Periods and Nori Motives

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Chapter 1: General Set-up

Chapter 1

General Set-up

In this chapter we collect some standard notation used throughout the book.

1.1 Varieties

Let k be field. It will almost always be of characteristic zero or even a subfield of $\mathbb{C}$.

By a *scheme* over k we mean a separated scheme of finite type over k. Let Sch be the category of k-schemes. By a *variety* over k we mean a quasi-projective reduced scheme of finite type over k. Let Var be the category of varieties over k. Let Sm and Aff be the full subcategories of smooth varieties and affine varieties, respectively.

1.1.1 Linearizing the category of varieties

We also need the additive categories generated by these categories of varieties. More precisely:

Definition 1.1.1. Let $\mathbb{Z}[\mathrm{Var}]$ be the category with objects the objects of Var. If $X = X_1 \cup \cdots \cup X_n$, $Y = Y_1 \cup \cdots \cup Y_m$ are varieties with connected components X_i, Y_j, we put

$$\mathrm{Mor}_{\mathbb{Z}[\mathrm{Var}]}(X,Y) = \left\{ \sum_{i,j} a_{ij} f_{ij} | a_{ij} \in \mathbb{Z},\ f_{ij} \in \mathrm{Mor}_{\mathrm{Var}}(X_i, Y_j) \right\}$$

with the addition of formal linear combinations. Composition of morphisms is defined by extending composition of morphisms of varieties $\mathbb{Z}$-linearly.

Analogously, we define $\mathbb{Z}[\mathrm{Sm}]$, $\mathbb{Z}[\mathrm{Aff}]$ from Sm and Aff. Moreover, let $\mathbb{Q}[\mathrm{Var}]$,

$\mathbb{Q}[\mathrm{Sm}]$ and $\mathbb{Q}[\mathrm{Aff}]$ be the analogous $\mathbb{Q}$-linear additive categories where morphisms are formal $\mathbb{Q}$-linear combinations of morphisms of varieties.

$\mathbb{Z}[\mathrm{Var}]$ is an additive category with direct sum given by the disjoint union of varieties. The zero object corresponds to the empty variety, or, if you prefer, has to be added formally.

We are also going to need the category of *smooth correspondences* SmCor. It has the same objects as Sm and as morphisms *finite correspondences*

$$\mathrm{Mor}_{\mathrm{SmCor}}(X,Y) = \mathrm{Cor}(X,Y),$$

where $\mathrm{Cor}(X,Y)$ is the free $\mathbb{Z}$-module with generators integral subschemes $\Gamma \subset X \times Y$ such that $\Gamma \to X$ is finite and dominant over a component of X.

1.1.2 Divisors with normal crossings

Definition 1.1.2. Let X be a smooth variety of dimension n and $D \subset X$ a closed subvariety. D is called *divisor with normal crossings* if for every point of D there is an affine neighbourhood U of x in X which is étale over $\mathbb{A}^n$ via coordinates $t_1, \ldots, t_n$ and such that $D|_U$ has the form

$$D|_U = V(t_1 t_2 \cdots t_r)$$

for some $1 \leq r \leq n$.

D is called a *simple divisor with normal crossings* if in addition the irreducible components of D are smooth.

In other words, D looks étale locally like an intersection of coordinate hyperplanes.

Example 1.1.3. Let $D \subset \mathbb{A}^2$ be the nodal curve, given by the equation $y^2 = x^2(x-1)$. It is smooth in all points different from $(0,0)$ and looks étale locally like $xy = 0$ in the origin. Hence it is a divisor with normal crossings but not a simple normal crossings divisor.

1.2 Complex analytic spaces

A classical reference for complex analytic spaces is the book of Grauert and Remmert [GR].

Definition 1.2.1. A *complex analytic space* is a locally ringed space $(X, \mathcal{O}_X^{\mathrm{hol}})$ with X paracompact and Hausdorff, and such that $(X, \mathcal{O}_X^{\mathrm{hol}})$ is locally isomorphic to the vanishing locus Z of a set S of holomorphic functions in some open $U \subset \mathbb{C}^n$ and $\mathcal{O}_Z^{\mathrm{hol}} = \mathcal{O}_U^{\mathrm{hol}}/\langle S \rangle$, where $\mathcal{O}_U^{\mathrm{hol}}$ is the sheaf of holomorphic functions on U.

A *morphism* of complex analytic spaces is a morphism $f : (X, \mathcal{O}_X^{\mathrm{hol}}) \to (Y, \mathcal{O}_Y^{\mathrm{hol}})$ of locally ringed spaces, which is given by a morphism of sheaves $\tilde{f} : \mathcal{O}_Y^{\mathrm{hol}} \to f_*\mathcal{O}_X^{\mathrm{hol}}$ that sends a germ $h \in \mathcal{O}_{Y,y}^{\mathrm{hol}}$ of a holomorphic function h near y to the germs $h \circ f$, which are holomorphic near x for all x with $f(x) = y$. A morphism will sometimes simply be called a holomorphic map, and will be denoted in short form as $f : X \to Y$.

Let An be the category of complex analytic spaces.

Example 1.2.2. Let X be a complex manifold. Then it can be viewed as a complex analytic space. The structure sheaf is defined via the charts.

Definition 1.2.3. A morphism $X \to Y$ between complex analytic spaces is called *proper* if the preimage of any compact subset in Y is compact.

1.2.1 Analytification

Polynomials over $\mathbb{C}$ can be viewed as holomorphic functions. Hence an affine variety immediately defines a complex analytic space. If X is smooth, it is even a complex submanifold. This assignment is well-behaved under gluing and hence it globalizes. A general reference for this is [SGA1], exposé XII by M. Raynaud.

Proposition 1.2.4. *There is a functor*

$$\cdot^{\mathrm{an}} : \mathrm{Sch}_{\mathbb{C}} \to \mathrm{An}$$

which assigns to a scheme of finite type over $\mathbb{C}$ its analytification. There is a natural morphism of locally ringed spaces

$$\alpha : (X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}^{\mathrm{hol}}) \to (X, \mathcal{O}_X)$$

and $\cdot^{\mathrm{an}}$ is universal with this property. Moreover, α is the identity on points.

If X is smooth, then X^{an} is a complex manifold. If $f : X \to Y$ is proper, the so ist f^{an}.

Proof. By the universal property it suffices to consider the affine case where the obvious construction works. Note that X^{an} is Hausdorff because X is separated, and it is paracompact because it has a finite cover by closed subsets of some $\mathbb{C}^n$. If X is smooth then X^{an} is smooth by [SGA1], Prop. 2.1 in exposé XII, or simply by the Jacobi criterion. The fact that f^{an} is proper if f is proper is shown in [SGA1], Prop. 3.2 in exposé XII. □

1.3 Complexes

1.3.1 Basic definitions

Let $\mathcal{A}$ be an additive category. If not specified otherwise, a complex will always mean a cohomological complex, i.e., a sequence A^i for $i \in \mathbb{Z}$ of objects of $\mathcal{A}$

with *ascending* differential $d^i : A^i \to A^{i+1}$ such that $d^{i+1}d^i = 0$ for all $i \in \mathbb{Z}$. The category of complexes is denoted by $C(\mathcal{A})$. We denote $C^+(\mathcal{A})$, $C^-(\mathcal{A})$ and $C^b(\mathcal{A})$ the full subcategories of complexes bounded below, bounded above and bounded, respectively.

If $K^\bullet \in C(\mathcal{A})$ is a complex, we define the *shifted* complex $K^\bullet[1]$ with

$$(K^\bullet[1])^i = K^{i+1} \ , \quad d^i_{K^\bullet[1]} = -d^{i+1}_{K^\bullet} \ .$$

If $f : K^\bullet \to L^\bullet$ is a morphism of complexes, its *cone* is the complex $\mathrm{Cone}(f)^\bullet$ with

$$\mathrm{Cone}(f)^i = K^{i+1} \oplus L^i, d^i_{\mathrm{Cone}(f)} = (-d^{i+1}_K, f^{i+1} + d^i_L) \ .$$

By construction there are morphisms

$$L^\bullet \to \mathrm{Cone}(f) \to K^\bullet[1] \ ,$$

Let $K(\mathcal{A})$, $K^+(\mathcal{A})$, $K^-(\mathcal{A})$ and $K^b(\mathcal{A})$ be the corresponding homotopy categories where the objects are the same and morphisms are homotopy classes of morphisms of complexes. Note that these categories are always triangulated with the above shift functor and the class of distinguished triangles are those homotopy equivalent to

$$K^\bullet \xrightarrow{f} L^\bullet \to \mathrm{Cone}(f) \to K^\bullet[1]$$

for some morphism of complexes f.

Recall:

Definition 1.3.1. Let $\mathcal{A}$ be an abelian category. A morphism $f^\bullet : K^\bullet \to L^\bullet$ of complexes in $\mathcal{A}$ is called *quasi-isomorphism* if

$$H^i(f) : H^i(K^\bullet) \to H^i(L^\bullet)$$

is an isomorphism for all $i \in \mathbb{Z}$.

We will always assume that an abelian category has enough injectives, or is essentially small, in order to avoid set-theoretic problems. If $\mathcal{A}$ is abelian, let $D(\mathcal{A})$, $D^+(\mathcal{A})$, $D^-(\mathcal{A})$ and $D^b(\mathcal{A})$ the induced derived categories where the objects are the same as in $K^?(\mathcal{A})$ and morphisms are obtained by localizing $K^?(\mathcal{A})$ with respect to the class of quasi-isomorphisms. A triangle is distinguished if it is isomorphic in $D^?(\mathcal{A})$ to a distinguished triangle in $K^?(\mathcal{A})$.

Remark 1.3.2. Let $\mathcal{A}$ be abelian. If $f : K^\bullet \to L^\bullet$ is a morphism of complexes, then

$$0 \to L^\bullet \to \mathrm{Cone}(f) \to K^\bullet[1] \to 0$$

is an exact sequence of complexes. Indeed, it is degreewise split-exact.

1.3.2 Filtrations

Filtrations on complexes are used in order to construct spectral sequences. We mostly need two standard cases.

Definition 1.3.3. 1. Let $\mathcal{A}$ be an additive category, $K^\bullet$ a complex in $\mathcal{A}$. The *stupid filtration* $F^{\geq p}K^\bullet$ on $K^\bullet$ is given by

$$F^{\geq p}K^\bullet = \begin{cases} K^i & i \geq p, \\ 0 & i < p. \end{cases}$$

The quotient $K^\bullet / F^{\geq p}K^\bullet$ is given by

$$F^{<p}K^\bullet = \begin{cases} 0 & i \geq p, \\ K^i & i < p. \end{cases}$$

2. Let $\mathcal{A}$ be an abelian category, $K^\bullet$ a complex in $\mathcal{A}$. The *canonical filtration* $\tau_{\leq p}K^\bullet$ on $K^\bullet$ is given by

$$F^{\leq p}K^\bullet = \begin{cases} K^i & i < p, \\ \mathrm{Ker}(d^p) & i = p, \\ 0 & i > p. \end{cases}$$

The quotient $K^\bullet / F^{\leq p}K^\bullet$ is given by

$$\tau_{>p}K^\bullet = \begin{cases} 0 & i < p, \\ K^p/\mathrm{Ker}(d^p) & i = p, \\ K^i & i > p. \end{cases}$$

The associated graded pieces of the stupid filtration are given by

$$F^{\geq p}K^\bullet / F^{\geq p+1}K^\bullet = K^p \ .$$

The associated graded pieces of the canonical filtration are given by

$$\tau_{\leq p}K^\bullet / \tau_{\leq p-1}K^\bullet = H^p(K^\bullet) \ .$$

1.3.3 Total complexes and signs

We return to the more general case of an additive category $\mathcal{A}$. We consider complexes in $K^{\bullet,\bullet} \in C(\mathcal{A})$, i.e., double complexes consisting of a set of objects $K^{i,j} \in \mathcal{A}$ for $i, j \in \mathbb{Z}$ with differentials

$$d_1^{i,j} : K^{i,j} \to K^{i,j+1} \ , \quad d_2^{i,j} : K^{i,j} \to K^{i+1,j}$$

such that $(K^{i,\bullet}, d_2^{i,\bullet})$ and $(K^{\bullet,j}, d_1^{\bullet,j})$ are complexes and the diagrams

$$\begin{array}{ccc} K^{i,j+1} & \xrightarrow{d_2^{i,j+1}} & K^{i+1,j+1} \\ d_1^{i,j}\uparrow & & \uparrow d_1^{i+1,j} \\ K^{i,j} & \xrightarrow{d_2^{i,j}} & K^{i+1,j} \end{array}$$

commute for all $i, j \in \mathbb{Z}$. The *associated simple complex* or *total complex* $\mathrm{Tot}(K^{\bullet,\bullet})$ is defined as

$$\mathrm{Tot}(K^{\bullet,\bullet})^n = \bigoplus_{i+j=n} K^{i,j} \ , \quad d^n_{\mathrm{Tot}(K^{\bullet,\bullet})} = \sum_{i+j=n} (d_1^{i,j} + (-1)^j d_2^{i,j}) \ .$$

In order to take the direct sum, either the category has to allow infinite direct sums or we have to assume boundedness conditions to make sure that only finite direct sums occur. This is the case if $K^{i,j} = 0$ unless $i, j \geq 0$.

Examples 1.3.4. 1. Our definition of the cone is a special case: for $f : K^\bullet \to L^\bullet$

$$\mathrm{Cone}(f) = \mathrm{Tot}(\tilde{K}^{\bullet,\bullet}) \ , \qquad \tilde{K}^{\bullet,-1} = K^\bullet, \tilde{K}^{\bullet,0} = L^\bullet \ .$$

2. Another example is given by the tensor product. Given two complexes $(F^\bullet, d_F)$ and $(G^\bullet, d_G)$, the tensor product

$$(F^\bullet \otimes G^\bullet)^n = \bigoplus_{i+j=n} F^i \otimes G^j$$

has a natural structure of a double complex with $K^{i,j} = F^i \otimes G^j$, and the differential is given by $d = \mathrm{id}_F \otimes d_G + (-1)^i d_F \otimes \mathrm{id}_F$.

Remark 1.3.5. There is a choice of signs in the definition of the total complex. See for example [Hu1] §2.2 for a discussion. We use the convention opposite to the one of loc. cit. For most formulae it does matter which choice is used, as long as it is used consistently. However, it does have an asymmetric effect on the formula for the compatibility of cup-products with boundary maps. We spell out the source of this assymmetry.

Lemma 1.3.6. *Let $F^\bullet$, $G^\bullet$ be complexes in an additive tensor category. Then:*

1. $F^\bullet \otimes (G^\bullet[1]) = (F^\bullet \otimes G^\bullet)[1]$.
2. $\epsilon : (F^\bullet[1] \otimes G^\bullet) \to (F^\bullet \otimes G^\bullet)[1]$ *with* $\epsilon = (-1)^j$ *on* $F^i \otimes G^j$ *(in degree* $i + j - 1$*) is an isomorphism of complexes.*

Proof. We compute the differential on $F^i \otimes G^i$ in all three complexes. Note that

$$F^i \otimes G^j = (F[1])^{i-1} \otimes G^j = F^i \otimes (G[1])^{j-1}.$$

For better readability, we drop $\otimes$id and id$\otimes$ and $|_{F^i\otimes G^j}$ everywhere. Hence we have

$$\begin{aligned}
d^{i+j-1}_{(F^\bullet\otimes G^\bullet)[1]} &= -d^{i+j}_{F^\bullet\otimes G^\bullet}\\
&= -\left(d^j_{G^\bullet} + (-1)^j d^i_{F^\bullet}\right)\\
&= -d^j_{G^\bullet} + (-1)^{j-1} d^i_{F^\bullet}\\
d^{i+j-1}_{F^\bullet\otimes(G^\bullet[1])} &= d^{j-1}_{G^\bullet[1]} + (-1)^{j-1} d^i_{F^\bullet}\\
&= -d^j_{G^\bullet} + (-1)^{j-1} d^i_{F^\bullet}\\
d^{i+j-1}_{(F^\bullet[1])\otimes G^\bullet} &= d^j_{G^\bullet} + (-1)^j d^{i-1}_{F^\bullet[1]}\\
&= d^j_{G^\bullet} + (-1)^{j-1} d^i_{F^\bullet}
\end{aligned}$$

We see that the first two complexes agree, whereas the differential of the third is different. Multiplication by $(-1)^j$ on the summand $F^i \otimes G^j$ is a morphism of complexes. □

1.4 Hypercohomology

Let X be a topological space and $\mathrm{Sh}(X)$ the abelian category of sheaves of abelian groups on X.

We want to extend the definition of sheaf cohomology on X, as explained in [Ha2], Chap. III, to complexes of sheaves.

1.4.1 Definition

Definition 1.4.1. Let $\mathcal{F}^\bullet$ be a bounded below complex of sheaves of abelian groups on X. An *injective resolution* of $\mathcal{F}^\bullet$ is a quasi-isomorphism

$$\mathcal{F}^\bullet \to \mathcal{I}^\bullet$$

where $\mathcal{I}^\bullet$ is a bounded below complex with $\mathcal{I}^n$ *injective* for all n, i.e., $\mathrm{Hom}(-, \mathcal{I}^n)$ is exact.

Sheaf cohomology of X with coefficients in $\mathcal{F}^\bullet$ is defined as

$$H^i(X, \mathcal{F}^\bullet) = H^i\left(\Gamma(X, \mathcal{I}^\bullet)\right) \qquad i \in \mathbb{Z}$$

where $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ is an injective resolution.

Remark 1.4.2. In the older literature, it is customary to write $\mathbb{H}^i(X, \mathcal{F}^\bullet)$ instead of $H^i(X, \mathcal{F}^\bullet)$ and call it *hypercohomology.* We do not see any need to distinguish. Note that in the special case $\mathcal{F}^\bullet = \mathcal{F}[0]$ a sheaf viewed as a complex concentrated in degree 0, the notion of an injective resolution in the above sense agrees with the ordinary one in homological algebra.

Remark 1.4.3. In the language of derived categories, we have

$$H^i(X, \mathcal{F}^\bullet) = \mathrm{Hom}_{D^+(\mathrm{Sh}(X))}(\mathbb{Z}, \mathcal{F}^\bullet[i])$$

because $\Gamma(X, \cdot) = \mathrm{Hom}_{\mathrm{Sh}(X)}(\mathbb{Z}, \cdot)$.

Lemma 1.4.4. *$H^i(X, \mathcal{F}^\bullet)$ is well-defined and functorial in $\mathcal{F}^\bullet$.*

Proof. We first need existence of injective resolutions. Recall that the category $\mathrm{Sh}(X)$ has enough injectives. Hence every sheaf has an injective resolution. This extends to bounded below complexes in $\mathcal{A}$ by [We] Lemma 5.7.2 (or rather, its analogue for injective rather than projective objects).

Let $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ and $\mathcal{G}^\bullet \to \mathcal{J}^\bullet$ be injective resolutions. By loc.cit. Theorem 10.4.8

$$\mathrm{Hom}_{D^+(\mathrm{Sh}(X))}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \mathrm{Hom}_{K^+(\mathrm{Sh}(X))}(\mathcal{I}^\bullet, \mathcal{J}^\bullet).$$

This means in particular that every morphism of complexes lifts to a morphism of injective resolutions and that the lift is unique up to homotopy of complexes. Hence the induced maps

$$H^i(\Gamma(X, \mathcal{I}^\bullet)) \to H^i(\Gamma(X, \mathcal{J}^\bullet))$$

agree. This implies that $H^i(X, \mathcal{F}^\bullet)$ is well-defined and a functor. □

Remark 1.4.5. Injective sheaves are abundant (by our general assumption that there are anough injectives), but not suitable for computations. Every injective sheaf is flasque [Ha1, III. Lemma 2.4], and there is a canonical flasque Godemont resolution. More generally, every flasque sheaf $\mathcal{F}$ is *acyclic*, i.e., $H^i(X, \mathcal{F}) = 0$ for $i > 0$. One may compute sheaf cohomology of $\mathcal{F}$ using any acyclic resolution $F^\bullet$. This follows from the hypercohomology spectral sequence

$$E_2^{p,q} = H^p(H^q(F^\bullet)) \Rightarrow H^{p+q}(X, \mathcal{F})$$

which is supported entirely on the $q = 0$-line.

Special acylic resolutions on X are the so-called *fine* resolutions. See [W, pg. 170] for a definition of fine sheaves involving partitions of unity. Their importance comes from the fact that sheaves of $\mathcal{C}^\infty$-functions and sheaves of $\mathcal{C}^\infty$-differential forms on X are fine sheaves.

1.4.2 Godement resolutions

For many purposes, it is useful to have functorial resolutions of sheaves. One such is given by the Godement resolution introduced in [God] chapter II §3.

Let X be a topological space. Recall that a sheaf on X vanishes if and only the stalks at all $x \in X$ vanish. For all $x \in X$ we denote $i_x : x \to X$ the natural inclusion.

Definition 1.4.6. Let $\mathcal{F} \in \mathrm{Sh}(X)$. Put

$$I(\mathcal{F}) = \prod_{x \in X} i_{x*}\mathcal{F}_x \ .$$

Inductively, we define the *Godement resolution* $Gd^\bullet(\mathcal{F})$ of $\mathcal{F}$ by

$$\begin{aligned} Gd^0(\mathcal{F}) &= I(\mathcal{F}) \ , \\ Gd^1(\mathcal{F}) &= I(\mathrm{Coker}(\mathcal{F} \to Gd^0(\mathcal{F}))) \ , \\ Gd^{n+1}(\mathcal{F}) &= I(\mathrm{Coker}(Gd^{n-1}(\mathcal{F}) \to Gd^n(\mathcal{F}))) \quad n > 0. \end{aligned}$$

Lemma 1.4.7. *1. Gd is an exact functor with values in* $C^+(\mathrm{Sh}(X))$*.*

2. The natural map $\mathcal{F} \to Gd^\bullet(\mathcal{F})$ *is a flasque resolution.*

Proof. Functoriality is obvious. The sheaf $I(\mathcal{F})$ is given by

$$U \mapsto \prod_{x \in U} i_{x*}\mathcal{F}_x \ .$$

All the sheaves involved are flasque, hence acyclic. In particular, taking the direct products is exact (it is not in general), turing $I(\mathcal{F})$ into an exact functor. $\mathcal{F} \to I(\mathcal{F})$ is injective, and hence by construction, $Gd^\bullet(\mathcal{F})$ is then a flasque resolution. □

Definition 1.4.8. Let $\mathcal{F}^\bullet \in C^+(\mathrm{Sh}(X))$ be a complex of sheaves. We call

$$Gd(\mathcal{F}^\bullet) := \mathrm{Tot}(Gd^\bullet(\mathcal{F}^\bullet))$$

the *Godement resolution* of $\mathcal{F}^\bullet$.

Corollary 1.4.9. *The natural map*

$$\mathcal{F} \to Gd(\mathcal{F}^\bullet)$$

is a quasi-isomorphism and

$$H^i(X, \mathcal{F}^\bullet) = H^i\left(\Gamma(X, Gd(\mathcal{F}^\bullet))\right) \ .$$

Proof. By Lemma 1.4.7, the first assertion holds if $\mathcal{F}^\bullet$ is concentrated in a single degree. The general case follows by the hypercohomology spectral sequence or by induction on the length of the complex using the studid filtration.

All terms in $Gd(\mathcal{F}^\bullet)$ are flasque, hence acyclic for $\Gamma(X, \cdot)$. □

We now study functoriality of the Godement resolution. For a continuous map $f : X \to Y$ be denote f^{-1} the pull-back functor on sheaves of abelian groups. Recall that it is exact.

Lemma 1.4.10. *Let $f : X \to Y$ be a continuous map between topological spaces, $\mathcal{F}^\bullet \in C^+(\mathrm{Sh}(Y))$. Then there is a natural quasi-isomorphism*

$$f^{-1}Gd_Y(\mathcal{F}^\bullet) \to Gd_X(f^{-1}\mathcal{F}^\bullet)\ .$$

Proof. Consider a sheaf $\mathcal{F}$ on Y. We want to construct

$$f^{-1}I(\mathcal{F}) \to I(f^{-1}\mathcal{F}) = \prod_{x\in X} i_{x*}(f^{-1}\mathcal{F})_x = \prod_{x\in X} i_{x*}\mathcal{F}_{f(x)}\ .$$

By the universal property of the direct product and adjunction for f^{-1}, this is equivalent to specifying for all $x \in X$

$$\prod_{y\in Y} i_{y*}\mathcal{F}_y = I(\mathcal{F}) \to f_* i_{x*}\mathcal{F}_{f(x)} = i_{f(x)*}\mathcal{F}_{f(x)}\ .$$

We use the natural projection map. By construction, we have a natural commutative diagram

$$\begin{array}{ccccc} f^{-1}\mathcal{F} & \longrightarrow & f^{-1}I(\mathcal{F}) & \longrightarrow & \mathrm{Coker}\big(f^{-1}\mathcal{F} \to f^{-1}I(\mathcal{F})\big) \\ {\scriptstyle =}\downarrow & & \downarrow & & \\ f^{-1}\mathcal{F} & \longrightarrow & I(f^{-1}\mathcal{F}) & \longrightarrow & \mathrm{Coker}\big(f^{-1}\mathcal{F} \to I(f^{-1}\mathcal{F})\big) \end{array}$$

It induces a map between the cokernels. Proceeding inductively, we obtain a morphism of complexes

$$f^{-1}Gd_Y^\bullet(\mathcal{F}) \to Gd_X^\bullet(f^{-1}\mathcal{F})\ .$$

It is a quasi-isomorphism because both are resolutions of $f^{-1}\mathcal{F}$. This transformation of functors extends to double complexes and hence defines a transformation of functors on $C^+(\mathrm{Sh}(Y))$. ☐

Remark 1.4.11. We are going to apply the theory of Godement resolutions in the case where X is a variety over a field k, a complex manifold or more generally a complex analytic space. The continuous maps that we need to consider are those in these categories, but also the maps of schemes $X_K \to X_k$ for the change of base field K/k and a variety over k; and the continuous map $X^{\mathrm{an}} \to X$ for an algebraic variety over $\mathbb{C}$ and its analytification.

1.4.3 Čech cohomology

Neither the definition of sheaf cohomology via injective resolutions nor Godement resolutions are convenient for concrete computations. We introduce Čech cohomology for this task. We follow [Ha2], Chap. III §4, but extend to hypercohomology.

Let k be a field. We work in the category of varieties over k. Let $I = \{1, \ldots, n\}$ as ordered set and $\mathfrak{U} = \{U_i | i \in I\}$ an affine open cover of X. For any subset $J \subset \{1, \ldots, n\}$ we denote

$$U_J = \bigcap_{j \in J} U_j \ .$$

As X is separated, they are all affine.

Definition 1.4.12. Let X and $\mathfrak{U}$ be as above. Let $\mathcal{F} \in \mathrm{Sh}(X)$. We define the *Čech complex* of $\mathcal{F}$ as

$$C^p(\mathfrak{U}, \mathcal{F}) = \prod_{J \subset I, |J| = p+1} \mathcal{F}(U_J) \qquad p \geq 0$$

with differential $\delta^p : C^p(\mathfrak{U}, \mathcal{F}) \to C^{p+1}(\mathfrak{U}, \mathcal{F})$

$$(\delta^p \alpha)_{i_0 < i_1 < \cdots < i_p} = \sum_{j=0}^{p+1} (-1)^p \alpha_{i_0 \cdots < \hat{i}_j < \cdots < i_{p+1}} |_{U_{i_0 \cdots i_{p+1}}}$$

where as usual $i_0 \cdots < \hat{i}_j < \cdots < i_{p+1}$ means the tuple with $\hat{i}_j$ removed.

We define the p-th *Čech cohomology* of X with coefficients in $\mathcal{F}$ as

$$\check{H}^p(\mathfrak{U}, \mathcal{F}) = H^p(C^\bullet(\mathfrak{U}, \mathcal{F}), \delta) \ .$$

Proposition 1.4.13 ([Ha2], chap. III Theorem 4.5)**.** *Let X be a variety, $\mathfrak{U}$ an affine open cover as before. Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules on X. Then there is a natural isomorphism*

$$H^p(X, \mathcal{F}) = \check{H}^p(\mathfrak{U}, \mathcal{F}) \ .$$

We now extend to complexes. We can apply the functor $C^\bullet(\mathfrak{U}, \cdot)$ to all terms in a complex $\mathcal{F}^\bullet$ and obtain a double complex $C^\bullet(\mathfrak{U}, \mathcal{F}^\bullet)$.

Definition 1.4.14. Let X and $\mathfrak{U}$ as before. Let $\mathcal{F}^\bullet \in C^+(\mathrm{Sh}(X))$. We define the *Čech complex* of $\mathfrak{U}$ with coefficients in $\mathcal{F}^\bullet$ as

$$C^\bullet(\mathfrak{U}, \mathcal{F}^\bullet) = \mathrm{Tot}\,(C^\bullet(\mathfrak{U}, \mathcal{F}^\bullet))$$

and *Čech cohomology* as

$$\check{H}^p(\mathfrak{U}, \mathcal{F}) = H^p(C^\bullet(\mathfrak{U}, \mathcal{F}^\bullet)) \ .$$

Proposition 1.4.15. *Let X be a variety, $\mathfrak{U}$ as before an open affine cover of X. Let $\mathcal{F}^\bullet \in C^+(\mathrm{Sh}(X))$ be complex such that all $\mathcal{F}^n$ are coherent sheaves of $\mathcal{O}_X$-modules. Then there is a natural isomorphism*

$$H^p(X, \mathcal{F}) \to \check{H}^p(\mathfrak{U}, \mathcal{F}^\bullet) \ .$$

Proof. The essential step is to define the map. We first consider a single sheaf $\mathcal{G}$. Let $\mathcal{C}^\bullet(\mathfrak{U}, \mathcal{G})$ be a sheafified version of the Čech complex for a sheaf $\mathcal{G}$. By [Ha2], chap. III Lemma 4.2, it is a resolution of $\mathcal{G}$. We apply the Godement resolution and obtain a flasque resolution of $\mathcal{G}$ by

$$\mathcal{G} \to \mathcal{C}^\bullet(\mathfrak{U}, \mathcal{G}) \to Gd\,(\mathcal{C}^\bullet(\mathfrak{U}, \mathcal{G}))\ .$$

By Proposition 1.4.13, the induced map

$$C^\bullet(\mathfrak{U}, \mathcal{G}) \to \Gamma(X, Gd\,(\mathcal{C}^\bullet(\mathfrak{U}, \mathcal{G}))$$

is a quasi-isomorphism as both compute $H^i(X, \mathcal{G})$.

The construction is functorial in $\mathcal{G}$, hence we can apply it to all components of a complex $\mathcal{F}^\bullet$ and obtain double complexes. We use the previous results for all $\mathcal{F}^n$ and take total complexes. Hence

$$\mathcal{F}^\bullet \to \mathrm{Tot}\mathcal{C}^\bullet(\mathfrak{U}, \mathcal{F}^\bullet) \to Gd\,(\mathcal{C}^\bullet(\mathfrak{U}, \mathcal{F}^\bullet))$$

are quasi-isomorphisms. Taking global sections we get a quasi-isomorphism

$$\mathrm{Tot}C^\bullet(\mathfrak{U}, \mathcal{F}^\bullet) \to \mathrm{Tot}\Gamma(X, Gd\,(\mathcal{C}^\bullet(\mathfrak{U}, \mathcal{F}^\bullet)))\ .$$

By definition, the complex on the left computes Čech cohomology of $\mathcal{F}^\bullet$ and the complex on right computes hypercohomology of $\mathcal{F}^\bullet$. □

Corollary 1.4.16. *Let X be an affine variety and $\mathcal{F}^\bullet \in C^+(\mathrm{Sh}(X))$ such that all $\mathcal{F}^n$ are coherent sheaves of $\mathcal{O}_X$-modules. Then*

$$H^i(\Gamma(X, \mathcal{F}^\bullet)) = H^i(X, \mathcal{F}^\bullet)\ .$$

Proof. We use the affine covering $\mathfrak{U} = \{X\}$ and apply Proposition 1.4.15. □

1.5 Simplicial objects

We introduce simplicial varieties in order to carry out some of the constructions in [D3]. Good general references on the topic are [May] or [We] Ch. 8.

Definition 1.5.1. Let Δ be the category whose objects are finite ordered sets

$$[n] = \{0, 1, \ldots, n\} \quad n \in \mathbb{N}_0$$

with morphisms nondecreasing monotone maps. Let Δ_N be the full subcategory with objects the $[n]$ with $n \leq N$.

If $\mathcal{C}$ is a category, we denote by $\mathcal{C}^\Delta$ the *category of simplicial objects* in $\mathcal{C}$ defined as contravariant functors

$$X_\bullet : \Delta \to \mathcal{C}$$

with transformation of functors as morphisms. We denote by $\mathcal{C}^{\Delta^\circ}$ the *category of cosimplicial objects* in $\mathcal{C}$ defined as covariant functors

$$X^\bullet : \Delta \to \mathcal{C} \ .$$

Similarly, we defined the categories $\mathcal{C}^{\Delta_N}$ and $\mathcal{C}^{\Delta_N^\circ}$ of *N-truncated simplicial and cosimplicial objects.*

Example 1.5.2. Let X be an object of $\mathcal{C}$. The constant functor

$$\Delta^\circ \to \mathcal{C}$$

which maps all objects to X and all morphism to the identity morphism is a simplicial object. It is called the *constant simplicial object* associated to X.

In Δ, we have in particular the *face maps*

$$\epsilon_i : [n-1] \to [n] \quad i = 0, \dots, n,$$

the unique injective map leaving out the index i, and the *degeneracy maps*

$$\eta_i : [n+1] \to [n] \quad i = 0, \dots, n,$$

the unique surjective map with two elements mapping to i. More generally, a map in Δ is called *face* or *degeneracy* if it is a composition of ϵ_i or η_i, respectively. Any morphism in Δ can be decomposed into a degeneracy followed by a face ([We] Lemma 8.12).

For all $m \geq n$, we denote $S_{m,n}$ the set of all degeneracy maps $[m] \to [n]$.

A simplicial object $X_\bullet$ is determined by a sequence of objects

$$X_0, X_1, \dots$$

and face and degeneracy morphisms between them. In particular, we write

$$\partial_i : X_n \to X_{n-1}$$

for the image of ϵ_i and

$$s_i : X_n \to X_{n+1}$$

for the image of η_i.

Example 1.5.3. For every n, there is a simplicial set $\Delta[n]$ with

$$\Delta[n]_m = \mathrm{Mor}_\Delta([m], [n])$$

and the natural face and degeneracy maps. It is called the *simplicial n-simplex.* For $n = 0$, this is the *simplicial point*, and for $n = 1$ the *simplicial interval.* Functoriality in the first argument induces maps of simplicial sets. In particular, there are

$$\delta_0 = \epsilon_0^*, \delta_1 = \epsilon_1^* : \Delta[1] \to \Delta[0] \ .$$

Definition 1.5.4. Let $\mathcal{C}$ be a category with finite products and coproducts. Let $\star$ be the final object. Let $X_\bullet$, $Y_\bullet$ simplicial objects in $\mathcal{C}$ and $S_\bullet$ a simplicial set

1. $X_\bullet \times Y_\bullet$ is the simplicial object with
$$(X_\bullet \times Y_\bullet)_n = X_n \times Y_n$$
with face and degeneracy maps induced from $X_\bullet$ and $Y_\bullet$.
2. $X_\bullet \times S_\bullet$ is the simplicial object with
$$(X_\bullet \times S_\bullet)_n = \coprod_{s \in S_n} X_n$$
with face and degeneracy maps induced from $X_\bullet$ and $S_\bullet$.
3. Let $f, g : X_\bullet \to Y_\bullet$ be morphisms of simplicial objects. Then f is called *homotopic* to g if there is a morphism
$$h : X_\bullet \times \Delta[1] \to Y_\bullet$$
such that $h \circ \delta_0 = f$ and $h \circ \delta_1 = g$.

The inclusion $\Delta_N \to \Delta$ induces a natural restriction functor
$$\mathrm{sq}_N : \mathcal{C}^\Delta \to \mathcal{C}^{\Delta_N} \ .$$
For a simplicial object $X_\bullet$, we call $\mathrm{sq}_N X_\bullet$ its N*-skeleton.* If $Y_\bullet$ is a fixed simplicial objects, we also denote sq_N the restriction functor from simplicial objects over $Y_\bullet$ to simplicial objects over $\mathrm{sq}_N Y_\bullet$.

Remark 1.5.5. The skeleta $\mathrm{sq}_k X_\bullet$ define the *skeleton filtration*, i.e., a chain of maps
$$\mathrm{sq}_0 X_\bullet \to \mathrm{sq}_1 X_\bullet \to \cdots \to \mathrm{sq}_N X_\bullet = X_\bullet.$$
Later, in section 2.3, we will define the topological realization $|X_\bullet|$ of a simplicial set $X_\bullet$. The skeleton filtration then defines a filtration of $|X_\bullet|$ by closed subspaces.

An important example in this book is the case when the simplicial set $X_\bullet$ is a finite set, i.e., all X_n are finite sets, and empty for $n > N$ sufficiently large. See section 2.3.

Lemma 1.5.6. *Let* $\mathcal{C}$ *be a category with finite limits. Then the functor* sq_N *has a right adjoint*
$$\mathrm{cosq}_N : \mathcal{C}^{\Delta_N} \to \mathcal{C}^\Delta \ .$$
If $Y_\bullet$ *is a fixed simplicial object, then*
$$\mathrm{cosq}_N^{Y_\bullet}(X_\bullet) = \mathrm{cosq}_N X_\times \times_{\mathrm{cosq}_N \mathrm{sq}_N Y_\bullet} Y_\bullet$$
is the right adjoint of the relative version of sq_N*.*

Proof. The existence of cosq_N is an instance of a Kan extension. We refer to [ML, chap. X] or [AM, chap. 2] for its existence. The relative case follows from the universal properties of fibre products. □

If $X_\bullet$ is an N-truncated simplicial object, we call $\mathrm{cosq}_N X_\bullet$ its *coskeleton*.

Remark 1.5.7. We apply this in particular to the case where $\mathcal{C}$ is one of the categories Var, Sm or Aff over a fixed field k. The disjoint union of varieties is a coproduct in these categories and the direct product a product.

Definition 1.5.8. Let S be a class of covering maps of varieties containing all identity morphisms. A morphism $f : X_\bullet \to Y_\bullet$ of simplicial varieties (or the simplicial variety $X_\bullet$ itself) is called an *S-hypercovering* if the adjunction morphisms

$$X_n \to (\mathrm{cosq}_{n-1}^{Y_\bullet} \mathrm{sq}_{n-1} X_\bullet)_n$$

are in S.

If S is the class of proper, surjective morphisms, we call $X_\bullet$ a *proper hypercover* of $Y_\bullet$.

Definition 1.5.9. Let $X_\bullet$ be a simplical variety. It is called *split* if for all $n \in \mathbb{N}_0$

$$N(X_n) = X_n \smallsetminus \bigcup_{i=0}^{n-1} s_i(X_{n-1})$$

is an open and closed subvariety of X_n.

We call $N(X_n)$ the non-degenerate part of X_n. If $X_\bullet$ is a split simplicial variety, we have a decomposition as varieties

$$X_n = N(X_n) \amalg \coprod_{m<n} \coprod_{s \in S_{m,n}} sN(X_m)$$

where $S_{m,n}$ is the set of degeneracy maps from X_m to X_n.

Theorem 1.5.10 (Deligne)**.** *Let k be a field and Y a variety over k. Then there is a split simplicial variety $X_\bullet$ with all X_n smooth and a proper hypercover $X_\bullet \to Y$.*

Proof. The construction is given in [D3] Section (6.2.5). It depends on the existence of resolutions of singularities. In positive characteristic, we may use de Jong's result on alterations instead. □

The other case we are going to need is the case of additive categories.

Definition 1.5.11. Let $\mathcal{A}$ be an additive category. We define a functor

$$C : \mathcal{A}^\Delta \to C^-(\mathcal{A})$$

by mapping a simplicial object $X_\bullet$ to the cohomological complex

$$\dots X_{-n} \xrightarrow{d^{-n}} X_{-(n-1)} \to \dots \to X_0 \to 0$$

with differential

$$d^{-n} = \sum_{i=0}^{n} (-1)^i \partial_i \ .$$

We define a functor

$$C : \mathcal{A}^{\Delta^\circ} \to C^+(\mathcal{A})$$

by mapping a cosimplicial object $X^\bullet$ to the cohomological complex

$$0 \to X^0 \to \dots X^n \xrightarrow{d^n} X_{n+1} \to \dots$$

with differential

$$d^n = \sum_{i=0}^{n} (-1)^i \partial_i \ .$$

Let $\mathcal{A}$ be an abelian category. We define a functor

$$N : \mathcal{A}^{\Delta^\circ} \to C^+(\mathcal{A})$$

by mapping a cosimplicial object $X^\bullet$ to the *normalized complex* $N(X^\bullet)$ with

$$N(X^\bullet)_n = \bigcap_{i=0}^{n-1} \operatorname{Ker}(s_i : X^n \to X^{n-1})$$

and differential $d^n|_{N(X^\bullet)}$.

Proposition 1.5.12 (Dold-Kahn correspondence)**.** *Let $\mathcal{A}$ be an abelian category, $X^\bullet \in \mathcal{A}^{\Delta^\circ}$ a cosimplicial object. Then the natural map*

$$N(X^\bullet) \to C(X^\bullet)$$

is a quasi-isomorphism.

Proof. This is the dual result of [We], Theorem 8.3.8. □

Remark 1.5.13. We are going to apply this in the case of cosimplicial complexes, i.e., to $C(\mathcal{A})^{\Delta^\circ}$, where $\mathcal{A}$ is abelian, e.g., a category of vector spaces.

1.6 Grothendieck topologies

Grothendieck topologies generalize the notion of open subsets in topological spaces. Using them one can define cohomology theories in more abstract settings. To define a Grothendieck topology, we need the notion of a *site, or situs.* Let $\mathcal{C}$ be a category. A basis for a Grothendieck topology on $\mathcal{C}$ is given by *covering families* in the category $\mathcal{C}$ satisfying the following definition.

Definition 1.6.1. A *site/situs* is a category $\mathcal{C}$ together with a collection of morphism in $\mathcal{C}$

$$(\varphi_i : V_i \longrightarrow U)_{i \in I},$$

the *covering families.*

The covering families satisfy the following axioms:

- An isomorphism $\varphi : V \to U$ is a covering family with an index set containing only one element.
- If $(\varphi_i : V_i \longrightarrow U)_{i \in I}$ is a covering family, and $f : V \to U$ a morphism in $\mathcal{C}$, then for each $i \in I$ there exists the pullback diagram

  $$\begin{array}{ccc} V \times_U V_i & \xrightarrow{F_i} & V_i \\ {\scriptstyle \Phi_i}\downarrow & & \downarrow{\scriptstyle \varphi_i} \\ V & \xrightarrow{f} & U \end{array}$$

  in $\mathcal{C}$, and $(\Phi_i : V \times_U V_i \to V)_{i \in I}$ is a covering family of V.
- If $(\varphi_i : V_i \longrightarrow U)_{i \in I}$ is a covering family of U, and for each V_i there is given a covering family $\left(\varphi_j^i : V_j^i \to V_i\right)_{j \in J(i)}$, then

  $$\left(\varphi_i \circ \varphi_j^i : V_j^i \to U\right)_{i \in I, j \in J(i)}$$

  is a covering family of U.

Example 1.6.2. Let X be a topological space. Then the category of open sets in X together with inclusions as morphisms form a site, where the covering maps are the families $(U_i)_{i \in I}$ of open subsets of U such that $\cup_{i \in I} U_i = U$. Thus each topological space defines a canonical site. For the Zariski open subsets of a scheme X this is called the *(small) Zariski site* of X.

Definition 1.6.3. A *presheaf* $\mathcal{F}$ of abelian groups on a situs $\mathcal{C}$ is a contravariant functor

$$\mathcal{F} : \mathcal{C} \to \mathrm{Ab}, U \mapsto \mathcal{F}(U).$$

A presheaf $\mathcal{F}$ is a *sheaf,* if for each covering family $(\varphi_i : V_i \longrightarrow U)_{i \in I}$, the difference kernel sequence

$$0 \to \mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(V_i) \rightrightarrows \prod_{(i,j) \in I \times I} \mathcal{F}(V_i \times_U V_j)$$

is exact. This means that a section $s \in \mathcal{F}(U)$ is determined by its restrictions to each V_i, and a tuple $(s_i)_{i \in I}$ of sections comes from a section on U, if one has $s_i = s_j$ on pullbacks to the fiber product $V_i \times_U V_j$.

Once we have a notion of sheaves in a certain Grothendieck topology, then we may define cohomology groups $H^*(X, \mathcal{F})$ by using injective resolutions as in section 1.4 as the right derived functor of the left-exact global section functor $X \mapsto \mathcal{F}(X) = H^0(X, \mathcal{F})$ in the presence of enough injectives.

Example 1.6.4. The *(small) étale site* over a smooth variety X consists of the category of all étale morphisms $\varphi : U \to X$ from a smooth variety U to X. See [Ha2, Chap. III] for the notion of étale maps. We just note here that étale maps are quasi-finite, i.e., have finite fibers, are unramified, and the image $\varphi(U) \subset X$ is a Zariski open subset.

A morphism in this site is given by a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & U \\ \downarrow & & \downarrow \\ X & \xrightarrow{\text{id}} & X. \end{array}$$

Let U be étale over X. A family $(\varphi_i : V_i \longrightarrow U)_{i \in I}$ of étale maps over X is called a covering family of U, if $\bigcup_{i \in I} \varphi_i(V_i) = U$, i.e., the images form a Zariski open covering of U.

This category has enough injectives by Grothendieck [SGA4.2], and thus we can define étale cohomology $H^*_{\text{et}}(X, \mathcal{F})$ for étale sheaves $\mathcal{F}$.

Example 1.6.5. In Section 2.7 we are going to introduce the h′-topology on the category of analytic spaces.

Definition 1.6.6. Let $\mathcal{C}$ and $\mathcal{C}'$ be sites. A *morphism of sites* $f : \mathcal{C} \to \mathcal{C}'$ consists of a functor $F : \mathcal{C}' \to \mathcal{C}$ (sic) which preserves fibre products and such that the F applied to a covering family of $\mathcal{C}'$ yields a covering family of $\mathcal{C}$.

A morphism of sites induces an adjoint pair of functors (f^*, f_*) between sheaves of sets on $\mathcal{C}$ and $\mathcal{C}'$.

Example 1.6.7.

1. Let $f : X \to Y$ be continuous map of topological spaces. As in Example 1.6.2 we view them as sites. Then the functor F mapping an open subset of Y to its preimage $f^{-1}(U)$.
2. Let X be a scheme. Then there is morphism of sites from the small étale site of X to the Zariki-site of X. The functor views an open subscheme $U \subset X$ as an étale X-scheme. Open covers are in particular étale covers.

Definition 1.6.8. Let $\mathcal{C}$ be a site. A $\mathcal{C}$-hypercover is an S-hypercover in the sense of Definition 1.5.8 with S the class of morphism

$$\coprod_{i \in I} U_i \to U$$

for all covering families $\{\phi_i : U_i \to U\}_{i \in I}$ in the site.

If $X_\bullet$ is a simplical object and $\mathcal{F}$ is a presheaf of abelian groups, then $\mathcal{F}(X_\bullet)$ is a cosimplicial abelian group. By applying the total complex functor C of Definition 1.5.11, we get a complex of abelian groups.

Proposition 1.6.9. *Let $\mathcal{C}$ be a site closed under finite products and fibre products, $\mathcal{F}$ a sheaf of abelian groups on $\mathcal{C}$, $X \in \mathcal{C}$. Then*

$$H^i(X, \mathcal{F}) = \lim_{X_\bullet \to X} H^i\left(C(\mathcal{F}(X_\bullet))\right)$$

where the direct limit runs through the system of all $\mathcal{C}$-hypercovers of X.

Proof. This is [SGA4V, Théorème 7.4.1] □