above, p a prime and l a prime different from p. We put

$$P_p(M,t)_l = \det(1 - \mathrm{Fr}_p t | M_l^{I_p}) \in \mathbb{Q}_l[t] \ .$$

It is conjectured that $P_p(M,t)_l$ is in $\mathbb{Q}[t]$, and independent of l. We denote this polynomial by $P_p(M,t)$.

Example 15.1.2. Let $M = H^i(X)$ for smooth projective variety over $\mathbb{Q}$ with good reduction at p. Then the conjecture holds by the Weil conjectures proved by Deligne. In the special case $X = \mathrm{Spec}(\mathbb{Q})$, we get

$$P_p(H(\mathrm{Spec}\mathbb{Q}), t) = 1 - t\ .$$

In the special case $X = \mathbb{P}^1$, $i = 1$, we get

$$P_p(H^1(\mathbb{P}^1), t) = 1 - pt\ .$$

Remark 15.1.3. There is a sign issue with the operation of Fr_p depending on the normalization of $\mathrm{Fr} \in \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ and whether it operates via geometric or arithmetic Frobenius. We refrain from working out all the details.

Definition 15.1.4. Let M be as above. We put

$$L(M, s) = \prod_{p \text{ prime}} \frac{1}{P_p(M, p^{-s})}$$

as function in the variable $s \in \mathbb{C}$. For $n \in \mathbb{Z}$, let

$$L(M, n)^*$$

be the leading coefficient of the Laurent expansion of $L(M, s)$ around n.

We conjecture that the infinite product converges for $\mathrm{Re}(s)$ big enough and that the function has a meromorphic continuation to all of $\mathbb{C}$.

Example 15.1.5. Let $M = H^i(X)$ for X a smooth projective variety over $\mathbb{Q}$. Then convergence follows from the Riemann hypothesis part of the Weil conjectures. (Note that X has good reduction at almost all p. It suffices to consider these. Then the zeros of $P_p(M, t)$ are known to have absolute value $p^{-\frac{i}{2}}$.

Analytic continuation is a very deep conjecture. It holds for all 0-dimensional X. Indeed, for any number field K, we have

$$L(H^0(\mathrm{Spec}K), s) = \zeta_K(s)$$

where $\zeta_K(s)$ is the Dedekind ζ-function. For $M = H^1(E)$ with E an elliptic curve over $\mathbb{Q}$, we have

$$L(H^1(E), s) = L(E, s)\ .$$

Analytic continuation holds, because E is modular.

Example 15.1.6. Let M be as above, $\mathbb{Q}(-1) = H^2(\mathbb{P}^1)$ be the Lefschetz motive. We put $M(-1) = M \otimes \mathbb{Q}(-1)$. Then

$$L(M(-1), s) = L(M, s - 1)$$

by the formula for $P_p(\mathbb{Q}(-1), t)$ above.

Hence, the Beilinson conjecture about $L(M,s)$ at $n \in \mathbb{Z}$ can be reduced to the Beilinson conjecture about $L(M(-n),s)$ at 0.

Conjecture 15.1.7 (Beilinson [Be3])**.** *Let M be as above. Then the vanishing order of $L(M,s)$ at $s=0$ is given by*

$$\dim H^1_{\mathcal{M},f}(\mathrm{Spec}\mathbb{Q}, M^*(1)) - \dim H^0_{\mathcal{M},f}(\mathrm{Spec}\mathbb{Q}, M),$$

where $H_{\mathcal{M},f}$ is unramified motivic cohomology.

For a conceptional discussion of unramified motivic cohomology and a comparison of the different possible definitions, see Scholbach's discussion in [Sch2].

In particular, we assume that unramified motivic cohomology is finite dimensional.

This conjecture is known for example when $M = H^0(\mathrm{Spec}K)(n)$ with K a number field, $n \in \mathbb{Z}$ or when $M = H^1(E)$ with E an elliptic curve with Mordell-Weil rank at most 1.

Definition 15.1.8. We call M *special* if the motivic cohomology groups

$$H^0_{\mathcal{M},f}(\mathrm{Spec}\mathbb{Q}, M),\ H^1_{\mathcal{M},f}(\mathrm{Spec}\mathbb{Q}, M),\ H^0_{\mathcal{M},f}(\mathrm{Spec}\mathbb{Q}, M^*(1)),\ H^1_{\mathcal{M},f}(\mathrm{Spec}\mathbb{Q}, M^*(1))$$

all vanish.

We are only going to state the Beilinson conjecture for special motives. In this case it is also known as Deligne conjecture. This suffices:

Proposition 15.1.9 (Scholl, [Scho])**.** *Let M be a motive as above. Assume all motivic cohomology groups over $\mathbb{Q}$ are finite-dimensional. Then there is a special motive M' such that*

$$L(M,0)^* = L(M',0)$$

and the Beilinson conjecture for M is equivalent to the Beilinson conjecture for M'.

Conjecture 15.1.10 (Beilinson [Be3], Deligne [D1])**.** *Let M be a special motive. Let M_B be its Betti-realization and M_{dR} its de Rham realization.*

1. *$L(M,0)$ is defined and non-zero.*
2. *The composition*

$$M_B^+ \otimes \mathbb{C} \to M_B \otimes \mathbb{C} \xrightarrow{\mathrm{per}} M_{\mathrm{dR}} \otimes \mathbb{C} \to M_{\mathrm{dR}} \otimes \mathbb{C}/F^0 M_{\mathrm{dR}} \otimes \mathbb{C}$$

   is an isomorphism. Here M_B^+ denotes the invariants under complex conjugation and F^0M_{dR} denotes the 0-step of the Hodge filtration.
3. *Up to a rational factor, the value $L(M,0)$ is given by the determinant of the above isomorphism in any choice of rational basis of M_B^+ and M_{dR}.*

Corollary 15.1.11. *Assume the Beilinson conjecture holds. Let M be a motive. Then $L(M,0)^*$ is a period number.*

Proof. By Scholl's reduction, it suffices to consider the case M special. The matrix of the morphism in the conjecture is a block in the matrix of

$$\mathrm{per} : M_B \otimes \mathbb{C} \to M_{\mathrm{dR}} \otimes \mathbb{C} \ .$$

All its entries are periods. Hence, the same is true for the determinant. □

15.2 Feynman periods

Standard procedures in quantum field theory (QFT) lead to loop amplitudes associated to certain graphs [BEK, MWZ2]. Although the foundations of QFT via path integrals are mathematically non-rigorous, Feynman and others have set up the so-called Feynman rules as axioms, leading to a mathematically precise definition of *loop integrals (or, amplitudes)*.

These are defined as follows. Associated to a graph G one defines the integral as

$$I_G = \frac{\prod_{j=1}^n \Gamma(\nu_j)}{\Gamma(\nu - \ell D/2)} \int_{\mathbb{R}^{D\ell}} \frac{\prod_{r=1}^{\ell} dk_r}{i\pi^{D/2}} \prod_{j=1}^{n} (-q_j^2 + m_j^2)^{-\nu_j}.$$

Here, D is the dimension of space-time (usually, but not always, $D = 4$), n is the number of internal edges of G, $\ell = h_1(G)$ is the loop number, ν_j are integers associated to each edge, ν is the sum of all ν_j, the m_j are masses, the q_j are combinations of external momenta and internal loop momenta k_r, over which one has to integrate [MWZ2, Sect. 2]. All occurring squares are scalar products in D-dimensional Minkowski space. The integrals usually do not converge in D-space, but standard renormalization procedures in physics, e.g. dimensional regularization, lead to explicit numbers as coefficients of Laurent series. In dimensional regularization, one views the integrals as analytic meromorphic functions in the paramter $\epsilon \in \mathbb{C}$ where $D = 4 - 2\epsilon$. The coefficients of the resulting Laurent expansion in the variable ϵ are then the relevant numbers. By a theorem of Belkale-Brosnan [BB] and Bogner-Weinzierl [BW], such numbers are periods, if all moments and masses in the formulas are rational numbers.

A process called Feynman-Schwinger trick [BEK] transforms the above integral into a period integral

$$I_G = \int_\sigma f\omega$$

with

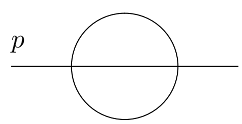
$$f = \frac{\prod_{j=1}^n x_j^{\nu_j - 1} \mathcal{U}^{\nu-(\ell+1)D/2}}{\mathcal{F}^{\nu - \ell D/2}}, \quad \omega = \sum_{j=1}^{n} (-1)^j dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n.$$

Here, $\mathcal{U}$ and $\mathcal{F}$ are homogenous graph polynomials of Kirchhoff type [MWZ2, Sect. 2], with only $\mathcal{F}$ depending on kinematical invariants, and σ is the standard real simplex in $\mathbb{P}^{n-1}(\mathbb{C})$. Since σ is a compact subset of $\mathbb{P}^{n-1}(\mathbb{C})$, this is almost a representation of I_G as a naive period, and it is indeed one as a Kontsevich-Zagier period, provided the external momenta p_i are rational numbers. The differential form $f\omega$ has poles along σ, but there is a canonical blow-up process to resolve this problem [BEK, MWZ2]. The period which emerges is the period of the relative cohomology group

$$H^n(P \setminus Y, B \setminus (B \cap Y)),$$

where P is a blow-up of projective space in linear coordinate subspaces, Y is the strict transform of the singularity set of the integrand, and B is the strict transform of the standard algebraic simplex $\Delta^{n-1} \subset \mathbb{P}^{n-1}$ [MWZ2, Sect. 2]. It is thus immediate that I_G is a Kontsevich-Zagier period, if it is convergent, and provided that all masses and momenta involved are rational. If I_G is not convergent, then, by a theorem of Belkale-Brosnan [BB] and Bogner-Weinzierl [BW], the same holds under these assumptions for the coefficients of the Laurent expansion in renormalization.

Example 15.2.1. A very popular graph with a divergent amplitude is the two-loop sunset graph



The corresponding amplitude in D dimensions is

$$\Gamma(3-D)\int_\sigma \frac{(x_1x_2+x_2x_3+x_3x_1)^{3-\frac{3}{2}D}(x_1dx_2\wedge dx_3 - x_2dx_1\wedge dx_3 + x_3dx_1\wedge dx_2)}{(-x_1x_2x_3p^2+(x_1m_1^2+x_2m_2^2+x_3m_3^2)(x_1x_2+x_2x_3+x_3x_1))^{3-D}},$$

where σ is the real 2-simplex in $\mathbb{P}^2$.

In $D=4$, this integral does not converge. One may, however, compute the integral in $D=2$ and study its dependence on the momentum p as an inhomogenous differential equation [MWZ1]. There is an obvious family of elliptic curves involved in the equations of the denominator of the integral, which gives rise to the homogenous Picard-Fuchs equation [MWZ1]. Then, a trick of Tarasov allows to compute the $D=4$ situation from that, see [MWZ1]. The extension of mixed Hodge structures

$$0 \to \mathbb{Z}(-1) \to H^2(P\setminus Y, B\setminus B\cap Y) \to H^2(P\setminus Y) \to 0$$

arising from this graph is already quite complicated [MWZ1, BV], as there are three different weights involved. The corresponding period functions when the momentum p varies are given by elliptic dilogarithm functions [BV, ABW]. There are generalizations to higher loop banana graphs [BKV].

In the literature, there are many more concrete examples of such periods, see the work of Broadhurst-Kreimer [BK] and subsequent work. Besides multiple zeta values, there are for examples graphs G where the integral is related to periods of K3 surfaces [BS].

15.3 Algebraic cycles and periods

In this section, we want to show how algebraic cycles in (higher) Chow groups give rise to Kontsevich-Zagier periods. Let us start with an example.

Example 15.3.1. Assume that $k \subset \mathbb{C}$, and let X be a smooth, projective curve of genus g, and $Z = \sum_{i=1}^k a_i Z_i \in CH^1(X)$ be a non-trivial zero-cycle on X with degree 0, i.e., $\sum_i a_i = 0$. Then we have a sequence of cohomology groups

$$0 \to H^1(X^{\mathrm{an}}) \to H^1(X^{\mathrm{an}} \backslash |Z|) \to H^2_{|Z|}(X^{\mathrm{an}}) \cong \bigoplus_i \mathbb{Z}(-1) \xrightarrow{\Sigma} H^2(X^{\mathrm{an}}, \mathbb{Z}) \cong \mathbb{Z}(-1) \; .$$

The cycle Z defines a non-zero vector $(a_1, ..., a_k) \in \bigoplus_i \mathbb{Z}(-1)$ mapping to zero in $H^2(X^{\mathrm{an}}, \mathbb{Z})$. Hence, by pulling back, we obtain an extension

$$0 \to H^1(X^{\mathrm{an}}) \to E \to \mathbb{Z}(-1) \to 0 \; .$$

The extension class of this sequence in the category of mixed Hodge structures is known to be the *Abel-Jacobi class* of Z [C]. One can compute it in several ways. For example, one can choose a continuous chain γ with $\partial\gamma = \sum_i a_i Z_i$ and a basis $\omega_1, ..., \omega_g$ of holomorphic 1-forms on X^{an}. Then the vector

$$\left(\int_\gamma \omega_1, \dots, \int_\gamma \omega_g \right)$$

defines the Abel-Jacobi class in the *Jacobian*

$$\mathrm{Jac}(X) = \frac{H^1(X^{\mathrm{an}}, \mathbb{C})}{F^1 H^1(X^{\mathrm{an}}, \mathbb{C}) + H^1(X^{\mathrm{an}}, \mathbb{Z})} \cong \frac{H^0(X^{\mathrm{an}}, \Omega^1_{X^{\mathrm{an}}})^\vee}{H_1(X^{\mathrm{an}}, \mathbb{Z})} \; .$$

If X and the cycle Z are both defined over k, then obviously the Abel-Jacobi class is defined by g period integrals in $\mathbb{P}^{\mathrm{eff}}(k)$. In the case of smooth, projective curves, the Abel-Jacobi map

$$\mathrm{AJ}^1 : CH^1(X)_{\mathrm{hom}} \to \mathrm{Jac}(X)$$

gives an isomorphism when $k = \mathbb{C}$.

One can generalize this construction to Chow groups. Let X be a smooth, projective variety over $k \subset \mathbb{C}$, and $Z \in CH^q(X)$ a cycle which is homologous to zero. Then the Abel-Jacobi map

$$\mathrm{AJ}^q : CH^q(X)_{\mathrm{hom}} \longrightarrow \frac{H^{2q-1}(X^{\mathrm{an}}, \mathbb{C})}{F^q + H^{2q-1}(X^{\mathrm{an}}, \mathbb{Z})} \cong \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(-q), H^{2q-1}(X^{\mathrm{an}}, \mathbb{Z})) ,$$

As in the example above, the cycle Z defines an extension of mixed Hodge structures

$$0 \to H^{2q-1}(X^{\mathrm{an}}) \to E \to \mathbb{Z}(-q) \to 0 ,$$

where E is a subquotient of $H^{2q-1}(X^{\mathrm{an}} \setminus |Z|)$. The Abel-Jacobi class is given by period integrals

$$\left(\int_\gamma \omega_1, ..., \int_\gamma \omega_g \right)$$

in *Griffiths' Intermediate Jacobian*

$$\begin{aligned} \mathrm{J}^q(X) &= \frac{H^{2q-1}(X^{\mathrm{an}}, \mathbb{C})}{F^q H^{2q-1}(X^{\mathrm{an}}, \mathbb{C}) + H^{2q-1}(X^{\mathrm{an}}, \mathbb{Z})} \\ &\cong \frac{F^q H^{2q-1}(X^{\mathrm{an}}, \mathbb{C})^\vee}{H_{2q-1}(X^{\mathrm{an}}, \mathbb{Z})} . \end{aligned}$$

Even more general, one may use Bloch's *higher Chow groups* [Bl]. Higher Chow groups are isomorphic to motivic cohomology in the smooth case by a result of Voevodsky. In the general case, they only form a Borel-Moore homology theory and not a cohomology theory [VSF]. Then the Abel-Jacobi map becomes

$$\mathrm{AJ}^{q,n} : CH^q(X,n)_{\mathrm{hom}} \longrightarrow \frac{H^{2q-n-1}(X^{\mathrm{an}}, \mathbb{C})}{F^q + H^{2q-n-1}(X^{\mathrm{an}}, \mathbb{Z})} \cong \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(-q), H^{2q-n-1}(X^{\mathrm{an}}, \mathbb{Z})) ,$$

There are explicit formulas for $\mathrm{AJ}^{q,n}$ in [KLM, KLM2, Wei] on the level of complexes. This shows that the higher Abel-Jacobi class is defined by period integrals which define numbers in $\mathbb{P}^{\mathrm{eff}}(k)$.

In analogy with the classical Chow groups, Spencer Bloch has found an explicit description of the extension of mixed Hodge structures associated to a cycle $Z \in CH^q(X,n)_{\mathrm{hom}}$. This is explained in [DS, Scho2]. The periods associated to this mixed Hodge structures can then be viewed as the periods associated to Z.

Let us describe this construction. We let $\square^n := (\mathbb{P}^1 \setminus \{1\})^n$. For varying n, this defines a cosimplicial object with face and degeneracy maps obtained by using the natural coordinate t on $\mathbb{P}^1$. Faces are given by setting $t_i = 0$ or $t_i = \infty$. By definition, a cycle Z in a higher Chow group $CH^q(X,n)$ is a subvariety of $X \times \square^n$ meeting all faces $F = X \times \square^m \subset X \times \square^n$ for $m < n$ properly, i.e., in codimension q. By looking at the normalized cycle complex, we may assume that Z has zero intersection with all faces of $X \times \square^n$. Removing the support of Z, let $U := X \times \square^n \setminus |Z|$, and define ∂U to be the union of the intersection of U

with the codimension 1 faces of $X \times \square^n$. Then one obtains an exact sequence [DS, Lemma A.2]

$$0 \to H^{2q-n-1}(X^{\mathrm{an}}) \to H^{2q-1}(U^{\mathrm{an}}, \partial U^{\mathrm{an}}) \to H^{2q-1}(U^{\mathrm{an}}) \to H^{2q-1}(\partial U^{\mathrm{an}}) \ ,$$

which can be pulled back to an extension E if Z is homologous to zero:

$$0 \to H^{2q-n-1}(X^{\mathrm{an}}) \to E \to \mathbb{Z}(-q) \to 0 \ .$$

Hence, E is a subquotient of the mixed Hodge structure $H^{2q-1}(U^{\mathrm{an}}, \partial U^{\mathrm{an}})$. This works for any cohomology satisfying certain axioms, see [DS]. In particular, applying it to singular or de Rham cohomology, we obtain an extension inside the category of Nori motives.

For the category of Nori motives, extension groups are not known in general, and have only been computed in the situation of 1-motives [AB]. The extension groups of any abelian category $\mathrm{MM}(k)$ of mixed motives over k are conjecturally supposed to be Adams eigenspaces of algebraic K-groups, or, equivalently, motivic cohomology groups. For example, one expects that

$$\mathrm{Ext}^1_{\mathrm{MM}(k)}(\mathbb{Q}(-q), H^{2q-n-q}(X)) = H^{2q-n}_M(X, \mathbb{Q}(-q)) = K_n(X)^{(q)}_{\mathbb{Q}}$$

for a smooth, projective variety X.

15.4 Periods of homotopy groups

In this section, we want to explain the periods associated to fundamental groups and higher homotopy groups.

The topological fundamental group $\pi_1^{\mathrm{top}}(X(\mathbb{C}), a)$ of an algebraic variety X (defined over $k \subset \mathbb{C}$) with base point a carries a MHS in the following sense.

First, look at the group algebra $\mathbb{Q}\pi_1^{\mathrm{top}}(X(\mathbb{C}), a)$, and the augmentation ideal $I := \mathrm{Ker}(\mathbb{Q}\pi_1^{\mathrm{top}}(X, a) \to \mathbb{Q})$. Then the Malcev-type object

$$\hat{\pi}_1(X(\mathbb{C}), a)_{\mathbb{Q}} := \lim_{n\to\infty} \mathbb{Q}[\pi_1^{\mathrm{top}}(X(\mathbb{C}), a)]/I^{n+1}$$

should carry an Ind-MHS, as we will explain now. Beilinson observed that each finite step $\mathbb{Q}\pi_1^{\mathrm{top}}(X(\mathbb{C}), a)/I^{n+1}$ can be obtained as a MHS of a certain algebraic variety defined over the same field k. This was known to experts for some time, and later worked out in [DG].

Theorem 15.4.1. *Let M be any connected complex manifold and $a \in M$ a point. Then there is an isomorphism*

$$H_n(\underbrace{M \times \cdots \times M}_{n}, D; \mathbb{Q}) \cong \mathbb{Q}\pi_1^{\mathrm{top}}(M, a)/I^{n+1},$$

and $H_k(\underbrace{M \times \cdots \times M}_{n}, D; \mathbb{Q}) = 0$ for $k < n$. Here $D = \cup D_i$ is a divisor, where $D_0 = \{a\} \times M^{n-1}$, $D_{n+1} = M^{n-1} \times \{a\}$, and, for $1 \le i \le n-1$, $D_i = M^{i-1} \times \Delta \times M^{n-i-1}$ with $\Delta \subset M \times M$ the diagonal.

Proof. The proof in loc. cit., which we will not give here, proceeds by induction on n, using the first projection $p_1 : M^n \to M$ and the Leray spectral sequence. □

In the framework of Nori motives, one can thus see that $\hat{\pi}_1(X,a)_{\mathbb{Q}}$ immediately carries the structure of an Ind-Nori motive over k, since the Betti realization is obvious. Deligne-Goncharov [DG] and F. Brown [B2, B1] work instead within the framework of the abelian category of mixed Tate motives over $\mathbb{Q}$ of Levine [L2]. From this it follows, that $\hat{\pi}_1(\mathbb{P}^1\setminus\{0,1,\infty\},a)_{\mathbb{Q}}$ is an Ind-mixed Tate motive over $\mathbb{Q}$ (in fact, over $\mathbb{Z}$ as explained in [B1]). There is also a description of the de Rham realization in [DG, B2, B1]. In particular, Brown showed that each MZV occurs as a period of this Ind-MHS [B2, B1, D3], as we explained in Section 14.5.

Theorem 15.4.2. *Every multiple zeta value occurs as a period of $\hat{\pi}_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\},a)_{\mathbb{Q}}$. Furthermore, every multiple zeta value is a polynomial with $\mathbb{Q}$-coefficients in multiple zate values with only 2 and 3 as entries.*

Proof. See [B1, B2]. □

The proof of this theorem also implies that every mixed Tate motive over $\mathbb{Z}$ occurs as a finite subquotient of the Ind-motive $\hat{\pi}_1(\mathbb{P}^1 \setminus \{0,1,\infty\},a)_{\mathbb{Q}}$.

Let us now look at higher homotopy groups $\pi_n(X^{\mathrm{an}})$ for $n \geq 2$ of an algebraic variety X over $k \subset \mathbb{C}$. They carry a MHS rationally by a theorem of Morgan [Mo] and Hain [H]:

Theorem 15.4.3. *The homotopy groups $\pi_n(X^{\mathrm{an}}) \otimes \mathbb{Q}$ of a simply connected and smooth projective variety over $\mathbb{C}$ carry a functorial mixed Hodge structure for $n \geq 2$.*

This theorem has a natural extension to the non-compact case using logarithmic forms, and to the singular case using cubical hyperresolutions, see [PS] and [Na].

Example 15.4.4. Let X be a simply connected, smooth projective 3-fold over $\mathbb{C}$. Then the MHS on $\pi_3(X^{\mathrm{an}})^\vee$ is given by an extension

$$0 \to H^3(X^{\mathrm{an}},\mathbb{Q}) \to \mathrm{Hom}(\pi_3(X^{\mathrm{an}}),\mathbb{Q}) \to \mathrm{Ker}\left(S^2H^2(X^{\mathrm{an}},\mathbb{Q}) \to H^4(X^{\mathrm{an}},\mathbb{Q})\right) \to 0$$

Carlson, Clemens, and Morgan [CCM] prove that this extension is given by the Abel-Jacobi class of a certain codimension 2 cycle $Z \in CH^2_{\mathrm{hom}}(X)$, and the extension class of this MHS in the sense of [C] is given by the Abel-Jacobi class

$$\mathrm{AJ}^2(Z) \in J^2(X) = \frac{H^3(X^{\mathrm{an}},\mathbb{C})}{F^2 + H^3(X^{\mathrm{an}},\mathbb{Z})} .$$

The proof of Morgan uses the theory of Sullivan [Su]. In the simply connected case, there is a differential graded Lie algebra $L(X,x)$ over $\mathbb{Q}$, concentrated in degrees 0, -1, ..., such that

$$H_*(L(X,x)) \cong \pi_{*+1}(X^{\mathrm{an}}) \otimes \mathbb{Q}.$$

One can then use the cohomological description of $L(X, x)$ and Deligne's mixed Hodge theory, to define the MHS on homotopy groups using a complex defined over k. We would like to mention that one can try to make this construction motivic in the Nori sense. At least for affine varieties, this was done in [Ga], see also [CG, pg. 22]. In [G4], a description of periods of homotopy groups is given in terms of Hodge correlators. This is not well understood yet.

From the approach in [Ga], one can see, at least in the affine case, that the periods of the MHS on $\pi_n(X^{\mathrm{an}})$ are defined over k, i.e., are contained in $\mathbb{P}^{\mathrm{eff}}(k)$, when X is defined over k, since all motives involved in the construction are defined over k.

15.5 Non-periods

The question whether a given transcendental complex number is a period number in $\mathbb{P}^{\mathrm{eff}}(\mathbb{Q})$, i.e., is a Kontsevich-Zagier period, is very difficult to answer in general, even though we know that there are only countably many of them. For example, we expect (but do not know) that the Euler number e is not a period. Also $1/\pi$ and Euler's γ are presumably not effective periods, although no proof is known.

When Kontsevich-Zagier wrote their paper, the situation was like at the beginning of the 19th century for the study of algebraic and trancendental numbers. It took a lot of effort to prove that Liouville numbers $\sum_i 10^{-i!}$, e (Hermite) and π (Lindemann) were transcendental.

In 2008, M. Yoshinaga [Y] first wrote down a non-period $\alpha = 0.77766444...$ in 3-adic expansion

$$\alpha = \sum_{i=1}^{\infty} \epsilon_i 3^{-i} \ .$$

We will now explain how to define this number, and why it is not a period. First, we have to explain the notions of computable and elementary computable numbers.

Computable numbers and equivalent notions of computable (i.e., equivalently, partial recursive) functions $f : \mathbb{N}_0^n \to \mathbb{N}_0$ were introduced by Turing [T], Kleene and Church around 1936 following the ideas from Gödel's famous paper [G], see the references in [Kl] . We refer to [Bri] for a modern treatment of such notions which is intended for mathematicians.

The modern theory of computable functions starts with the notion of certain classes $\mathcal{E}$ of functions $f : \mathbb{N}_0 \to \mathbb{N}_0$. For each class $\mathcal{E}$ there is then a notion of $\mathcal{E}$-computable real numbers. In the following definition we follow [Y], but this was defined much earlier, see for example [R, Spe].

Definition 15.5.1. A real number $\alpha > 0$ is called *$\mathcal{E}$-computable*, if there are sequences $a(n)$, $b(n)$, $c(n)$ in $\mathcal{E}$, such that

$$\left|\frac{a(n)}{b(n)+1} - \alpha\right| < \frac{1}{k},\ \text{for all } n \geq c(k)\ .$$

The set of $\mathcal{E}$-computable numbers, including 0 and closed under $\alpha \mapsto -\alpha$, is denoted by $\mathbb{R}_{\mathcal{E}}$.

Some authors use the bound 2^{-k} instead of $\frac{1}{k}$. This leads to an equivalent notion only for classes $\mathcal{E}$ which contain exponentials $n \mapsto 2^n$.

If $\mathcal{E}$ = comp is the class of *Turing computable* [T], or equivalently Kleene's *partial recursive functions* [Kl], then $\mathbb{R}_{\text{comp}}$ is the set of *computable real numbers.* Computable complex numbers $\mathbb{C}_{\text{comp}}$ are those complex numbers where the real- and imaginary part are computable reals.

Theorem 15.5.2. *$\mathbb{R}_{\text{comp}}$ is a countable subfield of $\mathbb{R}$, and $\mathbb{C}_{\text{comp}} = \mathbb{R}_{\text{comp}}(i)$ is algebraically closed.*

One can think of computable numbers as the set of all numbers that can be accessed with a computer.

There are some important levels of hierarchies inside the set of computable reals

$$\mathbb{R}_{\text{low}-\text{elem}} \subsetneq \mathbb{R}_{\text{elem}} \subsetneq \mathbb{R}_{\text{comp}}\ ,$$

induced by the elementary functions of Kalmár (1943) [Ka], and the lower elementary functions of Skolem (1962) [Sk]. There is also the related Grzegorczyk hierarchy [Gr]. In order to define such hierarchies of real numbers, we will now study functions $f : \mathbb{N}_0^n \to \mathbb{N}_0$ of several variables.

Definition 15.5.3. The class of *lower-elementary functions* is the smallest class of functions $f : \mathbb{N}_0^n \to \mathbb{N}_0$

- containing the zero-function, the successor function $x \mapsto x + 1$ and the projection function $P_i : (x_1, ..., x_n) \mapsto x_i$,
- containing the addition $x + y$, the multiplication $x \cdot y$, and the modified subtraction $\max(x - y, 0)$,
- closed under composition, and
- closed under bounded summation.

The class of *elementary functions* is the smallest class which is also closed under bounded products.

Here, bounded summation (resp. product) is defined as

$$g(x, x_1, ..., x_n) = \sum_{a \leq x} f(a, x_1, ..., x_n) \text{ resp. } \prod_{a \leq x} f(a, x_1, ..., x_n)\ .$$

Elementary functions contain exponentials 2^n, whereas lower elementary function do not. The levels of the above hierarchy are strict [TZ].

The main result about periods proven in [Y, TZ] is:

Theorem 15.5.4. *Real periods are lower elementary real numbers.*

In fact, Yoshinaga proved that periods are elementary computable numbers, and Tent-Ziegler made the refinement that periods are even lower-elementary numbers. The proofs are based on Hironaka's theorem on semi-algebraic sets which we have used already in chapter 2. The main idea is to reduce periods to volumes of bounded semi-algebraic sets, and then use Riemann sums to approximate the volumes inside the class of lower elementary computable functions.

Corollary 15.5.5. *One has inclusions:*

$$\bar{\mathbb{Q}} \subsetneq \mathbb{P}^{\mathrm{eff}}(\mathbb{Q}) \subset \mathbb{C}_{\mathrm{low-elem}} \subsetneq \mathbb{C}_{\mathrm{elem}} \subsetneq \mathbb{C}_{\mathrm{comp}} \ .$$

Hence, in order to construct a non-period, one needs to exhibit a computable number which is not elementary computable. By Tent-Ziegler, it would also be enough to write down an elementary computable number which is not lower elementary.

Here is how Yoshinaga proceeds. First, using a result of Mazzanti [Maz], one can show that elementary functions are generated by composition from the following functions:

- The successor function $x \mapsto x + 1$,
- the modified subtraction $\max(x - y, 0)$,
- the floor quotient $(x, y) \mapsto \lfloor \frac{x}{y+1} \rfloor$, and
- the exponential function $(x, y) \mapsto x^y$.

Using this, there is an explicit enumeration $(f_n)_{n\in\mathbb{N}_0}$ of all elementary functions $f : \mathbb{N}_0 \to \mathbb{N}_0$. Together with the standard enumeration of $\mathbb{Q}_{>0}$, we obtain an explicit enumeration $(g_n)_{n\in\mathbb{N}_0}$ of all elementary maps $g : \mathbb{N}_0 \to \mathbb{Q}_{>0}$. Using a trick, see [Y, pg. 9], one can "speed up" each function g_n, so that $g_n(m)$ is a Cauchy sequence (hence, convergent) in m for each n.

Following [Y], we therefore obtain

$$\mathbb{R}_{\mathrm{elem}} = \{\beta_0, \beta_1, ...\}, \text{ where } \beta_n = \lim_{m\to\infty} g_n(m) \ .$$

Finally, Yoshinaga defines

$$\alpha := \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \sum_{i=1}^{n} \epsilon_i 3^{-i} \ ,$$

where $\epsilon_0 = 0$, and recursively

$$\epsilon_{n+1} := \begin{cases} 0, & \text{if } g_n(n) > \alpha_n + \frac{1}{2\cdot 3^n} \\ 1, & \text{if } g_n(n) \leq \alpha_n + \frac{1}{2\cdot 3^n} \end{cases} \quad .$$

Now, it is quite easy to show that α does not occur in the list $\mathbb{R}_{\text{elem}} = \{\beta_0, \beta_1, ...\}$, see [Y, Prop. 17]. Note that the proof is essentially a version of Cantor's diagonal argument.

Part V

Bibliography

Bibliography

[Ae] A. Aeppli: Modifikation von reellen und komplexen Mannigfaltigkeiten, Comm. Math. Helv, 31, 219–301 (1956).

[ABW] L. Adams, C. Bogner, S. Weinzierl: The two-loop sunrise graph in two space-time dimensions with arbitrary masses in terms of elliptic dilogarithms, J. Math. Phys., Vol. 55, xxx–xxx (2014).

[A1] Y. André: Galois theory, motives and transcendental numbers, in: Renormalization and Galois theories, IRMA Lect. Math. Theor. Phys. 15, Eur. Math. Soc., Zürich, 165–177 (2009).

[A2] Y. André: Une introduction aux motifs, Panoramas et Synthèse 17, Société Mathématique de France (2004).

[AF] A. Andreotti, T. Fraenkel: The Lefschetz theorem on hyperplane sections, Ann. of Math. (2) 69, 713–717 (1959).

[Ara] D. Arapura: An abelian category of motivic sheaves, Advances in Mathematics Volume 233, Issue 1, 135–195 (2013).

[Ar] V. I. Arnold: Huygens and Barrow, Newton and Hook, Birkhäuser Verlag, Basel-Boston-Berlin (1990).

[A] M. Artin: Théorème de finitude pour un morphisme propre; dimension cohomologique des schémas algébriques affines. In SGA 4, tome 3, XIV, Lect. Notes Math., Vol. 305, Berlin-Heidelberg-New York, Springer Verlag, 145–168 (1973).

[AM] M. Artin, B. Mazur: Étale homotopy, Lecture Notes in Mathematics, No. 100, Springer-Verlag, Berlin-New York (1969).

[AH] M. Atiyah, W. Hodge: Integrals of the second kind on an algebraic variety, Ann. of Math. 62, 56–91 (1955).

[Ay] J. Ayoub: Periods and the conjectures of Grothendieck and Kontsevich-Zagier, Newsletter of the European Mathematical Society, Vol. 91, (2014).

[AB] J. Ayoub, L. Barbieri-Viale: Nori 1-motives, Mathematische Annalen, Vol. 361(1-2), 367–402.(2014).

[Ba] R. Baer: Zur Einführung des Scharbegriffs, Crelle Journal Band 160, Heft 4, 199–207 (1929).

[Be1] A. Beilinson: On the derived category of perverse sheaves, in: K-Theory, Arithmetic, and Geometry (Moscow, 1984-1986), Springer LNM 1289, Berlin, 27–41 (1987).

[Be2] A. Beilinson: Notes on absolute Hodge cohomology, in: Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983), Contemp. Math., 55, Amer. Math. Soc., Providence, RI, 35–88 (1986).

[Be3] A. Beilinson: Higher regulators and values of L-functions, J. Soviet Math. 30, 2036–2070 (1985).

[BBD] A.A. Beilinson, J. Bernstein, P. Deligne: Faisceaux pervers. Analysis and topology on singular spaces, I (Luminy, 1981), 5-171, Astrisque, 100, Soc. Math. France, Paris, 1982.

[BB] P. Belkale, P. Brosnan: Periods and Igusa local zeta functions, Int. Math. Res. Not., no. 49, 2655–2670 (2003).

[BEK] S. Bloch, H. Esnault, D. Kreimer: On motives associated to graph polynomials, Comm. Math. Phys. 267, no. 1, 181–225 (2006).

[Bl] S. Bloch: Algebraic cycles and higher K-theory, Advances in Mathematics, Vol. 61(3), 267–304 (1986).

[BV] S. Bloch, P. Vanhove: The elliptic dilogarithm for the sunset graph, arXiv:1309.5865 (2013).

[BKV] S. Bloch, M. Kerr, P. Vanhove: A Feynman integral via higher normal functions, arXiv:1406.2664 (2014).

[BBV] J. Blümlein, D. J. Broadhurst, J. A. M. Vermaseren: The multiple zeta value data mine, Comput. Phys. Commun. 181, 582-625 (2010).

[BW] C. Bogner, S. Weinzierl: Periods and Feynman integrals, J. Math. Phys., Vol. 50, 1–16 (2009).

[Bo] M.V. Bondarko: Weight structures vs. t-structures; weight filtrations, spectral sequences, and complexes (for motives and in general). J. K-Theory 6 (2010), no. 3, 387-504.

[BLR] S. Bosch, W. Lütkebohmert, M. Raynaud: Néron models, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 21. Springer-Verlag, Berlin (1990).

[BC] J.-B. Bost, F. Charles: Some remarks concerning the Grothendieck period conjecture, arXiv:1307.1045 (2013).

[Bre] G. Bredon: Topology and Geometry, Graduate Texts in Mathematics, 139. Springer-Verlag, New York (1993).

[Bri] D. S. Bridges: Computability - a mathematical sketchbook, Springer GTM 146 (1994).

[BK] D. J. Broadhurst, D. Kreimer: Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops, Physics Letters B 393, 403–412 (1997).

[B1] F. Brown: Mixed Tate motives over $\mathbb{Z}$, Ann. of Math. (2) 175, no. 2, 949–976 (2012).

[B2] F. Brown: Motivic periods and $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, Proceedings of the ICM Seoul 2014, see arXiv:1407.5165 (2014).

[B3] F. Brown: Multiple zeta values and periods of moduli spaces $\bar{M}_{0,n}$, Ann. Sci. Éc. Norm. Supér. (4) 42 , no. 3, 371–489 (2009).

[B4] F. Brown, A. Levin: Multiple elliptic polylogarithms, arXiv:1110.6917 (2011).

[BS] F. Brown, O. Schnetz: A K3 in ϕ^4, Duke Math. Journal, Vol. 161, 1817–1862 (2012).

[Bru] A. Brughuières: On a tannakian theorem due to Nori, Preprint, see http://www.math.univ-montp2.fr/˜bruguieres/docs/ntan.pdf (2004).

[Br] J. H. Bruinier: Harmonic Maass forms and periods, Math. Annalen 357, 1363–1387 (2013).

[C] J. Carlson: Extensions of mixed Hodge structures, Journées de géométrie algébrique d'Angers, 107–128 (1979).

[CCM] J. Carlson, H. Clemens, J. Morgan: On the mixed Hodge structure associated to π_3 of a simply connected complex projective manifold, Ann. Sci. École Norm. Sup., Vol. (4) 14, 323–338 (1981).

[CMP] J. Carlson, Ch. Peters, S. Müller-Stach: Period mappings and period domains, Cambridge Studies in Pure Math. Vol. 85, Cambridge Univ. Press first edition 2003, second edition to appear.

[CG] U. Choudhury, M. Gallauer Alvez de Souza: An isomorphism of motivic Galois groups, arXiv:1410.6104 (2014).

[Ch] U. Choudhury: Motives of Deligne-Mumford stacks. Adv. Math. 231 (2012), no. 6, 3094-3117.

[CS] S. Chowla, A. Selberg: On Epstein's zeta function (I), Proc. Nat. Acad. Sc. Vol 35, 371–374 (1949).

[Ch] G. V. Chudnovsky: Algebraic independence of values of exponential and ellitpic functions, Proc. of the Intern. Congress Helsinki 1978, 339-350 (1978).

[D1] P. Deligne: Valeurs de fonctions L et périodes d'intégrales, Proc. Sympos. Pure Math., XXXIII, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Amer. Math. Soc., Providence, R.I., 313–346 (1979).

[D2] P. Deligne: Hodge cycles on abelian varieties, in: P. Deligne, J. Milne, A. Ogus and K.Y. Shih: Hodge Cycles, Motives, and Shimura Varieties, Springer LNM 900, 9–100 (1982).

[DG] P. Deligne, A. Goncharov: Groupes fondamentaux motiviques de Tate mixte, Ann. Sci. École Norm. Sup. (4) 38, no. 1, 1–56 (2005).

[DM] P. Deligne, J. Milne: Tannakian categories, in: P. Deligne, J. Milne, A. Ogus and K.Y. Shih: Hodge Cycles, Motives, and Shimura Varieties, Springer LNM 900, 101–228 (1982).

[D3] P. Deligne: Multizetas, d'après Francis Brown, Séminaire Bourbaki, No. 1048, 1–26 (2012).

[D4] P. Deligne: Théorie des Hodge II, Publication Mathématique IHES 40, 5–57 (1971).

[D5] P. Deligne: Théorie des Hodge III, Publication Mathématique IHES 44, 5–77 (1974).

[DMOS] P. Deligne, J.S. Milne, A. Ogus, K. Shih: Hodge cycles, motives, and Shimura varieties, Springer LNM 900 (1982).

[DS] C. Deninger, A. J. Scholl: The Beilinson conjectures, London Math. Soc. Lecture Notes, Vol. 153, 173–209 (1991).

[Dur] A. Durfee: A naive guide to mixed Hodge theory, Singularities, Part 1 (Arcata, Calif., 1981), Proc. Sympos. Pure Math., 40, Amer. Math. Soc., Providence, RI, 313–320 (1983).

[Eb] D. Eisenbud: Commutative Algebra with a View Toward Algebraic Geometry, Graduate Texts in Mathematics, 150, Springer Verlag, New York (1994).

[Fi] G. Fichtenholz: Differential- und Integralrechnung II, Tenth edition, translated from the Russian by Brigitte Mai and Walter Mai, Hochschulbücher für Mathematik Nr. 62, VEB Deutscher Verlag der Wissenschaften, Berlin (1990).

[F] E. Friedlander: Étale homotopy of simplicial schemes, Annals of Math. Studies Nr. 104, Princeton Univ. Press (1982).

[Fri] G. Friedman: Survey article: an elementary illustrated introduction to simplicial sets, Rocky Mountain J. Math. 42, no. 2, 353–423 (2012).

[Fr] B. Friedrich: Periods and algebraic de Rham cohomology, Diplomarbeit Univ. Leipzig 2004, see arXiv:math.AG/0506113v1 (2005).

[Fu] W. Fulton: Intersection theory, Ergebnisse der Mathematik, 3. Folge, Band 2, Springer Verlag, Berlin (1984).

[Ga] K. M. Gartz: A construction of a differential graded Lie algebra in the category of effective homological motives, arXiv:math/0602287v1 (2006).

[Ge] T. Geisser: Arithmetic cohomology over finite fields and special values of ζ-functions, Duke Math. J. 133, no. 1, 27–57 (2006).

[GM] S. Gelfand, Y. Manin: Homological Algebra, Springer Verlag, Berlin (1999).

[God] R. Godement: Topologie algébrique et théorie des faisceaux, Publ. Math. Univ. Strasbourg. No. 13, Hermann, Paris (1958).

[G] K. Gödel: Über formal unentscheidbare Sätze der Principia Mathematica und verwandte Systeme I, Monatshefte Math. Phys. 38, 173–198 (1931).

[GM] A. Goncharov, Y. Manin: Multiple ζ-motives ad moduli spaces $\bar{M}_{0,n}$, Compos. Math. 140, no. 1, 1–14 (2004).

[G1] A. B. Goncharov: The double logarithm and Manin's complex for modular curves, Math. Res. Let. 4, 617–636 (1997).

[G2] A. B. Goncharov: The dihedral lie algebra and Galois symmetries of $\pi_1^l(\mathbb{P}^1 - (\{0,\infty\} \cup \mu_n))$, Duke Math. J. 110, no. 3, 397–487 (2001).

[G3] A. B. Goncharov: Multiple polylogarithms and mixed Tate motives, see arXiv:math.AG/0103059v4 (2001).

[G4] A. B. Goncharov: Hodge correlators II, Moscow Math. Journal, Vol. 10, 139–188 (2010).

[GR] H. Grauert, R. Remmert: Theorie der Steinschen Räume, Grundlehren der Mathematischen Wissenschaften, Vol. 227, Springer Verlag, Berlin-New York (1977).

[GGK] M. Green, Ph. Griffiths, M. Kerr: Mumford-Tate groups and domains, Annals of Math. Studies Vol. 183, Princeton Univ. Press (2012).

[GH] P. Griffiths, J. Harris: Principles of Algebraic Geometry, Wiley Interscience, New York (1978).

[Gr] B. Gross: On the periods of abelian integrals and a formula of Chowla-Selberg (with an appendix by D. Rohrlich), Inventiones Math. Vol. 45, 193–211 (1978).

[Gro1] A. Grothendieck: On the de Rham cohomology of algebraic varieties, Publication Mathématique IHES 29, 95–103 (1966).

[Gr] A. Grzegorczyk: Computable functionals, Fund. Math., Vol. 42, 168–202 (1955).

[Gro2] A. Grothendieck: EGA V, unpublished manuscript, scan available at http://www.jmilne.org/math/Documents/.

[Gun] R. Gunning: Introduction to holomorphic functions of several variables, Vol. 1, Wadsworth, Pacific Grove, CA (1990).

[H] R. M. Hain: Classical polylogarithms, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., 55, Part 2, Amer. Math. Soc., Providence, RI, 3–42 (1994).

[HZ] R. M. Hain, S. Zucker: A guide to unipotent variations of mixed Hodge structures, Hodge theory (Sant Cugat, 1985), Lecture Notes in Math., 1246, Springer, Berlin, 92–106 (1987).

[Ham1] H. Hamm: Zum Homotopietyp Steinscher Räume, Journal für reine und angewandte Mathematik, Vol. 338, 121–135 (1983).

[Ham2] H. Hamm: Zum Homotopietyp q-vollständiger Räume, J. Reine Angew. Math. 364 (1986), 1-9.

[Ha] D. Harrer: Thesis in preparation, Freiburg (2015).

[Ha1] R. Hartshorne: On the de Rham cohomology of algebraic varieties, Publication Mathématique IHES 45, 1–99 (1971).

[Ha2] R. Hartshorne: Algebraic Geometry, Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg (1977).

[Hat] A. Hatcher: Algebraic topology, Cambridge Univ. Press, Cambridge (2002).

[He] R. Henderson: The algebra of multiple zeta-values, final year project, university of Durham (2012).

[Hi1] H. Hironaka: Resolution of singularities, Ann. of Math. 79, 109–326 (1964).

[Hi2] H. Hironaka: Triangulation of algebraic sets, Triangulations of algebraic sets. Algebraic geometry (Proc. Sympos. Pure Math., Vol. 29, Humboldt State Univ., Arcata, Calif., 1974), Amer. Math. Soc., Providence, R.I., 165–185 (1975).

[Hof] M. E. Hoffman: The algebra of multiple harmonic series, Journ. of Algebra 194, 477–495 (1997).

[HO] M. E. Hoffman, Y. Ohno: Relation of multiple zeta values and their algebraic expression, J. Algebra 262, 332–347 (2003).

[Hu1] A. Huber: Mixed Motives and their Realization in Derived Categories, Lecture Notes in Mathematics, 1604, Springer-Verlag, Berlin (1995).

[Hu2] A. Huber: Realization of Voevodsky's motives, J. Algebraic Geom. 9, no. 4, 755–799 (2000).

[Hu3] A. Huber: Corrigendum to: Realization of Voevodsky's motives [J. Algebraic Geom. 9, no. 4, 755–799 (2000)], J. Algebraic Geom. 13, no. 1, 195–207 (2004).

[HMS] A. Huber, S. Müller-Stach: On the relation between Nori motives and Kontsevich periods, Preprint, see arXiv:1105.0865 (2011).

[HJ] A. Huber, C. Jörder: Differential forms in the h-topology, Algebr. Geom. 1, no. 4, 449–478 (2014).

[HW] A. Huber, J. Wildeshaus: The classical polylogarithm, abstract of a series of lectures given at the workshop on polylogs in Essen, 1997, see arXiv:1210.2358 (2012).

[Hui] J. Huisman: Real algebraic differential forms on complex algebraic varieties, Indag. Math. (N.S.) 11, no. 1, 63–71 (2000).

[IKZ] K. Ihara, M. Kaneko, D. Zagier: Derivation and double shuffle relations for multiple zeta values, Compositio Mathematica 142, 307-338 (2006).

[Iv] F. Ivorra: Perverse Nori motives, Preprint (2014).

[Ja1] U. Jannsen: Mixed motives and algebraic K-theory, with appendices by S. Bloch and C. Schoen, Springer LNM 1400, Berlin (1990).

[Ja2] U. Jannsen: Motives, numerical equivalence, and semi-simplicity. Invent. Math. 107, no. 3, 447–452 (1992).

[Jo] J.-P. Jouanolou: Théorèmes de Bertini at applications, Progress in Math. vol 42, Birkhäuser, Boston (1983).

[Joe] C. Jörder: On the Poincaré Lemma for reflexive differential forms, Dissertation, Albert-Ludwigs-Universität Freiburg, arXiv:1401.7495 (2014).

[dJ] A.J. de Jong: Smoothness, semi-stability and alterations, Inst. Hautes tudes Sci. Publ. Math. No. 83, 51–93 (1996).

[Joh] P. T. Johnstone: Affine categories and naturally Malcev categories, Journal of Pure and Applied Algebra 61, 251–256 (1989).

[Ka] L. Kalmár: Egyszerü pelda eldönthetetlen aritmetikai problémára, Matematikai és fizikai lapok 50, 1–23 (1943).

[KS] M. Kashiwara, P. Schapira,: Sheaves on manifolds, with a chapter in French by Christian Houzel, Grundlehren der Mathematischen Wissenschaften, Vol. 292, Springer-Verlag, Berlin (1990).

[KLM] M. Kerr, J. Lewis, S. Müller-Stach: The Abel-Jacobi map for higher Chow groups, Compositio Math., Vol. 142, 374-396 (2006).

[KLM2] M. Kerr, J. Lewis: The Abel-Jacobi map for higher Chow groups II, Inventiones Math., Vol. 170, 49-58 (2007).

[Kl] S. Kleene: The theory of recursive functions, approaching its centennial, Bull. of the AMS 5, 43–61 (1981).

[Kj] T. Kleinjung: Gerahmte gemischte Tate-Motive und die Werte von Zetafunktionen zu Zahlkörpern an den Stellen 2 und 3, Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonner Mathematische Schriften, Nr. 340 (2000).

[Kn] C.M. Knighten: Differentials on quotients of algebraic varieties, Trans. Am. Math. Soc. 177, 65–89 (1973).

[K1] M. Kontsevich: Operads and motives in deformation quantisation, Letters in Mathematical Physics 48, 35–72 (1999).

[K2] M. Kontsevich: Intersection theory on the moduli space of curves and the matrix Airy function, Comm. Math. Phys. Volume 147, Number 1, 1–23 (1992).

[KZ] M. Kontsevich, D. Zagier: Periods, In: B. Engquist and W. Schmid, editors, Mathematics unlimited – 2001 and beyond, Springer Verlag, Berlin, 771–808 (2001).

[KB] B. Koopman, A. Brown: On the covering of analytic loci by complexes, Trans. Amer. Math. Soc. 34, 231–251 (1932).

[L] M. Lerch: Bemerkungen zur Theorie der elliptischen Functionen, Jahrbuch ber die Fortschritte der Mathematik, Vol. 24, 442–445 (1892).

[L1] M. Levine: Mixed motives, K-theory Handbook, Vol. 1, Springer Verlag Berlin, 429–521 (2005).

[L2] M. Levine: Tate motives and the vanishing conjectures for algebraic K-theory, in: Algebraic K-Theory and Algebraic Topology, Lake Louise 1991, Nato ASI Ser., Kluwer, Vol. 407, 167–188 (1993)

[LW] F. Lecomte, N. Wach: Le complexe motivique de Rham, Manuscripta Math. 129, no. 1, 75–90 (2009).

[ML] S. MacLane: Categories for the working mathematician, Graduate Texts in Mathematics, Vol. 5, Springer-Verlag, New York-Berlin (1971).

[Man] Y.I. Manin: Correspondences, motifs and monoidal transformations, Mat. Sb. (N.S.) 77 (119), 475–507 (1968), english translation: Math. USSR-Sb. 6, 439–470 (1968).

[Mat] H. Matsumura: Commutative Algebra, W.A. Benjamin, New York (1970).

[May] J.P. May: Simplicial objects in algebraic topology, Van Nostrand Mathematical Studies, No. 11 D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London (1967).

[Maz] S. Mazzanti: Plain bases for classes of primitive recursive functions, Math. Logic Quat. 48, 93–104 (2002).

[Me] W. Messing: Differentials of the first, second, and third kinds, Proc. Arcata conference, AMS Proc. of Symposia Vol. 29, 547–562 (1974).

[Mi] J. Milne: Étale cohomology, Princeton Mathematical Series, 33, Princeton University Press, Princeton, N.J. (1980).

[MM] J. Milnor, J. Moore: On the structure of Hopf algebras, Ann. of Math. 81(2), 211–264 (1965).

[Mo] J. W. Morgan: The algebraic topology of smooth algebraic varieties, Publ. Math. de l'IHÉS Vol. 48, 137–204 (1978).

[MWZ1] S. Müller-Stach, S. Weinzierl, R. Zayadeh: A second order differential equation for the two-loop sunrise graph with arbitrary masses, Communications in Number Theory and Physics, Vol. 6 Nr. 1, 203–222 (2012).

[MWZ2] S. Müller-Stach, S. Weinzierl, R. Zayadeh: Picard-Fuchs equations for Feynman integrals, Communications in Math. Physics 326, 237–249 (2014).

[M] D. Mumford: Families of abelian varieties, in: Algebraic Groups and Discontinous Subgroups, Proc. of Symposia Vol. IX, 347-351 (1966).

[Na] V. Navarro Aznar: Sur la théorie de Hodge des varietés algébriques à singularités isolées, Asterisque 130, 272–307 (1985).

[N] M. Nori: Lectures at TIFR, LaTeX notes by N. Fakhruddin, unpublished, 32 pages (2000).

[N1] M. Nori: Handwritten notes titled "Motives", distributed around 1997 in the Esnault-Viehweg seminar at Essen, unpublished, 27 pages.

[N2] M. Nori: Constructible sheaves, Proc. Int. Conf. on Algebra, Arithmetic and Geometry (2000), Part I, II, Mumbai, Tata Inst. Fund. Res. Stud. Math. Vol. 16, 471–491 (2002).

[N3] M. Nori: Unpublished handwritten notes titled "Periods", sent to us by M. Nori in Sept. 2011 (21 pages).

[Pa] I. Panin: Push-forwards in oriented cohomology theories of algebraic varieties: II, after I. Panin and A. Smirnov, Preprint Nr. 619, K-theory preprint archive, see http://www.math.uiuc.edu/K-theory/0619/ (2003).

[PS] C. Peters, J. Steenbrink: Mixed Hodge structures, Ergebnisse der Mathematik Vol. 52, Springer Verlag (2008).

[Pr] H. Prüfer: Theorie der Abelschen Gruppen, Math. Zeitschrift Band 20, 165–187 (1924).

[Re] L. E. Renner: Linear Algebraic Monoids, Encyclopaedia of mathematical sciences, Vol. 134, Springer Verlag (2005).

[R] H. G. Rice: Recursive real numbers, Proc. of the AMS 5, 784–791 (1954).

[Sa] N. Saavedra Rivano: Catégories Tannakiennes, Lecture Notes in Mathematics, Vol. 265. Springer-Verlag, Berlin-New York (1972).

[S] Th. Schneider: Transzendenzuntersuchungen periodischer Funktionen, Teile 1 und 2, Journal für die Reine und Angewandte Mathematik, Bd. 172, 65–69 und 70–74 (1934).

[Sb] A. Seidenberg: A new decision method for elementary algebra, Annals of Math. 60, no. 2, 365–374 (1954).

[Sch1] J. Scholbach: Geometric motives and the h-topology, Mathematische Zeitschrift 272, 956–986 (2012).

[Sch2] J. Scholbach: f-cohomology and motives over number rings, Kodai Math. J. 35, no. 1, 1–32 (2012).

[Scho] A.J. Scholl: Remarks on special values of L-functions, In: L-functions and arithmetic (Durham, 1989), London Math. Soc. Lecture Note Ser. 153, Cambridge Univ. Press, 373–392 (1991).

[Scho2] A.J. Scholl: Extensions of motives and higher Chow groups, unpublished notes.

[Schü] J. Schürmann, Topology of Singular Spaces and Constructible Sheaves, Instytut Matematyczny Polskiej Akademii Nauk, Monografie Matematyczne (New Series) 63, Birkhäuser Verlag, Basel (2003).

[Se1] J. P. Serre: Géométrie algébrique et géométrie analytic, Ann. Inst. Fourier 6, 1–42 (1956).

[Se2] J. P. Serre: A Course in Arithmetic, Graduate Texts in Mathematics, No. 7. Springer-Verlag, New York-Heidelberg (1973).

[Sil] J. Silverman: The Arithmetic of Elliptic Curves, Graduate Texts in Mathematics, 106, Springer-Verlag, New York (1992).

[Sk] Th. Skolem: Proofs of some theorems on recursively enumerable sets, Notre Dame Journ. of Formal Logic III, 65–74 (1962).

[Spa] E. H. Spanier: Algebraic Topology, Springer Verlag (1966).

[Spe] E. Specker: Nicht konstruktiv beweisbare Sätze der Analysis, Journ. Symb. Logic 14, 145–158 (1949).

[St] J. Steenbrink: A summary of mixed Hodge theory, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., 55, Part 1, Amer. Math. Soc., Providence, RI, 31–41 (1994).

[Str] R. Street: Quantum groups, Australian Mathematical Society Lecture Series, Cambridge Univ. Press, Cambridge (2007).

[Su] D. Sullivan: Infinitesimal computations in topology, Publ. Math. IHES 47, 269–331 (1977).

[Tei] B. Teissier: Variétés polaires II, Multiplicites polaires, sections planes, et conditions de whitney, Springer LNM 961, 314–491 (1982).

[TZ] K. Tent, M. Ziegler: Computable functions of reals, Münster J. Math, Vol. 3, 43–66 (2010).

[Te] T. Terasoma: Mixed Tate motives and multiple zeta values, Invent. Math. 149, 339–369 (2002).

[T] A. Turing: On computable numbers, with an application to the Entscheidungsproblem, Proc. London Math. Soc. 42, 230–265 (1936).

[Vil] K. Vilonen: Master Thesis, Harvard University, unpublished.

[Vis] A. Vistoli: Grothendieck topologies, fibered categories and descent theory, Fundamental algebraic geometry, Math. Surveys Monogr., Vol. 123, Amer. Math. Soc., Providence, RI, 1–104 (2005).

[Voe] V. Voevodsky: Homology of schemes, Selecta Math. (N.S.) 2, no. 1, 111–153 (1996).

[VSF] V. Voevodsky, A. Suslin, E. Friedlander: Cycles, transfers, and motivic homology theories, Annals of Mathematics Studies 143, Princeton University Press, Princeton (2000).

[Wa] M. Waldschmidt: Elliptic functions and transcendence, In: Surveys in number theory, Springer New York, 1–46 (2008).

[vW] J. von Wangenheim: Nori-Motive und Tannaka-Theorie, Diplomarbeit, Universität Freiburg, see arXiv:1111.5146 (2011).

[Wa] F. Warner: Foundations of Differentiable Manifolds and Lie Groups, Graduate Texts in Mathematics 94, Springer-Verlag, New York-Berlin (1983).

[Wed] T. Wedhorn: On Tannakian duality over valuation rings, J. Algebra 282, 575–609 (2004).

[We] C. Weibel: An introduction to homological algebra, Cambridge Studies in Advanced Mathematics 38, Cambridge Univ. Press, Cambridge (1994).

[Wei] Th. Weißschuh: A strictly commutative higher cycle map into Deligne-Beilinson cohomology, Dissertation Mainz (2015), arXiv:1410.4686 (2014).

[Wu] G. Wüstholz: Leibniz's conjecture, periods, and motives, in: U. Zannier ed., De Giorgi Colloquium 2009, Scuola Normale Superiore, 33–42 (2012).

[Y] M. Yoshinaga: Periods and elementary real numbers, arXiv:0805.0349 (2008).

[Z] D. Zagier: Values of zeta functions and their applications, Progr. Math. 120, 497–512 (1994),

[Z1] J.Zhao: Multiple polylogarithms: analytic continuation, monodromy, and variations of mixed Hodge structures, Contemporary trends in algebraic geometry and algebraic topology (Tianjin, 2000), Nankai Tracts Math., 5, World Sci. Publ., River Edge, NJ, 167–193 (2002).

[Z2a] J. Zhao: Analytic continuation of multiple polylogarithms, Anal. Math. 33, no. 4, 301–323 (2007).

[Z2b] J. Zhao: Variations of Mixed Hodgestructures of Multiple Polylogarithms, Canad. J. Math. 56, no. 6, 1308–1338 (2004).

[Z2] J. Zhao: Analytic continuation of multiple zeta functions, Proc. of the AMS, Vol. 128, Nr. 5, 1275–1283 (1999).

[SGA1] A. Grothendieck: Séminaire de Géométrie Algébrique du Bois Marie - 1960-61 - Revtements tales et groupe fondamental - (SGA 1), Lecture notes in mathematics 224, Springer-Verlag, Berlin-New York (1971).

[SGA3] M. Demazure, A. Grothendieck: Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3), Schémas en groupes, Vol 1, Lecture Notes in Mathematics 151, Springer Verlag, Berlin-New York (1970).

[SGA4.2] Théorie des topos et cohomologie étale des schémas, Tome 2, Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), dirigé par M. Artin, A. Grothendieck et J. L. Verdier, avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat, Springer LNM 270 (1970).

[SGA4V] J.-L. Verdier: Cohomologie dans les topos, Exposé V, 1–82, in: Théorie des topos et cohomologie étale des schémas, Tome 2, Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), dirigé par M. Artin, A. Grothendieck et J. L. Verdier.

[SGA4Vbis] B. Saint-Donat: Techniques de descent cohomologique, Exposé Vbis, 83–162, in: Théorie des topos et cohomologie étale des schémas, Tome 2, Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), dirigé par M. Artin, A. Grothendieck et J. L. Verdier.

[SGA 4 1/2] P. Deligne: Cohomologie étale, Séminaire de Géométrie Algébrique du Bois-Marie SGA 4 1/2, avec la collaboration de J. F. Boutot, A. Grothendieck, L. Illusie et J. L. Verdier, Lecture Notes in Mathematics, Vol. 569, Springer-Verlag, Berlin-New York (1977).