Periods and Nori Motives

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Chapter 2: Singular cohomology

Chapter 2

Singular Cohomology

In this chapter we give a short introduction to singular cohomology. Many properties are only sketched, as this theory is considerably easier than de Rham cohomology for example.

2.1 Sheaf cohomology

Let X be a topological space. Sometimes, if indicated, X will be the underlying topological space of an analytic or algebraic variety also denoted by X. To avoid technicalities, X will always be assumed to be a paracompact space, i.e., locally compact, Hausdorff, and satisfying the second countability axiom.

From now on, let $\mathcal{F}$ be a constant sheaf on X, e.g, $\mathcal{F} = \mathbb{Z}$. Later, we will consider more general coefficient rings $R \supset \mathbb{Z}$, but this will not change the following topological statements.

Definition 2.1.1 (Relative Cohomology)**.** Let $A \subset X$ be a closed subset, $U = X \setminus A$ the open complement, $i : A \hookrightarrow X$ and $j : U \hookrightarrow X$ be the inclusion maps. We define relative cohomology as

$$H^i(X, A; \mathbb{Z}) := H^i(X, j_!\mathbb{Z}),$$

where $j_!$ is the extension by zero, i.e., the sheafification of the presheaf $V \mapsto \mathbb{Z}$ for $V \subset U$ and $V \mapsto 0$ else.

Corollary 2.1.2. *There is a long exact sequence*

$$\cdots \to H^i(X, A; \mathbb{Z}) \to H^i(X, \mathbb{Z}) \to H^i(A, \mathbb{Z}) \xrightarrow{\delta} H^{i+1}(X, A; \mathbb{Z}) \to \cdots$$

Proof. This follows from the exact sequence of sheaves

$$0 \to j_!\mathbb{Z} \to \mathbb{Z} \to i_*\mathbb{Z} \to 0.$$

□

Note that by our definition of cones, see section 1.3, one has a quasi-isomorphism $j_!\mathbb{Z} = \mathrm{Cone}(\mathbb{Z} \to i_*\mathbb{Z})[-1]$. For Nori motives we need a version for triples, which can be proved using iterated cones by the same method:

Corollary 2.1.3. *Let $X \supset A \supset B$ be a triple of relative closed subsets. Then there is a long exact sequence*

$$\cdots \to H^i(X,A;\mathbb{Z}) \to H^i(X,B;\mathbb{Z}) \to H^i(A,B;\mathbb{Z}) \xrightarrow{\delta} H^{i+1}(X,A;\mathbb{Z}) \to \cdots$$

Here, δ is the connecting homomorphism, which in the cone picture is explained in Section 1.3.

Proposition 2.1.4 (Mayer-Vietoris)**.** *Let $\{U,V\}$ be an open cover of X. Let $A \subset X$ be closed. Then there is a natural long exact sequence*

$$\begin{aligned}\cdots \to H^i(X,A;\mathbb{Z}) \to H^i_{\mathrm{dR}}(U,U\cap A;\mathbb{Z}) \oplus H^i(V,V\cap A;\mathbb{Z})\\ \to H^i(U\cap V,U\cap V\cap A;\mathbb{Z}) \to H^{i+1}(X,A;\mathbb{Z}) \to \cdots\end{aligned}$$

Proof. Pairs (U,V) of open subsets form an excisive couple in the sense of [Sp, pg. 188], and therefore the Mayer-Vietoris property holds by [Sp, pg. 189-190]. □

Theorem 2.1.5 (Proper base change)**.** *Let $\pi : X \to Y$ be proper (i.e., the preimage of a compact subset is compact). Let $\mathcal{F}$ be a sheaf on X. Then the stalk in $y \in Y$ is computed as*

$$(R^i\pi_*\mathcal{F})_y = H^i(X_y, \mathcal{F}|_{X_y}).$$

Proof. See [KS] Proposition 2.6.7. As π is proper, we have $R\pi_* = R\pi_!$. □

As an immediate consequence we get:

Proposition 2.1.6 (Excision, or abstract blow-up)**.** *Let $f : (X',D') \to (X,D)$ be a proper, surjective morphism of algebraic varieties over $\mathbb{C}$, which induces an isomorphism $F : X' \setminus D' \to X \setminus D$. Then*

$$f^* : H^*(X,D;\mathbb{Z}) \cong H^*(X',D';\mathbb{Z}).$$

Proof. This fact goes back to A. Aeppli [Ae]. It is a special case of proper-base change: Let $j : U \to X$ be the complement of D and $j' : U \to X'$ its inclusion into X'. For all $x \in X$, we have

$$R^i\pi_*j'_!\mathbb{Z} = H^i(X_x, j'_!\mathbb{Z}|_{X'_x}).$$

For $x \in U$, the fibre is one point. It has no higher cohomology. For $x \in D$, the restriction of $j'_!\mathbb{Z}$ to X'_x is zero. Together this means

$$R\pi_*j'_!\mathbb{Z} = j_!\mathbb{Z}.$$

The statement follows from the Leray spectral sequence. □

We will later prove a slightly more general proper base change theorem for singular cohomology, see Theorem 2.5.10.

2.2 Singular (co)homology

Let X be a topological space. The definition of singular homology and cohomology uses simplexes.

Definition 2.2.1. The topological *n-simplex* Δ_n is defined as

$$\Delta_n := \{(t_0, ..., t_n) \mid \sum_{i=0}^{n} t_i = 1,\ t_i \geq 0\} \ .$$

Δ_n has natural codimension one faces defined by $t_i = 0$.

Singular (co)homology is defined by looking at all possible continuous maps from simplices to X.

Definition 2.2.2. A singular *n-simplex* σ is a continuous map

$$f \colon \Delta_n \to X.$$

In the case where X is a differentiable manifold, a singular simplex σ is called *differentiable*, if the map f can be extended to a $\mathcal{C}^\infty$-map from a neighbourhood of $\Delta_n \subset \mathbb{R}^{n+1}$ to X. The *group of singular n-chains* is the free abelian group

$$S_n(X) := \mathbb{Z}[f \colon \Delta_n \to X \mid f \text{ singular chain }].$$

In a similar way, we denote by $S_n^\infty(X)$ the free abelian *group of differentiable chains*. The boundary map $\partial_n : S_n(X) \to S_{n-1}(X)$ is defined as

$$\partial_n(f) := \sum_{i=0}^{n} (-1)^i f|_{t_i=0}.$$

The *group of singular n-cochains* is the free abelian group

$$S^n(X) := \mathrm{Hom}_{\mathbb{Z}}(S_n(X), \mathbb{Z}).$$

The *group of differentiable singular n-cochains* is the free abelian group

$$S^n(X) := \mathrm{Hom}_{\mathbb{Z}}(S_n^\infty(X), \mathbb{Z}).$$

The adjoint of ∂_{n+1} defines the boundary map

$$d_n : S^n(X) \to S^{n+1}(X).$$

Lemma 2.2.3. *One has $\partial_{n-1}\partial_n = 0$ and $d_{n+1}d_n = 0$, i.e., the groups $S_\bullet(X)$ and $S^\bullet(X)$ define complexes of abelian groups.*

The proof is left to the reader as an exercise.

Definition 2.2.4. Singular homology and cohomology with values in $\mathbb{Z}$ is defined as

$$H^i_{\text{sing}}(X,\mathbb{Z}) := H^i(S^\bullet(X), d_\bullet),\ H_i^{\text{sing}}(X,\mathbb{Z}) := H_i(S_\bullet(X), \partial_\bullet)\ .$$

In a similar way, we define (for X a manifold) the differentiable singular (co)homology as

$$H^i_{\text{sing},\infty}(X,\mathbb{Z}) := H^i(S^\bullet_\infty(X), d_\bullet),\ H_i^{\text{sing},\infty}(X,\mathbb{Z}) := H_i(S^\infty_\bullet(X), \partial_\bullet)\ .$$

Theorem 2.2.5. *Singular cohomology $H^i_{\text{sing}}(X,\mathbb{Z})$ agrees with sheaf cohomology $H^i(X,\mathbb{Z})$ with coefficients in $\mathbb{Z}$. If Y is a manifold, differentiable singular (co)homology agrees with singular (co)homology.*

Proof. Let $\mathcal{S}^n$ be the sheaf associated to the presheaf $U \mapsto S^n(U)$. One shows that $\mathbb{Z} \to \mathcal{S}^\bullet$ is a fine resolution of the constant sheaf $\mathbb{Z}$ [W, pg. 196]. If X is a manifold, differentiable cochains also define a fine resolution [W, pg. 196]. Therefore, the inclusion of complexes $S^\infty_\bullet(X) \hookrightarrow S_\bullet(X)$ induces isomorphisms

$$H^i_{\text{sing},\infty}(X,\mathbb{Z}) \cong H^i_{\text{sing}}(X,\mathbb{Z}) \text{ and } H_i^{\text{sing},\infty}(X,\mathbb{Z}) \cong H_i^{\text{sing}}(X,\mathbb{Z})\,.$$

□

2.3 Simplicial cohomology

In this section we want to introduce simplicial (co)homology and its relation to singular (co)homology. Simplicial (co)homology can be defined for topological spaces with an underlying combinatorial structure.

In the literature there are various notions of such spaces. In increasing order of generality, these are: (geometric) simplicial complexes and topological realizations of abstract simplicial complexes, of Δ-complexes (sometimes also called semi-simplicial complexes), and of simplicial sets. A good reference with a discussion of various definitions is the book by Hatcher [Hat], or the introductory paper [Fri] by Friedman.

By construction, such spaces are built from topological simplices Δ_n in various dimensions n, and the faces of each simplex are of the same type. Particularly nice examples are polyhedra, for example a tetrahedron, where the simplicial structure is obvious.

Geometric simplicial complexes come up more generally in geometric situations in the triangulations of manifolds with certain conditions. An example is the case of an analytic space X^{an} where X is an algebraic variety defined over $\mathbb{R}$. There one can always find a semi-algebraic triangulation by a result of Lojasiewicz, cf. Hironaka [Hi2, p. 170] and Prop. 2.6.8.

In this section, we will think of a simplicial space as the topological realization of a finite simplicial set:

Definition 2.3.1. Let $X_\bullet$ be a finite simplicial set in the sense of Remark 1.5.5. One has the face maps

$$\partial_i : X_n \to X_{n-1} \quad i = 0, \ldots, n,$$

and the degeneracy maps

$$s_i : X_n \to X_{n+1} \quad i = 0, \ldots, n.$$

The topological realization $|X_\bullet|$ of $X_\bullet$ is defined as

$$|X_\bullet| := \coprod_{n=0}^{\infty} X_n \times \Delta_n / \sim,$$

where each X_n carries the discrete topology, Δ_n is the topological n-simplex, and the equivalence relation is given by the two relations

$$(x, \partial_i(y)) \sim (\partial_i(x), y), \quad (x, s_i(y)) \sim (s_i(x), y), \quad x \in X_{n-1},\ y \in \Delta_n.$$

(Note that we denote the face and degeneracy maps for the n-simplex by the same letters ∂_i, s_i.)

In this way, every finite simplicial set gives rise to a topological space $|X_\bullet|$. It is known that $|X_\bullet|$ is a compactly generated CW-complex [Hat, Appendix]. In fact, every finite CW-complex is homotopy equivalent to a finite simplical complex of the same dimension by [Hat, Thm. 2C.5]. Thus, our restriction to realizations of finite simplicial sets is not a severe restriction.

The skeleton filtration from Remark 1.5.5 defines a filtration of $|X_\bullet|$

$$|\mathrm{sq}_0 X_\bullet| \subseteq |\mathrm{sq}_1 X_\bullet| \subseteq \cdots \subseteq |\mathrm{sq}_N X_\bullet| = |X_\bullet|$$

by closed subspaces, if X_n is empty for $n > N$.

There is finite number of simplices in each degree n. Associated to each of them is a continuous map $\sigma : \Delta_n \to |X_\bullet|$. We denote the free abelian group of all such σ of degree n by $C_n^\Delta(X_\bullet)$

$$\partial_n : C_n^\Delta(X_\bullet) \to C_{n-1}^\Delta(X_\bullet)$$

are given by alternating restriction maps to faces, as in the case of singular homology. Note that the vertices of each simplex are ordered, so that this is well-defined.

Definition 2.3.2. Simplicial homology of the topological space $X = |X_\bullet|$ is defined as

$$H_n^{\mathrm{simpl}}(X, \mathbb{Z}) := H_n(C_*^\Delta(X_\bullet), \partial_*),$$

and simplicial cohomology as

$$H^n_{\mathrm{simpl}}(X;\mathbb{Z}) := H^n(C^*_\Delta(X_\bullet), d_*),$$

where $C^n_\Delta(X_\bullet) = \mathrm{Hom}(C^\Delta_n(X_\bullet), \mathbb{Z})$ and d_n is adjoint to ∂_n.

Example 2.3.3. A tetrahedron arises from a simplicial set with four vertices (0-simplices), six edges (1-simplices), and four faces (2-simplices). A computation shows that $H_n = \mathbb{Z}$ for $i = 0, 2$ and zero otherwise (this was a priori clear, since it is topologically a sphere).

A torus T^2 can be obtained from a square by identifying opposite sides, called a and b. If we look at the diagonal of the square, we see that there is a simplicial complex with one vertex (!), three edges, and two faces. A computation shows that $H_1(T^2, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$ as expected, and $H_0(T^2, \mathbb{Z}) = H_2(T^2, \mathbb{Z}) = \mathbb{Z}$.

This definition does not depend on the representation of a topological space X as the topological realization of a simplicial set, since one can prove that simplicial (co)homology coincides with singular (co)homology:

Theorem 2.3.4. *Singular and simplicial (co)homology of X are equal.*

Proof. (For homology only.) The chain of closed subsets

$$|\mathrm{sq}_0 X_\bullet| \subseteq |\mathrm{sq}_1 X_\bullet| \subseteq \cdots \subseteq |\mathrm{sq}_N X_\bullet| = |X_\bullet|$$

gives rise to long exact sequences of simplicial homology groups

$$\cdots \to H_n^{\mathrm{simpl}}(|\mathrm{sq}_{n-1}X_\bullet|, \mathbb{Z}) \to H_n^{\mathrm{simpl}}(|\mathrm{sq}_n X_\bullet|, \mathbb{Z}) \to H_n^{\mathrm{simpl}}(|\mathrm{sq}_{n-1}X_\bullet|, |\mathrm{sq}_n X_\bullet|; \mathbb{Z}) \to \cdots$$

A similar sequence holds for singular homology, and there is a canonical map $C_n^\Delta(X) \to C_n(X)$ from simplicial to singular chains. The result is then proved by induction on n. We use that the relative complex $C_n(|\mathrm{sq}_{n-1}X_\bullet|, |\mathrm{sq}_n X_\bullet|)$ has zero differential and is a free abelian group of rank equal to the cardinality of X_n. Therefore, one concludes by observing a computation of the singular (co)homology of Δ_n, i.e., $H^i(\Delta_n, \mathbb{Z}) = \mathbb{Z}$ for $i = 0$ and zero otherwise. $\square$

In a similar way, one can define the simplicial (co)homology of a pair $(X, D) = (|X_\bullet|, |D_\bullet|)$, where $D_\bullet \subset X_\bullet$ is a simplicial subobject. The associated chain complex is given by the quotient complex $C_*(X_\bullet)/C_*(D_\bullet)$. The same proof will then show that the singular and simplicial (co)homology of pairs coincide.

From the definition of the topological realization, we see that X is a CW-complex. In the special case, when X is the topological space underlying an affine algebraic variety X over $\mathbb{C}$, or more generally a Stein manifold, then one can show:

Theorem 2.3.5 (Artin vanishing)**.** *Let X be an affine variety over $\mathbb{C}$ of dimension n. Then $H^q(X^{\mathrm{an}}, \mathbb{Z}) = 0$ for $q > n$. In fact, X^{an} is homotopy equivalent to a finite simplicial complex where all cells are of dimension $\leq n$.*

Proof. The proof was first given by Andreotti and Fraenkel [AF] for Stein spaces. An algebraic proof was given by M. Artin [A, Cor. 3.5, tome 3]. □

Corollary 2.3.6 (Good topological filtration)**.** *Let X be an affine variety over $\mathbb{C}$ of dimension n. Then the skeleton filtration of X^{an} is given by*

$$X^{\mathrm{an}} = X_n \supset X_{n-1} \supset \cdots \supset X_0$$

where the pairs (X_i, X_{i-1}) have only cohomology in degree i.

Remark 2.3.7. The Basic Lemma of Nori and Beilinson, see Thm. 2.5.6, shows that there is even an algebraic variant of this topological skeleton filtration.

The following theorem is strongly related to the Artin vanishing theorem.

Theorem 2.3.8 (Lefschetz hyperplane theorem)**.** *Let $X \subset \mathbb{P}^N_{\mathbb{C}}$ be an integral projective variety of dimension n, and $H \subset \mathbb{P}^N_{\mathbb{C}}$ a hyperplane section such that $H \cap X$ contains the singularity set X_{sing} of X. Then the inclusion $H \cap X \subset X$ is $(n-1)$-connected. In particular, one has $H^q(X, \mathbb{Z}) = H^q(X \cap H, \mathbb{Z})$ for $q \leq n$.*

Proof. See for example [AF]. □

2.4 Künneth formula and Poincaré duality

Assume that we have given two topological spaces X and Y, and two closed subsets $j : A \hookrightarrow X$, and $j' : C \hookrightarrow Y$. By the above, we have

$$H^*(X, A; \mathbb{Z}) = H^*(X, j_! \mathbb{Z})$$

and

$$H^*(Y, C; \mathbb{Z}) = H^*(Y, j'_! \mathbb{Z}) \ .$$

The relative cohomology group

$$H^*(X \times Y, X \times C \cup A \times Y; \mathbb{Z})$$

can be computed as $H^*(X \times Y, \tilde{j}_! \mathbb{Z})$, where

$$\tilde{j} : X \times C \cup A \times Y \hookrightarrow X \times Y$$

is the inclusion map. One has $\tilde{j}_! = j_! \boxtimes j'_!$. Hence, we have a natural exterior product map

$$H^i(X, A; \mathbb{Z}) \otimes H^j(Y, C; \mathbb{Z}) \xrightarrow{\times} H^{i+j}(X \times Y, X \times C \cup A \times Y; \mathbb{Z}).$$

This is related to the so-called *Künneth formula*:

Theorem 2.4.1 (Künneth formula for pairs)**.** *Let $A \subset X$ and $C \subset Y$ be closed subsets. The exterior product map induces a natural isomorphism*

$$\bigoplus_{i+j=n} H^i(X,A;\mathbb{Q}) \otimes H^j(Y,C;\mathbb{Q}) \xrightarrow{\cong} H^n(X \times Y, X \times C \cup A \times Y;\mathbb{Q}).$$

The same result holds with $\mathbb{Z}$-coefficients, provided all cohomology groups of (X,A) and (Y,C) in all degrees are free.

Proof. Using the sheaves of singular cochains, see the proof of theorem 2.2.5, one has fine resolutions $j_!\mathbb{Z} \to F^\bullet$ on X, and $j'_!\mathbb{Z} \to G^\bullet$ on Y. The tensor product $F^\bullet \boxtimes G^\bullet$ thus is a fine resolution of $\tilde{j}_!\mathbb{Z} = j_!\mathbb{Z} \boxtimes j'_!\mathbb{Z}$. Here one uses that the tensor product of fine sheaves is fine [W, pg. 193]. The cohomology of the tensor product complex $F^\bullet \otimes G^\bullet$ induces a short exact sequence

$$0 \to \bigoplus_{i+j=n} H^i(X,A;\mathbb{Z}) \otimes H^j(Y,C;\mathbb{Z}) \to H^n(X \times Y, X \times C \cup A \times Y;\mathbb{Z})$$

$$\to \bigoplus_{i+j=n+1} \mathrm{Tor}_1^{\mathbb{Z}}(H^i(X,A;\mathbb{Z}), H^j(Y,C;\mathbb{Z})) \to 0$$

by [God, thm. 5.5.1] or [We, thm. 3.6.3]. If all cohomology groups are free, the last term vanishes. $\square$

In later constructions, we will need a certain compatibility of the exterior product with coboundary maps. Assume that $X \supset A \supset B$ and $Y \supset C$ are closed subsets.

Proposition 2.4.2. *The diagram involving coboundary maps*

$$\begin{array}{ccc} H^i(A,B;\mathbb{Z}) \otimes H^j(Y,C;\mathbb{Z}) & \longrightarrow & H^{i+j}(A \times Y, A \times C \cup B \times Y;\mathbb{Z}) \\ \downarrow{\scriptstyle \delta\otimes \mathrm{id}} & & \downarrow{\scriptstyle \delta} \\ H^{i+1}(X,A;\mathbb{Z}) \otimes H^j(Y,C;\mathbb{Z}) & \longrightarrow & H^{i+j+1}(X \times Y, X \times C \cup A \times Y;\mathbb{Z}) \end{array}$$

commutes up to a sign $(-1)^j$. The diagram

$$\begin{array}{ccc} H^i(Y,C;\mathbb{Z}) \otimes H^j(A,B;\mathbb{Z}) & \longrightarrow & H^{i+j}(Y \times A, Y \times B \cup C \times A;\mathbb{Z}) \\ \downarrow{\scriptstyle \mathrm{id}\otimes\delta} & & \downarrow{\scriptstyle \delta} \\ H^i(Y,C;\mathbb{Z}) \otimes H^{j+1}(X,A;\mathbb{Z}) & \longrightarrow & H^{i+j+1}(Y \times X, Y \times A \cup C \times X;\mathbb{Z}) \end{array}$$

commutes (without a sign).

Proof. We indicate the argument, without going into full details. Let $F^\bullet$ be a complex computing $H^\bullet(Y,C;\mathbb{Z})$ Let $G_1^\bullet$ and $G_2^\bullet$ be complexes computing $H^\bullet(A,B;\mathbb{Z})$ and $H^\bullet(X,A;\mathbb{Z})$. Let $K_1^\bullet$ and $K_2^\bullet$ be the complexes computing cohomology of the corresponding product varieties. Cup product is induced from maps of complexes $F_i^\bullet \otimes G^\bullet \to K_i^\bullet$. In order to get compatibility with the boundary map, we have to consider the diagram of the form

$$\begin{array}{ccc} F_1 \otimes G & \longrightarrow & K_1 \\ \downarrow & & \downarrow \\ (F_2[1]) \otimes G & \longrightarrow & K_2[1] \end{array}$$

However, by Lemma 1.3.6, the complexes $(F_2[1]) \otimes G$ and $(F_2 \otimes G)[1]$ are not equal. We need to introduce the sign $(-1)^j$ in bidegree (i, j) to make the identification and get a commutative diagram.

The argument for the second type of boundary map is the same, but does not need the introduction of signs by Lemma 1.3.6. □

Assume now that $X = Y$ and $A = C$. Then, $j_!\mathbb{Z}$ has an algebra structure, and we obtain the *cup product* maps:

$$H^i(X, A; \mathbb{Z}) \otimes H^j(X, A; \mathbb{Z}) \longrightarrow H^{i+j}(X, A; \mathbb{Z})$$

via the multiplication maps

$$H^{i+j}(X \times X, \tilde{j}_!\mathbb{Z}) \to H^{i+j}(X, j_!\mathbb{Z}),$$

induced by

$$\tilde{j}_! = j_! \boxtimes j_! \to j_! \ .$$

In the case where $A = \emptyset$, the cup product induces Poincaré duality:

Proposition 2.4.3 (Poincaré Duality)**.** *Let X be a compact, orientable topological manifold of dimension m. Then the cup product pairing*

$$H^i(X, \mathbb{Q}) \times H^{m-i}(X, \mathbb{Q}) \longrightarrow H^m(X, \mathbb{Q}) \cong \mathbb{Q}$$

is non-degenerate in both factors. With $\mathbb{Z}$-coefficients, the same result holds for the two left groups modulo torsion.

Proof. We will give a proof of a slightly more general statement in the algebraic situation below. A proof of the stated theorem can be found in [GH, pg. 53]. There it is shown that $H^{2n}(X)$ is torsion-free of rank one, and the cup-product pairing is unimodular modulo torsion, using simplicial cohomology, and the relation between Poincaré duality and the dual cell decomposition. □

We will now prove a relative version in the algebraic case. It implies the version above in the case where $A = B = \emptyset$. By abuse of notation, we again do not distinguish between an algebraic variety over $\mathbb{C}$ and its underlying topological space.

Theorem 2.4.4 (Poincaré duality for algebraic pairs)**.** *Let X be a smooth and proper complex variety of dimension n over $\mathbb{C}$ and $A, B \subset X$ two normal crossing divisors, such that $A \cup B$ is also a normal crossing divisor. Then there is a non-degenerate duality pairing*

$$H^d(X \backslash A, B \backslash (A \cap B); \mathbb{Q}) \times H^{2n-d}(X \backslash B, A \backslash (A \cap B); \mathbb{Q}) \longrightarrow H^{2n}(X, \mathbb{Q}) \cong \mathbb{Q}(-n).$$

Again, with $\mathbb{Z}$-coefficients this is true modulo torsion by unimodularity of the cup-product pairing.

Proof. We give a sheaf theoretic proof using Verdier duality and some formulas and ideas of Beilinson (see [Be1]). Look at the commutative diagram:

$$\begin{tikzcd} U = X \setminus (A \cup B) \arrow[r, "\ell_U"] \arrow[d, "k_U"'] & X \setminus A \arrow[d, "k"] \\ X \setminus B \arrow[r, "\ell"] & X. \end{tikzcd}$$

Assuming $A \cup B$ is a normal crossing divisor, we want to show first that the natural map

$$\ell_! k_{U*} \mathbb{Q}_U \longrightarrow k_* \ell_{U!} \mathbb{Q}_U,$$

extending $\mathrm{id} : \mathbb{Q}_U \to \mathbb{Q}_U$, is an isomorphism. This is a local computation. We look without loss of generality at a neighborhood of an intersection point $x \in A \cap B$, since the computation at other points is even easier. If we work in the analytic topology, we may choose a polydisk neighborhood D in X around x such that D decomposes as

$$D = D_A \times D_B$$

and such that

$$A \cap D = A_0 \times D_B, \quad B \cap D = D_A \times B_0$$

for some suitable topological spaces A_0, B_0. Using the same symbols for the maps as in the above diagram, the situation looks locally like

$$\begin{tikzcd} (D_A \setminus A_0) \times (D_B \setminus B_0) \arrow[r, "\ell_U"] \arrow[d, "k_U"'] & (D_A \setminus A_0) \times D_B \arrow[d, "k"] \\ D_A \times (D_B \setminus B_0) \arrow[r, "\ell"] & D = D_A \times D_B. \end{tikzcd}$$

Using the Künneth formula, one concludes that both sheaves $\ell_! k_{U*} \mathbb{Q}_U$ and $k_* \ell_{U!} \mathbb{Q}_U$ are isomorphic to

$$k_{U*} \mathbb{Q}_{D_A \setminus A_0} \otimes \ell_! \mathbb{Q}_{D_B \setminus B_0}$$

near the point x, and the natural map provides an isomorphism.

Now, one has

$$H^d(X \setminus A, B \setminus (A \cap B)); \mathbb{Q}) = H^d(X, \ell_! k_{U*}\mathbb{Q}_U),$$

(using that the maps involved are affine), and

$$H^{2n-d}(X \setminus B, A \setminus (A \cap B)); \mathbb{Q}) = H^{2n-d}(X, k_! \ell_{U*}\mathbb{Q}_U).$$

We have to show that there is a perfect pairing

$$H^d(X \setminus A, B \setminus (A \cap B)); \mathbb{Q}) \times H^{2n-d}(X \setminus B, A \setminus (A \cap B)); \mathbb{Q}) \to \mathbb{Q}(-n).$$

However, by Verdier duality, we have a perfect pairing

$$\begin{aligned} H^{2n-d}(X \setminus B, A \setminus (A \cap B)); \mathbb{Q})^\vee &= H^{2n-d}(X, k_! \ell_{U*}\mathbb{Q}_U)^\vee \\ &= H^{-d}(X, k_! \ell_{U*}\mathbb{D}\mathbb{Q}_U)(-n) \\ &= H^{-d}(X, \mathbb{D}(k_* \ell_{U!}\mathbb{Q}_U))(-n) \\ &= H^d(X, k_* \ell_{U!}\mathbb{Q}_U)(-n) \\ &= H^d(X, \ell_! k_{U*}\mathbb{Q}_U)(-n) \\ &= H^d(X \setminus A, B \setminus (A \cap B)); \mathbb{Q}). \end{aligned}$$

□

Remark 2.4.5. The normal crossing condition is necessary, as one can see in the example of $X = \mathbb{P}^2$, where A consists of two distinct lines meeting in a point, and B a line going through the same point.

2.5 Basic Lemma

In this section we prove the basic lemma of Nori [N, N1, N2], a topological result, which was also known to Beilinson [Be1] and Vilonen (unpublished). Let $k \subset \mathbb{C}$ be a subfield. The proof of Beilinson works more generally in positive characteristics as we will see below.

Convention 2.5.1. We fix an embedding $k \to \mathbb{C}$. All sheaves and all cohomology group in the following section is to be understood in the complex topology on $X(\mathbb{C})$.

Theorem 2.5.2 (Basic Lemma I)**.** *Let $k \subset \mathbb{C}$. Let X be an affine variety over k of dimension n and $W \subset X$ be a Zariski closed subset with* $\dim(W) < n$. *Then there exists a Zariski closed subset $Z \supset W$ with* $\dim(Z) < n$ *and $H^q(X, Z; \mathbb{Z}) = 0$ for $q \neq n$. The cohomology group $H^n(X, Z; \mathbb{Z})$ is a free $\mathbb{Z}$-module.*

We formulate the Lemma for coefficients in $\mathbb{Z}$, but by the universal coefficient theorem it will hold with other coefficients as well.

Example 2.5.3. There is an example where there is an easy way to obtain Z. Assume that X is of the form $\bar{X} \setminus H$ for some smooth projective $\bar{X}$ and a hyperplane H. Let $W = \emptyset$. Then take another hyperplane section H' meeting $\bar{X}$ and H transversally. Then $Z := H' \cap X$ will have the property that $H^q(X, Z; \mathbb{Z}) = 0$ for $q \neq n$ by the Lefschetz hyperplane theorem, see Thm. 2.3.8. This argument will be generalized in Beilinson's proof below.

An inductive application of this Basic Lemma in the case $Z = \emptyset$ yields a filtration of X by closed subsets

$$X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset$$

with $\dim(X_i) = i$ such that the complex of free $\mathbb{Z}$-modules

$$\cdots \xrightarrow{\delta_{i-1}} H^i(X_i, X_{i-1}) \xrightarrow{\delta_i} H^{i+1}(X_{i+1}, X_i) \xrightarrow{\delta_{i+1}} \cdots ,$$

where the maps $\delta_\bullet$ arise from the coboundary in the long exact sequence associated to the triples $X_{i+1} \supset X_i \supset X_{i-1}$, computes the cohomology of X.

Remark 2.5.4. This means that we can understand this filtration as algebraic analogue of the skeletal filtraton of simplicial complexes, see Corollary 2.3.6. Note that the filtration is not only algebraic, but even defined over the base field k.

The Basic Lemma is deduced from the following variant, which was also known to Beilinson [Be1]. To state it, we need the notion of a (weakly) constructible sheaf.

Definition 2.5.5. A sheaf of abelian groups on a variety X over k is *weakly constructible*, if there is a stratification of X into a disjoint union of finitely many Zariski locally closed subsets Y_i, and such that the restriction of F to Y_i is locally constant.

We will also need some basic facts about sheaf cohomology. If $j : U \hookrightarrow X$ is a Zariski open subset with closed complement $i : W \hookrightarrow X$ and F a sheaf of abelian groups on X, then there is an exact sequence of sheaves

$$0 \to j_! j^* F \to F \to i_* i^* F \to 0.$$

In addition, for F the constant sheaf $\mathbb{Z}$, one has $H^q(X, j_! j^* F) = H^q(X, W; \mathbb{Z})$ and $H^q(X, i_* i^* F) = H^q(W, \mathbb{Z})$, see section 2.1.

Theorem 2.5.6 (Basic Lemma II)**.** *Let X be an affine variety over k of dimension n and F be a weakly constructible sheaf on X. Then there exists a Zariski open subset $j : U \hookrightarrow X$ such the following three properties hold:*

1. $\dim(X \setminus U) < n$.
2. $H^q(X, F') = 0$ *for* $q \neq n$*, where* $F' := j_! j^* F \subset F$.

3. *There exists a finite subset $E \subset U(\mathbb{C})$ such that $H^{\dim(X)}(X, F')$ is isomorphic to a direct sum $\oplus_x F_x$ of stalks of F at points of E.*

Version II of the Lemma implies version I. Let $V = X \setminus W$ with open immersion $h : V \hookrightarrow X$, and the sheaf $F = h_! h^* \mathbb{Z}$ on X. Version II for F gives an open subset $\ell : U \hookrightarrow X$ such that the sheaf $F' = \ell_! \ell^* F$ has non-vanishing cohomology only in degree n. Let $W' = X \setminus U$. Since F was zero on W, we have that F' is zero on $Z = W \cup W'$ and it is the constant sheaf on $X \setminus Z$, i.e., $F' = j_! j^* F$ for $j : X \setminus Z \hookrightarrow X$. In particular, F' computes the relative cohomology $H^q(X, Z; \mathbb{Z})$ and it vanishes for $q \neq n$. Freeness follows from property (3). □

Now we will give two proofs of the Basic Lemma II. The first proof by Nori will prove all three assertions, the second proof of Beilinson we give below, proves (1) and (2).

2.5.1 Nori's proof

We first present the proof of Basic Lemma II published by Nori in [N2].

We start with a couple of lemmas on weakly constructible sheaves.

Lemma 2.5.7. *Let $0 \to F_1 \to F_2 \to F_3 \to 0$ be a short exact sequence of sheaves on X with F_1, F_3 weakly constructible. Then F_2 is weakly constructible.*

Proof. By assumption, there are stratifications of X such that F_1 and F_3 become locally constant, respectively. We take a common refinement. We replace X by one of the strata and are now in the situation that F_1 and F_3 are locally constant. Then F_3 is also locally constant. Indeed, by passing to a suitable open cover (in the analytic topology), F_1 and F_3 become even constant. If $V \subset U$ is an inclusion of open connected subsets, then the restrictions $F_1(U) \to F_1(V)$ and $F_3(U) \to F_3(U)$ are isomorphisms. This implies the same statement for F_2 because $H^1(U, F_1) = H^1(V, F_1) = 0$ because constant sheaves do not have higher cohomology. □

Lemma 2.5.8. *The notion of weak constructibility is stable under $j_!$ for j an open immersion and π_* for π finite.*

Proof. The assertion of $j_!$ is obvious, same as for i_* for closed immersions.

Now assume $\pi : X \to Y$ is finite and in addition étale. Let F be weakly constructible on X. Let $X_0, \ldots, X_n \subset X$ be the stratification such that $F|_{X_i}$ is locally constant. Let Y_i be the image of X_i. These are locally closed subvarieties of Y because π is closed and open. We refine them into a stratification of Y. As π is finite étale, it is locally in the analytic topology of the form $I \times B$ with I finite and $B \subset Y(\mathbb{C})$ an open in the analytic topology. Obviously $\pi_* F|_B$ is locally constant on the strata we have defined.

Now let π be finite. As we have already discussed closed immersion, it suffices to assume π surjective. There is an open dense subscheme $U \subset Y$ such π is étale above U. Let $U' = \pi^{-1}U$, $Z = Y \setminus U$ and $Z' = X \setminus V$. We consider the exact sequence on X

$$0 \to j_{U'!}j_{U'}^*F \to F \to i_{Z'*}i_{Z'}^*F \to 0.$$

As π is finite, the functor π_* is exact and hence

$$0 \to \pi_* j_{U'!}j_{U'}^*F \to \pi_*F \to \pi_* i_{Z'*}i_{Z'}^*F \to 0.$$

By Lemma 2.5.7 it suffices to consider the outer terms. We have

$$\pi_* j_{U'!}j_{U'}^*F = j_{U*}\pi|_{U'*}j_{U'}^*F$$

and this is weakly constructible by the etale case and the assertion on open immersions. We also have

$$\pi_* i_{Z'*}i_{Z'}^*F = i_{Z*}\pi|_{Z'*}i_{Z'}^*F$$

and this is weakly constructible by noetherian induction and the case of closed immersions. $\square$

Nori's proof of Basic Lemma II. Let $n := \dim(X)$. In the first step, we reduce to $X = \mathbb{A}^n$. We use Noether normalization to obtain a finite morphism $\pi : X \to \mathbb{A}^n$. By Lemma 2.5.8 the sheaf π_*F is weakly constructible.

Let then $v : V \hookrightarrow \mathbb{A}^n$ be a Zariski open set with the property that $F' := v_!v^*\pi_*F$ satisfies the Basic Lemma II on $\mathbb{A}^n$. Let $U := \pi^{-1}(V) \overset{j}{\hookrightarrow} X$ be the preimage in X. One has an equality sheaves:

$$\pi_* j_! j^* F = v_! v^* \pi_* F.$$

Therefore, $H^q(X, j_!j^*F) = H^q(\mathbb{A}^n, v_!v^*\pi_*F)$ and the latter vanishes for $q \neq n$.

So let us now assume that F is weakly constructible on $X = \mathbb{A}^n$. We argue by induction on n and all F. The case $n = 0$ is trivial.

By replacing F by $j_!j^*F$ for an appropriate open $j : U \to \mathbb{A}^n$ we may assume that F is locally constant on U and that $\mathbb{A}^n \setminus U = V(f)$. By Noether normalization or its proof, there is a surjective projection map $p : \mathbb{A}^n \to \mathbb{A}^{n-1}$ such that $p|_{V(f)} : V(f) \to \mathbb{A}^{n-1}$ is surjective and finite.

We will see in Lemma 2.5.9 that $R^qp_*F = 0$ for $q \neq 1$ and R^1p_*F is weakly constructible. The Leray spectral sequence now gives that

$$H^q(\mathbb{A}^n, F) = H^{q-1}(\mathbb{A}^{n-1}, R^1\pi_*F).$$

In the induction procedure we apply the Basic Lemma II to R^1p_*F on $\mathbb{A}^{n-1}$. By induction, there exists a Zariski open $h : V \hookrightarrow \mathbb{A}^{n-1}$ such that $h_!h^*R^1\pi_*F$ has cohomology only in degree $n-1$. Let $U := \pi^{-1}(V)$ and $j : U \hookrightarrow \mathbb{A}^n$ be the inclusion. Then $j_!j^*F$ has cohomology only in degree n. $\square$

Lemma 2.5.9. *Let p be as in the above proof. Then $R^q\pi_*F = 0$ for $q \neq 0$ and $R^1\pi_*F$ is weakly constructible.*

Proof. This is a standard fact, but Nori gives a direct proof.

The stalk of R^qp_*F at $y \in \mathbb{A}^{n-1}$ is given by $H^q(\{y\} \times \mathbb{A}^1, F_{\{y\}\times\mathbb{A}^1})$ by the variation of proper base change in Theorem 2.5.10 below.

Let, more generally, G be a sheaf on $\mathbb{A}^1$ which is locally constant outside a finite, non-empty set S. Let T be a tree in $\mathbb{A}^1(\mathbb{C})$ with vertex set S. Then the restriction map to the tree defines a retraction isomorphism $H^q(\mathbb{A}^1, G) \cong H^q(T, G_T)$ for all $q \geq 0$. Using Cech cohomology we can compute that $H^q(T, G_T)$: For each vertex $v \in S$, let U_s be the star of half edges of length more than half the length of all outgoing edges at the vertex s. Then U_a and U_b only intersect if the vertices a and b have a common edge $e = e(a, b)$. The intersection $U_a \cap U_b$ is contractible and contains the center $t(e)$ of the edge e. There are no triple intersections. Therefore $H^q(T, G_T) = 0$ for $q \geq 2$. Since G is zero on S, U_s is simply connected, and G is locally constant, $G(U_s) = 0$ for all s. Therefore also $H^0(T, G_T) = 0$ and $H^1(T, G_T)$ is isomorphic to $\bigoplus_e G_{t(e)}$.

This implies already that $R^qp_*F = 0$ for $q \neq 1$.

To show that R^1p_*F is weakly constructible, means to show that it is locally constant on some stratification. We see that the stalks $(R^1p_*F)_y$ depend only on the set of points in $\{y\} \times \mathbb{A}^1 = p^{-1}(y)$ where $F_{\{y\}\times\mathbb{A}^1}$ vanishes. But the sets of points where the vanishing set has the same degree (cardinality) defines a suitable stratification. Note that the stratification only depends on the branching behaviour of $V(f) \to \mathbb{A}^{n-1}$, hence the stratification is algebraic and defined over k. □

Theorem 2.5.10 (Variation of Proper Base Change)**.** *Let $p : X \to Y$ be a continuous map between locally compact, locally contractible topological spaces which is a fiber bundle and let G be a sheaf on X. Assume $W \subset X$ is closed and such that G is locally constant on $X \setminus W$ and p restricted to W is proper. Then $(R^qp_*G)_y \cong H^q(p^{-1}(y), G_{p^{-1}(y)})$ for all q and all $y \in Y$.*

Proof. The statement is local on Y, so we may assume that $X = T \times Y$ is a product with p the projection. Since Y is locally compact and locally contractible, we may assume that Y is compact by passing to a compact neighbourhood of y. As $W \to Y$ is proper, this implies that W is compact. By enlarging W, we may assume that $W = K \times Y$ is a product of compact sets for some compact subset $K \subset X$. Since Y is locally contractible, we replace Y be a contractible neighbourhood. (We may loose compactness, but this does not matter anymore.) Let $i : K \times Y \to X$ be the inclusion and and $j : (T \setminus K) \times Y \to X$ the complement.

Look at the exact sequence

$$0 \to j_!G_{(T\setminus K)\times Y} \to G \to i_*G_{K\times Y} \to 0.$$

The result holds for $G_{K\times Y}$ by the usual proper base change.

Since Y is contractible, we may assume that $G_{(T\setminus K)\times Y}$ is the pull-back of constant sheaf on $T \setminus K$. Now the result for $j_! G_{(T\setminus K)\times Y}$ follows from the Künneth formula. $\square$

2.6 Triangulation of Algebraic Varieties

If X is a variety defined over $\mathbb{Q}$, we may ask whether any singular homology class $\gamma \in H_\bullet^{\text{sing}}(X^{\text{an}}; \mathbb{Q})$ can be represented by an object described by polynomials. This is indeed the case: for a precise statement we need several definitions. The result will be formulated in Proposition 2.6.8.

This section follows closely the Diploma thesis of Benjamin Friedrich, see [Fr]. The results are due to him.

We work over $k = \widetilde{\mathbb{Q}}$, i.e., the integral closure of $\mathbb{Q}$ in $\mathbb{R}$. Note that $\widetilde{\mathbb{Q}}$ is a field.

In this section, we use X to denote a variety over $\widetilde{\mathbb{Q}}$, and X^{an} for the associated analytic space over $\mathbb{C}$ (cf. Subsection 1.2).

2.6.1 Semi-algebraic Sets

Definition 2.6.1 ([Hi2, Def. 1.1., p.166])**.** A subset of $\mathbb{R}^n$ is said to be $\widetilde{\mathbb{Q}}$-*semi-algebraic*, if it is of the form

$$\{\underline{x} \in \mathbb{R}^n | f(\underline{x}) \geq 0\}$$

for some polynomial $f \in \widetilde{\mathbb{Q}}[x_1, \ldots, x_n]$, or can be obtained from sets of this form in a finite number of steps, where each step consists of one of the following basic operations:

1. complementary set,
2. finite intersection,
3. finite union.

We need also a definition for maps:

Definition 2.6.2 ($\widetilde{\mathbb{Q}}$-semi-algebraic map [Hi2, p. 168])**.** A continuous map f between $\widetilde{\mathbb{Q}}$-semi-algebraic sets $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ is said to be $\widetilde{\mathbb{Q}}$-*semi-algebraic*, if its graph

$$\Gamma_f := \big\{ \big(a, f(a)\big) \,|\, a \in A \big\} \subseteq A \times B \subseteq \mathbb{R}^{n+m}$$

is $\widetilde{\mathbb{Q}}$-semi-algebraic.

Example 2.6.3. Any polynomial map

$$\begin{aligned} f : A &\longrightarrow B \\ (a_1, \ldots, a_n) &\mapsto (f_1(a_1, \ldots, a_n), \ldots, f_m(a_1, \ldots, a_n)) \end{aligned}$$

between $\widetilde{\mathbb{Q}}$-semi-algebraic sets $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ with $f_i \in \widetilde{\mathbb{Q}}[x_1, \ldots, x_n]$ for $i = 1, \ldots, m$ is $\widetilde{\mathbb{Q}}$-semi-algebraic, since it is continuous and its graph $\Gamma_f \subseteq \mathbb{R}^{n+m}$ is cut out from $A \times B$ by the polynomials

$$y_i - f_i(x_1, \ldots, x_n) \in \widetilde{\mathbb{Q}}[x_1, \ldots, x_n, y_1, \ldots, y_m] \quad \text{for} \quad i = 1, \ldots, m. \tag{2.1}$$

We can even allow f to be a rational map with rational component functions

$$f_i \in \widetilde{\mathbb{Q}}(x_1, \ldots, x_n), \quad i = 1, \ldots, m$$

as long as none of the denominators of the f_i vanish at a point of A. The argument remains the same except that the expression (2.1) has to be multiplied by the denominator of f_i.

Fact 2.6.4 ([Hi2, Prop. II, p. 167], [Sb, Thm. 3, p. 370])**.**
By a result of Seidenberg-Tarski, the image (respectively preimage) of a $\widetilde{\mathbb{Q}}$-semi-algebraic set under a $\widetilde{\mathbb{Q}}$-semi-algebraic map is again $\widetilde{\mathbb{Q}}$-semi-algebraic.

As the name suggests, any algebraic set should be in particular $\widetilde{\mathbb{Q}}$-semi-algebraic.

Lemma 2.6.5. *Let X be a quasi-projective algebraic variety defined over $\widetilde{\mathbb{Q}}$. Then we can regard the complex analytic space X^{an} associated to the base change $X_{\mathbb{C}} = X \times_{\widetilde{\mathbb{Q}}} \mathbb{C}$ as a bounded $\widetilde{\mathbb{Q}}$-semi-algebraic subset*

$$X^{\mathrm{an}} \subseteq \mathbb{R}^N \tag{2.2}$$

for some N. Moreover, if $f : X \to Y$ is a morphism of varieties defined over $\widetilde{\mathbb{Q}}$, we can consider $f^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$ as a $\widetilde{\mathbb{Q}}$-semi-algebraic map.

Remark 2.6.6. We will mostly need the case when X is even *affine*. Then $X \subset \mathbb{C}^n$ is defined by polynomial equations with coefficients in $\widetilde{\mathbb{Q}}$. We identify $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and rewrite the equations for the real and imaginary part. Hence X is obviously $\widetilde{\mathbb{Q}}$-semialgebraic. In the lemma, we will show in addition that X can be embedded as a *bounded* $\widetilde{\mathbb{Q}}$-semialgebraic set.

Proof of Lemma 2.6.5. First step $X = \mathbb{P}^n_{\widetilde{\mathbb{Q}}}$: Consider

- $\mathbb{P}^n_{\mathbb{C}} = (\mathbb{P}^n_{\widetilde{\mathbb{Q}}} \times_{\widetilde{\mathbb{Q}}} \mathbb{C})^{\mathrm{an}}$ with homogenous coordinates $x_0, \ldots, x_n$, which we split as $x_m = a_m + ib_m$ with $a_m, b_m \in \mathbb{R}$ in real and imaginary part, and
- $\mathbb{R}^N$, $N = 2(n+1)^2$, with coordinates $\{y_{kl}, z_{kl}\}_{k,l=0,\ldots,n}$.

We define a map

$$\psi : \underset{[x_0:\ldots:x_n]}{\mathbb{P}^n_{\mathbb{C}}} \longrightarrow \underset{(y_{00},z_{00},\ldots,y_{nn},z_{nn})}{\mathbb{R}^N}$$

$$[x_0 : \ldots : x_n] \mapsto \Bigg(\ldots, \underbrace{\frac{\operatorname{Re} x_k \overline{x}_l}{\sum_{m=0}^n |x_m|^2}}_{y_{kl}}, \underbrace{\frac{\operatorname{Im} x_k \overline{x}_l}{\sum_{m=0}^n |x_m|^2}}_{z_{kl}}, \ldots\Bigg)$$

$$[a_0 + ib_0 : \ldots : a_n + ib_n] \mapsto \Bigg(\ldots, \underbrace{\frac{a_k a_l + b_k b_l}{\sum_{m=0}^n a_m^2 + b_m^2}}_{y_{kl}}, \underbrace{\frac{b_k a_l - a_k b_l}{\sum_{m=0}^n a_m^2 + b_m^2}}_{z_{kl}}, \ldots\Bigg).$$

Rewriting the last line (with the convention $0 \cdot \cos(\substack{\text{indeterminate}\\ \text{angle}}) = 0$) as

$$[r_0 e^{i\phi_0} : \ldots : r_n e^{i\phi_n}] \mapsto \Bigg(\ldots, \frac{r_k r_l \cos(\phi_k - \phi_l)}{\sum_{m=0}^n r_m^2}, \frac{r_k r_l \sin(\phi_k - \phi_l)}{\sum_{m=0}^n r_m^2}, \ldots\Bigg) \tag{2.3}$$

shows that ψ is injective: Assume

$$\psi\big([r_0 e^{i\phi_0} : \ldots : r_n e^{i\phi_n}]\big) = (y_{00}, z_{00}, \ldots, y_{nn}, z_{nn})$$

where $r_k \neq 0$, or equivalently $y_{kk} \neq 0$, for a fixed k. We find

$$\frac{r_l}{r_k} = \frac{\sqrt{y_{kl}^2 + z_{kl}^2}}{y_{kk}}, \quad \text{and}$$

$$\phi_k - \phi_l = \begin{cases} \arctan(z_{kl}/y_{kl}) & \text{if } y_{kl} \neq 0, \\ \pi/2 & \text{if } y_{kl} = 0, z_{kl} > 0, \\ \text{indeterminate} & \text{if } y_{kl} = z_{kl} = 0, \\ -\pi/2 & \text{if } y_{kl} = 0, z_{kl} < 0; \end{cases}$$

that is, the preimage of $(y_{00}, z_{00}, \ldots, y_{nn}, z_{nn})$ is uniquely determined.

Therefore, we can consider $\mathbb{P}^n_{\mathbb{C}}$ via ψ as a subset of $\mathbb{R}^N$. It is bounded since it is contained in the unit sphere $S^{N-1} \subset \mathbb{R}^N$. We claim that $\psi(\mathbb{P}^n_{\mathbb{C}})$ is also $\widetilde{\mathbb{Q}}$-semi-algebraic. The composition of the projection

$$\begin{aligned} \pi : \mathbb{R}^{2(n+1)} \setminus \{(0, \ldots, 0)\} &\longrightarrow \mathbb{P}^n_{\mathbb{C}} \\ (a_0, b_0, \ldots, a_n, b_n) &\mapsto [a_0 + ib_0 : \ldots : a_n + ib_n] \end{aligned}$$

with the map ψ is a polynomial map, hence $\widetilde{\mathbb{Q}}$-semi-algebraic by Example 2.6.3. Thus

$$\operatorname{Im} \psi \circ \pi = \operatorname{Im} \psi \subseteq \mathbb{R}^N$$

is $\widetilde{\mathbb{Q}}$-semi-algebraic by Fact 2.6.4.

Second step (zero set of a polynomial): We use the notation

$$V(g) := \{\underline{x} \in \mathbb{P}^n_{\mathbb{C}} \mid g(\underline{x}) = 0\} \quad \text{for} \quad g \in \mathbb{C}[x_0, \ldots, x_n] \text{ homogenous}, \quad \text{and}$$
$$W(h) := \{\underline{t} \in \mathbb{R}^N \mid h(\underline{t}) = 0\} \quad \text{for} \quad h \in \mathbb{C}[y_{00}, \ldots, z_{nn}].$$

Let $X^{\mathrm{an}} = V(g)$ for some homogenous $g \in \widetilde{\mathbb{Q}}[x_0, \ldots, x_n]$. Then $\psi(X^{\mathrm{an}}) \subseteq \mathbb{R}^N$ is a $\widetilde{\mathbb{Q}}$-semi-algebraic subset, as a little calculation shows. Setting for $k = 0, \ldots, n$

$$\begin{aligned} g_k &:= \text{“}g(\underline{x}\,\overline{x}_k)\text{”} \\ &= g(x_0\overline{x}_k, \ldots, x_n\overline{x}_k) \\ &= g\big((a_0a_k + b_0b_k) + i(b_0a_k - a_0b_k), \ldots, (a_na_k + b_nb_k) + i(b_na_k - a_nb_k)\big), \end{aligned}$$

where $x_j = a_j + ib_j$ for $j = 0, \ldots, n$, and

$$h_k := g(y_{0k} + iz_{0k}, \ldots, y_{nk} + iz_{nk}),$$

we obtain

$$\begin{aligned} \psi(X^{\mathrm{an}}) &= \psi(V(g)) \\ &= \bigcap_{k=0}^{n} \psi(V(g_k)) \\ &= \bigcap_{k=0}^{n} \psi(\mathbb{P}^n_{\mathbb{C}}) \cap W(h_k) \\ &= \bigcap_{k=0}^{n} \psi(\mathbb{P}^n_{\mathbb{C}}) \cap W(\operatorname{Re} h_k) \cap W(\operatorname{Im} h_k). \end{aligned}$$

Final step: We can choose an embedding

$$X \subseteq \mathbb{P}^n_{\widetilde{\mathbb{Q}}},$$

thus getting

$$X^{\mathrm{an}} \subseteq \mathbb{P}^n_{\mathbb{C}}.$$

Since X is a locally closed subvariety of $\mathbb{P}^n_{\widetilde{\mathbb{Q}}}$, the space X^{an} can be expressed in terms of subvarieties of the form $V(g)$ with $g \in \widetilde{\mathbb{Q}}[x_0, \ldots, x_n]$, using only the following basic operations

1. complementary set,

2. finite intersection,

3. finite union.

Now $\widetilde{\mathbb{Q}}$-semi-algebraic sets are stable under these operations as well and the first assertion is proved.

Second assertion: The first part of the lemma provides us with $\widetilde{\mathbb{Q}}$-semi-algebraic inclusions

$$\psi : X^{\mathrm{an}} \subseteq \underset{\underline{x}=[x_0:\ldots:x_n]}{\mathbb{P}^n_{\mathbb{C}}} \subseteq \underset{(y_{00},z_{00},\ldots,y_{nn},z_{nn})}{\mathbb{R}^N},$$
$$\phi : Y^{\mathrm{an}} \subseteq \underset{\underline{u}=[u_0:\ldots:u_m]}{\mathbb{P}^m_{\mathbb{C}}} \subseteq \underset{(v_{00},w_{00},\ldots,v_{mm},w_{mm})}{\mathbb{R}^M},$$

and a choice of coordinates as indicated. We use the notation

$$\begin{aligned} V(g) &:= \{(\underline{x},\underline{u}) \in \mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^m_{\mathbb{C}} \mid g(\underline{x},\underline{u}) = 0\}, \\ &\qquad \text{for } g \in \mathbb{C}[x_0,\ldots,x_n,u_0,\ldots,u_m] \text{ homogenous in both } \underline{x} \text{ and } \underline{u}, \quad \text{and} \\ W(h) &:= \{\underline{t} \in \mathbb{R}^{N+M} \mid h(\underline{t}) = 0\}, \quad \text{for } h \in \mathbb{C}[y_{00},\ldots,z_{nn},v_{00},\ldots,w_{mm}]. \end{aligned}$$

Let $\{U_i\}$ be a finite open affine covering of X such that $f(U_i)$ satisfies

- $f(U_i)$ does not meet the hyperplane $\{u_j = 0\} \subset \mathbb{P}^m_{\widetilde{\mathbb{Q}}}$ for some j, and
- $f(U_i)$ is contained in an open affine subset V_i of Y.

This is always possible, since we can start with the open covering $Y \cap \{u_j \neq 0\}$ of Y, take a subordinated open affine covering $\{V_{i'}\}$, and then choose a finite open affine covering $\{U_i\}$ subordinated to $\{f^{-1}(V_{i'})\}$. Now each of the maps

$$f_i := f^{\mathrm{an}}{}_{|U_i} : U_i^{\mathrm{an}} \longrightarrow Y^{\mathrm{an}}$$

has image contained in V_i^{an} and does not meet the hyperplane $\{\underline{u} \in \mathbb{P}^m_{\mathbb{C}} \mid u_j = 0\}$ for an appropriate j

$$f_i : U_i^{\mathrm{an}} \longrightarrow V_i^{\mathrm{an}}.$$

Being associated to an algebraic map between affine varieties, this map is rational

$$f_i : \underline{x} \mapsto \left[\frac{g_0'(\underline{x})}{g_0''(\underline{x})} : \ldots : \frac{g_{j-1}'(\underline{x})}{g_{j-1}''(\underline{x})} : \underset{j}{1} : \frac{g_{j+1}'(\underline{x})}{g_{j+1}''(\underline{x})} : \ldots : \frac{g_m'(\underline{x})}{g_m''(\underline{x})}\right],$$

with $g_k', g_k'' \in \widetilde{\mathbb{Q}}[x_0,\ldots,x_n]$, $k = 0,\ldots,\widehat{j},\ldots,m$. Since the graph $\Gamma_{f^{\mathrm{an}}}$ of f^{an} is the finite union of the graphs Γ_{f_i} of the f_i, it is sufficient to prove that $(\psi \times \phi)(\Gamma_{f_i})$ is a $\widetilde{\mathbb{Q}}$-semi-algebraic subset of $\mathbb{R}^{N+M}$. Now

$$\Gamma_{f_i} = (U_i^{\mathrm{an}} \times V_i^{\mathrm{an}}) \cap \bigcap_{\substack{k=0\\k\neq j}}^{n} V\left(\frac{y_k}{y_j} - \frac{g_k'(\underline{x})}{g_k''(\underline{x})}\right) = (U_i^{\mathrm{an}} \times V_i^{\mathrm{an}}) \cap \bigcap_{\substack{k=0\\k\neq j}}^{n} V(y_k g_k''(\underline{x}) - y_j g_k'(\underline{x})),$$

so all we have to deal with is

$$V(y_k g_k''(\underline{x}) - y_j g_k'(\underline{x})).$$

Again a little calculation is necessary. Setting

$$\begin{aligned}
g_{pq} &:= \text{“}u_k\overline{u}_q g_k''(\underline{x}\,\overline{x}_p) - u_j\overline{u}_q g_k'(\underline{x}\,\overline{x}_p)\text{”}\\
&= u_k\overline{u}_q g_k''(x_0\overline{x}_p,\dots,x_n\overline{x}_p) - u_j\overline{u}_q g_k'(x_0\overline{x}_p,\dots,x_n\overline{x}_p)\\
&= \big((c_kc_q+d_kd_q)+i(d_kc_q-c_kd_q)\big)\\
&\qquad g_k''\big((a_0a_p+b_0b_p)+i(b_0a_p-a_0b_p),\dots,(a_na_p+b_nb_p)+i(b_na_p-a_nb_p)\big)\\
&\quad -\big((c_jc_q+d_jd_q)+i(d_jc_q-c_jd_q)\big)\\
&\qquad g_k'\big((a_0a_p+b_0b_p)+i(b_0a_p-a_0b_p),\dots,(a_na_p+b_nb_p)+i(b_na_p-a_nb_p)\big),
\end{aligned}$$

where $x_l = a_l + ib_l$ for $l = 0,\dots,n$, $u_l = c_l + id_l$ for $l = 0,\dots,m$, and

$$h_{pq} := (v_{kq}+iw_{kq})g_k''(y_{0p}+iz_{0p},\dots,y_{np}+iz_{np})-(v_{jq}+iw_{jq})g_k'(y_{0p}+iz_{0p},\dots,y_{np}+iz_{np}),$$

we obtain

$$\begin{aligned}
(\psi\times\phi)\Big(V\big(y_kg_k''(\underline{x})-y_jg_k'(\underline{x})\big)\Big) &=\\
&=\bigcap_{p=0}^{n}\bigcap_{q=0}^{m}(\psi\times\phi)(V(g_{pq}))\\
&=\bigcap_{p=0}^{n}\bigcap_{q=0}^{m}(\psi\times\phi)(U_i^{\mathrm{an}}\times V_j^{\mathrm{an}})\cap W(h_{pq})\\
&=\bigcap_{p=0}^{n}\bigcap_{q=0}^{m}(\psi\times\phi)(U_i^{\mathrm{an}}\times V_j^{\mathrm{an}})\cap W(\operatorname{Re}h_{pq})\cap W(\operatorname{Im}h_{pq}).
\end{aligned}$$

□

2.6.2 Semi-algebraic singular chains

We need further prerequisites in order to state the promised Proposition 2.6.8.

Definition 2.6.7 ([Hi2, p. 168]). By an *open simplex* $\triangle^\circ$ we mean the interior of a simplex (= the convex hull of $r+1$ points in $\mathbb{R}^n$ which span an r-dimensional subspace). For convenience, a point is considered as an open simplex as well.

The notation $\triangle_d^{\mathrm{std}}$ will be reserved for the *closed standard simplex* spanned by the standard basis

$$\{e_i = (0,\dots,0,\underset{i}{1},0,\dots,0)\mid i=1,\dots,d+1\}$$

of $\mathbb{R}^{d+1}$.

Consider the following data $(*)$:

- X a variety defined over $\widetilde{\mathbb{Q}}$,

- D a divisor in X with normal crossings,
- and finally $\gamma \in H_p^{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q}),\ p \in \mathbb{N}_0$.

As before, we have denoted by X^{an} (resp. D^{an}) the complex analytic space associated to the base change $X_{\mathbb{C}} = X \times_{\widetilde{\mathbb{Q}}} \mathbb{C}$ (resp. $D_{\mathbb{C}} = D \times_{\widetilde{\mathbb{Q}}} \mathbb{C}$).

By Lemma 2.6.5, we may consider both X^{an} and D^{an} as bounded $\widetilde{\mathbb{Q}}$-semi-algebraic subsets of $\mathbb{R}^N$.

We are now able to formulate our proposition.

Proposition 2.6.8. *With data $(*)$ as above, we can find a representative of γ that is a rational linear combination of singular simplices each of which is $\widetilde{\mathbb{Q}}$-semi-algebraic.*

The proof of this proposition relies on the following proposition due to Lojasiewicz which has been written down by Hironaka.

Proposition 2.6.9 ([Hi2, p. 170])**.** *For $\{X_i\}$ a finite system of bounded $\widetilde{\mathbb{Q}}$-semi-algebraic sets in $\mathbb{R}^n$, there exists a simplicial decomposition*

$$\mathbb{R}^n = \coprod_j \overset{\circ}{\triangle}_j$$

by open simplices $\overset{\circ}{\triangle}_j$ and a $\widetilde{\mathbb{Q}}$-semi-algebraic automorphism

$$\kappa : \mathbb{R}^n \to \mathbb{R}^n$$

such that each X_i is a finite union of some of the $\kappa(\overset{\circ}{\triangle}_j)$.

Note 2.6.10. Although Hironaka considers $\mathbb{R}$-semi-algebraic sets, we can safely replace $\mathbb{R}$ by $\widetilde{\mathbb{Q}}$ in this result (including the fact he cites from [Sb]). The only problem that could possibly arise concerns a "good direction lemma":

Lemma 2.6.11 (Good direction lemma for $\mathbb{R}$, [Hi2, p. 172], [KB, Thm. 5.I, p. 242])**.** *Let Z be a $\mathbb{R}$-semi-algebraic subset of $\mathbb{R}^n$, which is nowhere dense. A direction $v \in \mathbb{P}_{\mathbb{R}}^{n-1}(\mathbb{R})$ is called* good*, if any line l in $\mathbb{R}^n$ parallel to v meets Z in a discrete (maybe empty) set of points; otherwise v is called* bad*. Then the set $B(Z)$ of bad directions is a Baire category set in $\mathbb{P}_{\mathbb{R}}^{n-1}(\mathbb{R})$.*

This gives immediately good directions $v \in \mathbb{P}_{\mathbb{R}}^{n-1}(\mathbb{R}) \setminus B(Z)$, but not necessarily $v \in \mathbb{P}_{\widetilde{\mathbb{Q}}}^{n-1}(\widetilde{\mathbb{Q}}) \setminus B(Z)$. However, in Remark 2.1 of [Hi2], which follows directly after the lemma, the following statement is made: If Z is compact, then $B(Z)$ is closed in $\mathbb{P}_{\mathbb{R}}^{n-1}(\mathbb{R})$. In particular $\mathbb{P}_{\widetilde{\mathbb{Q}}}^{n-1}(\widetilde{\mathbb{Q}}) \setminus B(Z)$ will be non-empty. Since we only consider *bounded* $\widetilde{\mathbb{Q}}$-semi-algebraic sets Z', we may take $Z := \overline{Z'}$ (which is compact by Heine-Borel), and thus find a good direction $v \in \mathbb{P}_{\widetilde{\mathbb{Q}}}^{n-1}(\widetilde{\mathbb{Q}}) \setminus B(Z')$ using $B(Z') \subseteq B(Z)$. Hence:

Lemma 2.6.12 (Good direction lemma for $\widetilde{\mathbb{Q}}$)**.** *Let Z' be a bounded $\widetilde{\mathbb{Q}}$-semi-algebraic subset of $\mathbb{R}^n$, which is nowhere dense. Then the set $\mathbb{P}^{n-1}_{\mathbb{R}}(\mathbb{R}) \setminus B(Z)$ of good directions is non-empty.*

Proof of Proposition 2.6.8. Applying Proposition 2.6.9 to the two-element system of $\widetilde{\mathbb{Q}}$-semi-algebraic sets $X^{\text{an}}, D^{\text{an}} \subseteq \mathbb{R}^N$, we obtain a $\widetilde{\mathbb{Q}}$-semi-algebraic decomposition

$$\mathbb{R}^N = \coprod_j \triangle^\circ_j$$

of $\mathbb{R}^N$ by open simplices $\triangle^\circ_j$ and a $\widetilde{\mathbb{Q}}$-semi-algebraic automorphism

$$\kappa : \mathbb{R}^N \to \mathbb{R}^N.$$

We write $\triangle_j$ for the closure of $\triangle^\circ_j$. The sets

$$K := \{\triangle^\circ_j \,|\, \kappa(\triangle^\circ_j) \subseteq X^{\text{an}}\} \quad \text{and} \quad L := \{\triangle^\circ_j \,|\, \kappa(\triangle^\circ_j) \subseteq D^{\text{an}}\}$$

can be thought of as finite simplicial complexes, but built out of open simplices instead of closed ones. We define their *geometric realizations*

$$|K| := \bigcup_{\triangle^\circ_j \in K} \triangle^\circ_j \quad \text{and} \quad |L| := \bigcup_{\triangle^\circ_j \in L} \triangle^\circ_j.$$

Then Proposition 2.6.9 states that κ maps the pair of topological spaces $(|K|, |L|)$ homeomorphically to $(X^{\text{an}}, D^{\text{an}})$.

Easy case: If X is complete, so is $X_{\mathbb{C}}$ (by [Ha2, Cor. II.4.8(c), p. 102]), hence X^{an} and D^{an} will be compact [Ha2, B.1, p. 439]. In this situation,

$$\overline{K} := \{\triangle_j \,|\, \kappa(\triangle_j) \subseteq X^{\text{an}}\} \quad \text{and} \quad \overline{L} := \{\triangle_j \,|\, \kappa(\triangle_j) \subseteq D^{\text{an}}\}$$

are (ordinary) simplicial complexes, whose geometric realizations coincide with those of K and L, respectively. In particular

$$\begin{aligned} H^{\text{simpl}}_\bullet(\overline{K}, \overline{L}; \mathbb{Q}) &\cong H^{\text{sing}}_\bullet(\left|\overline{K}\right|, \left|\overline{L}\right|; \mathbb{Q}) \\ &\cong H^{\text{sing}}_\bullet(|K|, |L|; \mathbb{Q}) \\ &\cong H^{\text{sing}}_\bullet(X^{\text{an}}, D^{\text{an}}; \mathbb{Q}). \end{aligned} \tag{2.4}$$

Here $H^{\text{simpl}}_\bullet(\overline{K}, \overline{L}; \mathbb{Q})$ denotes simplicial homology of course.

We write $\gamma_{\text{simpl}} \in H^{\text{simpl}}_p(\overline{K}, \overline{L}; \mathbb{Q})$ and $\gamma_{\text{sing}} \in H^{\text{sing}}_p(\left|\overline{K}\right|, \left|\overline{L}\right|; \mathbb{Q})$ for the image of γ under this isomorphism. Any representative Γ_{simpl} of γ_{simpl} is a rational linear combination

$$\Gamma_{\text{simpl}} = \textstyle\sum_j a_j \triangle_j, \quad a_j \in \mathbb{Q}$$

of oriented closed simplices $\triangle_j \in \overline{K}$. We can choose orientation-preserving affine-linear maps of the standard simplex $\triangle^{\text{std}}_p$ to $\triangle_j$

$$\sigma_j : \triangle^{\text{std}}_p \longrightarrow \triangle_j \quad \text{for} \quad \triangle_j \in \Gamma_{\text{simpl}}.$$

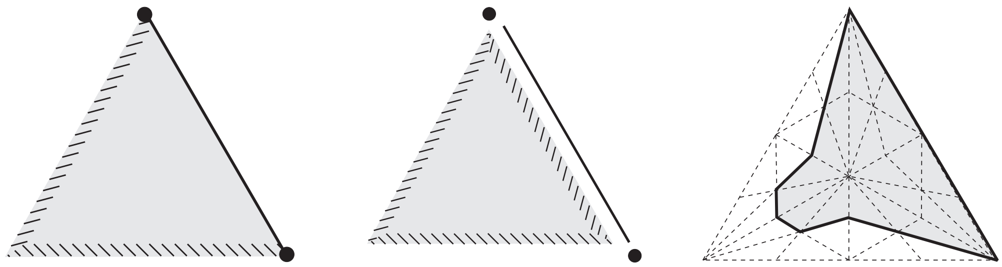
These maps yield a representative

$$\Gamma_{\text{sing}} := \textstyle\sum_j a_j \, \sigma_j$$

of γ_{sing}. Composing with κ yields $\Gamma := \kappa_* \Gamma_{\text{sing}} \in \gamma$, where Γ has the desired properties.

In the *general case*, we perform a barycentric subdivision $\mathcal{B}$ on K twice (once is not enough) and define $|K|$ and $|L|$ not as the "closure" of K and L, but as follows (see Figure 2.1)

$$\begin{aligned} \overline{K} &:= \{\triangle \mid \triangle^\circ \in \mathcal{B}^2(K) \text{ and } \triangle \subseteq |K|\}, \\ \overline{L} &:= \{\triangle \mid \triangle^\circ \in \mathcal{B}^2(K) \text{ and } \triangle \subseteq |L|\}. \end{aligned} \tag{2.5}$$



$\kappa^{-1}(X^{\text{an}}) \cap \triangle_j$ — Intersection of $\kappa^{-1}(X^{\text{an}})$ with a closed 2-simplex $\triangle_j$, where we assume that part of the boundary $\partial\triangle_j$ does not belong to $\kappa^{-1}(X^{\text{an}})$

$K \cap \triangle_j$ — Open simplices of K contained in $\triangle_j$

$\left|\overline{K}\right| \cap \triangle_j$ — Intersection of $\left|\overline{K}\right|$ with $\triangle_j$ (the dashed lines show the barycentric subdivision)

Figure 2.1: Definition of $\overline{K}$

The point is that the pair of topological spaces $(\left|\overline{K}\right|, \left|\overline{L}\right|)$ is a strong deformation retract of $(|K|, |L|)$. Assuming this, we see that in the general case with $\overline{K}$, $\overline{L}$ defined as in (2.5), the isomorphism (2.4) still holds and we can proceed as in the easy case to prove the proposition.

We define the retraction map

$$\rho : (|K| \times [0,1], |L| \times [0,1]) \to (\left|\overline{K}\right|, \left|\overline{L}\right|)$$

as follows: Let $\triangle_j^\circ \in K$ be an open simplex which is not contained in the boundary of any other simplex of K and set

$$\textit{inner} := \triangle_j \cap \overline{K}, \qquad \textit{outer} := \triangle_j \setminus \overline{K}.$$

Note that *inner* is closed. For any point $p \in \textit{outer}$ the ray $\overrightarrow{cp}$ from the center c of $\triangle_j^\circ$ through p "leaves" the set *inner* at a point q_p, i.e. $\overrightarrow{cp} \cap \textit{inner}$ equals

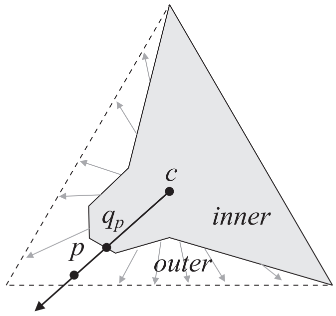


Figure 2.2: Definition of q_p

the line segment $c\,q_p$; see Figure 2.2. The map

$$\rho_j : \triangle_j \times [0,1] \to \triangle_j$$
$$(p,t) \mapsto \begin{cases} p & \text{if } p \in inner, \\ q_p + t\cdot(p - q_p) & \text{if} \quad p \in outer \end{cases}$$

retracts $\triangle_j$ onto *inner*.

Now these maps ρ_j glue together to give the desired homotopy ρ. □

We want to state one of the intermediate results of this proof explicitly:

Corollary 2.6.13. *Let X and D be as above. Then the pair of topological spaces $(X^{\mathrm{an}}, D^{\mathrm{an}})$ is homotopy equivalent to a pair of (realizations of) simplicial complexes $(|X^{\mathrm{simpl}}|, |D^{\mathrm{simpl}}|)$.*

2.7 Singular cohomology via the h′-topology

In order to give a simple description of the period isomorphism for singular varieties, we are going to need a more sophisticated description of singular cohomology.

We work in the category of complex analytic spaces An.

Definition 2.7.1. Let X be a complex analytic space. The h′-*topology* on the category $(\mathrm{An}/X)_{\mathrm{h}'}$ of complex analytic spaces over X is the smallest Grothendieck topology such that the following are covering maps:

1. proper surjective morphisms;
2. open covers.

If $\mathcal{F}$ is a presheaf of An/X we denote $\mathcal{F}_{\mathrm{h}'}$ its sheafification in the h′-topology.

Remark 2.7.2. This definition is inspired by Voevodsky's h-topology on the category of schemes, see Section 3.2. We are not sure if it is the correct analogue in the analytic setting. However, it is good enough for our purposes.

Lemma 2.7.3. *For $Y \in \mathrm{An}$ let $\mathbb{C}_Y$ be the (ordinary) sheaf associated to the presheaf $\mathbb{C}$. Then*

$$Y \mapsto \mathbb{C}_Y(Y)$$

is an h'-sheaf on An.

Proof. We have to check the sheaf condition for the generators of the topology. By assumption it is satisfied for open covers. Let $\tilde{Y} \to Y$ be proper surjective. Without loss of generality Y is connected. Let $\tilde{Y}_i$ for $i \in I$ be the collection of connected components of $\tilde{Y}$. Then

$$\tilde{Y} \times_Y \tilde{Y} = \bigcup_{i,j \in I} \tilde{Y}_i \times_Y \tilde{Y}_j$$

We have to compute the kernel of

$$\prod_{i \in I} \mathbb{C} \to \prod_{i,j} \mathbb{C}(\tilde{Y}_i \times_Y \tilde{Y}_j)$$

via the difference of the two natural restriction maps. Comparing a_i and a_j in $\mathbb{C}(\tilde{Y}_i \times_Y \tilde{Y}_j)$ we see that they agree. Hence the kernel is just one copy of $\mathbb{C} = \mathbb{C}_Y(Y)$. □

Proposition 2.7.4. *Let X be an analytic space and $i : Z \subset X$ a closed subspace. Then there is a morphism of sites $\rho : (\mathrm{An}/X)_{\mathrm{h}'} \to X$. It induces an isomorphism*

$$H^i_{\mathrm{sing}}(X, Z; \mathbb{C}) \to H^i_{\mathrm{h}'}((\mathrm{An}/X)_{\mathrm{h}'}, \mathrm{Ker}(\mathbb{C}_{\mathrm{h}'} \to i_*\mathbb{C}_{\mathrm{h}'}))$$

compatible with long exact sequences and products.

Remark 2.7.5. This statement and the following proof can be extended to more general sheaves $\mathcal{F}$.

The argument uses the notion of a hypercover, see Definition 1.5.8.

Proof. We first treat the absolute case with $Z = \emptyset$. We use the theory of cohomological descent as developed in [SGA4Vbis]. Singular cohomology satisfies cohomological descent for open covers and also for proper surjective maps (see Theorem 2.7.6). (The main ingredient for the second case is the proper base change theorem.) Hence it satisfies cohomological descent for h'-covers. This implies that singular cohomology can be computed as a direct limit

$$\lim_{\mathfrak{X}_\bullet} \mathbb{C}(\mathfrak{X}_\bullet),$$

where $\mathfrak{X}_\bullet$ runs through all h$'$-hypercovers. On the other hand, the same limit computes h$'$-cohomology, see Proposition 1.6.9 For the general case, recall that we have a short exact sequence

$$0 \to j_!\mathbb{C} \to \mathbb{C} \to i_*\mathbb{C} \to 0$$

of sheaves on X. Its pull-back to An/X maps naturally to the short exact sequence

$$0 \to \mathrm{Ker}(\mathbb{C}_{\mathrm{h}'} \to i_*\mathbb{C}_{\mathrm{h}'})) \to \mathbb{C}_{\mathrm{h}'} \to i_*\mathbb{C}_{\mathrm{h}'} \to 0\ .$$

This reduces the comparison in the relative case to the absolute case once we have shown that $Ri_*\mathbb{C}_{\mathrm{h}'} = i_*\mathbb{C}_{\mathrm{h}'}$. The sheaf $R^n i_*\mathbb{C}_{\mathrm{h}'}$ is given by the h$'$-sheafification of the presheaf

$$X' \mapsto H^n_{\mathrm{h}'}(Z \times_X X', \mathbb{C}_{\mathrm{h}'}) = H^n_{\mathrm{sing}}(Z \times_X X', \mathbb{C})$$

for $X' \to X$ in An/X. By resolution of singularities for analytic spaces we may assume that X' is smooth and $Z' = X' \times_X Z$ a divisor with normal crossings. By passing to an open cover, we may assume that Z' an open ball in a union of coordinate hyperplanes, in particular contractible. Hence its singular cohomology is trivial. This implies that $R^n i_*\mathbb{C}_{\mathrm{h}'} = 0$ for $n \geq 1$. □

Theorem 2.7.6 (Descent for proper hypercoverings)**.** *Let $D \subset X$ be a closed subvariety and $D_\bullet \to D$ a proper hypercover(see Definition 1.5.8), such that there is a commutative diagram*

$$\begin{tikzcd} D_\bullet \arrow[r] \arrow[d] & X_\bullet \arrow[d] \\ D \arrow[r] & X \end{tikzcd}$$

Then one has cohomological descent for singular cohomology:

$$H^*(X, D; \mathbb{Z}) = H^*\left(\mathrm{Cone}(\mathrm{Tot}(X_\bullet) \to \mathrm{Tot}(D_\bullet))[-1]; \mathbb{Z}\right).$$

Here, $\mathrm{Tot}(-)$ *denotes the total complex in* $\mathbb{Z}[\mathrm{Var}]$ *associated to the corresponding simplicial variety, see Definition 1.5.11.*

Proof. The relative case follows from the absolute case. The essential ingredient is proper base change, which allows to reduce to the case where X is a point. The statement then becomes a completely combinatorial assertion on contractibility of simplicial sets. The results are summed up in [D3] (5.3.5). For a complete reference see [SGA4Vbis], in particular Corollaire 4.1.6. □