Periods and Nori Motives

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Chapter 6: Nori's diagram category

Chapter 6

Nori's diagram category

We explain Nori's construction of an abelian category attached to the representation of a diagram and establish some properties for it. The construction is completely formal. It mimicks the standard construction of the Tannakian dual of a rigid tensor category with a fibre functor. Only, we do not have a tensor product or even a category but only what we should think of as the fibre functor.

The results are due to Nori. Notes from some of his talks are available [N, N1]. There is a also a sketch in Levine's survey [Le] §5.3. In the proofs of the main results we follow closely the diploma thesis of von Wangenheim in [vW].

6.1 Main results

6.1.1 Diagrams and representations

Let R be a noetherian, commutative ring with unit.

Definition 6.1.1. A *diagram* D is a directed graph on a set of vertices $V(D)$ and edges $E(D)$. A *diagram* with *identities* is a diagram with a choice of a distinguished edge $\mathrm{id}_v : v \to v$ for every $v \in D$. A diagram is called *finite* if it has only finitely many vertices. A *finite full subdiagram* of a diagram D is a diagram containing a finite subset of vertices of D and all edges (in D) between them.

By abuse of notation we often write $v \in D$ instead of $v \in V(D)$. The set of all directed edges between $p, q \in D$ is denoted by $D(p, q)$.

Remark 6.1.2. One may view a diagram as a category where composition of morphisms is not defined. The notion of a diagram with identity edges is not standard. The notion is useful later when we consider multiplicative structures.

Example 6.1.3. Let $\mathcal{C}$ be a small category. Then we can associate a diagram $D(\mathcal{C})$ with vertices the set of objects in $\mathcal{C}$ and edges given by morphisms. It is even a diagram with identities. By abuse of notation we usually also write $\mathcal{C}$ for the diagram.

Definition 6.1.4. A *representation* T of a diagram D in a small category $\mathcal{C}$ is a map T of directed graphs from D to $D(\mathcal{C})$. A *representation* T of a diagram D with identities is a representation such that id is mapped to id.

For $p, q \in D$ and every edge m from p to q we denote their images simply by Tp, Tq and $Tm : Tp \to Tq$ (mostly without brackets).

Remark 6.1.5. Alternatively, a representation is defined as a functor from the *path category* $\mathcal{P}(D)$ to $\mathcal{C}$. Recall that the objects of the path category are the vertices of D, and the morphisms are sequences of directed edges $e_1e_2 \dots e_n$ for $n \geq 0$ with the edge e_i starting in the end point of e_{i-1} for $i = 2, \dots, n$. Morphisms are composed by concatenating edges.

We are particularly interested in representations in categories of modules.

Definition 6.1.6. Let R be a noetherian commutative ring with unit. By $R-$Mod we denote the category of finitely generated R-modules. By $R-$Proj we denote the subcategory of finitely generated projective R-modules.

Note that these categories are essentially small by passing to isomorphic objects, so we will not worry about smallness from now on.

Definition 6.1.7. Let S be a commutative unital R-algebra and $T : D \to R-$Mod a representation. We denote T_S the representation

$$D \xrightarrow{T} R\text{−Mod} \xrightarrow{\otimes_R S} S\text{−Mod} .$$

Definition 6.1.8. Let T be a representation of D in $R-$Mod. We define the *ring of endomorphisms* of T by

$$\mathrm{End}(T) := \left\{ (e_p)_{p\in D} \in \prod_{p\in D} \mathrm{End}_R(Tp) | e_q \circ Tm = Tm \circ e_p \ \forall p, q \in D \ \forall m \in D(p, q) \right\} .$$

Remark 6.1.9. In other words, an element of $\mathrm{End}(T)$ consists of tuples $(e_p)_{p\in V(D)}$ of endomorphisms of Tp, such that all diagrams of the following form commute:

$$\begin{tikzcd} Tp \arrow[r, "Tm"] \arrow[d, "e_p"] & Tq \arrow[d, "e_q"] \\ Tp \arrow[r, "Tm"] & Tq \end{tikzcd}$$

Note that the ring of endomorphisms does not change when we replace D by the path category $\mathcal{P}(D)$.

6.1.2 Explicit construction of the diagram category

The diagram category can be characterized by a universal property, but it also has a simple explicit description that we give first.

Definition 6.1.10 (Nori)**.** Let R be a noetherian commutative ring with unit. Let T be a representation of D in $R-$Mod.

1. Assume D is finite. Then we put
$$\mathcal{C}(D,T) = \mathrm{End}(T)-\mathrm{Mod}$$
the category of finitely generated R-modules equipped with an R-linear operation of the algebra $\mathrm{End}(T)$.

2. In general let
$$\mathcal{C}(D,T) = 2-\mathrm{colim}_F \mathcal{C}(F,T|_F)$$
where F runs through the system of finite subdiagrams of D.

   More explicitly: the objects of $\mathcal{C}(D,T)$ are the objects of $\mathcal{C}(F,T|_F)$ for some finite subdiagram F. For $X \in \mathcal{C}(F,T|_F)$ and $F \subset F'$ we write $X_{F'}$ for the image of X in $\mathcal{C}(F',T|_{F'})$. For objects $X,Y \in \mathcal{C}(D,T)$, we put
$$\mathrm{Mor}_{\mathcal{C}(D,T)}(X,Y) = \varinjlim_F \mathrm{Mor}_{\mathcal{C}(F,T|_F)}(X_F,Y_F)\ .$$

The category $\mathcal{C}(D,T)$ is called the *diagram category.* With
$$f_T : \mathcal{C}(D,T) \longrightarrow R-\mathrm{Mod}$$
we denote the forgetful functor.

Remark 6.1.11. The representation $T : D \longrightarrow \mathcal{C}(D,T)$ extends to a functor on the path category $\mathcal{P}(D)$. By construction the diagram categories $\mathcal{C}(D,T)$ and $\mathcal{C}(\mathcal{P}(D),T)$ agree. The point of view of the path category will be useful Chapter 7, in particular in Definition 7.2.1.

In section 6.5 we will prove that under additional conditions for R, satisfied in the cases of most interest, there is the following even more direct description of $\mathcal{C}(D,T)$ as comodules over a coalgebra.

Theorem 6.1.12. *If the representation T takes values in free modules over a field or Dedekind domain R, the diagram category is equivalent to the category of finitely generated comodules (see Definition 6.5.4) over the coalgebra $A(D,T)$ where*
$$A(D,T) = \mathrm{colim}_F A(F,T) = \mathrm{colim}_F \mathrm{End}(T|_F)^\vee$$
with F running through the system of all finite subdiagrams of D and $^\vee$ the R-dual.

The proof of this theorem is given in Section 6.5.

6.1.3 Universal property: Statement

Theorem 6.1.13 (Nori)**.** *Let D be a diagram and*

$$T : D \longrightarrow R\text{−Mod}$$

a representation of D.

Then there exists an R-linear abelian category $\mathcal{C}(D,T)$, together with a representation

$$\tilde{T} : D \longrightarrow \mathcal{C}(D,T),$$

and a faithful, exact, R-linear functor f_T, such that:

1. *T factorizes over $D \xrightarrow{\tilde{T}} \mathcal{C}(D,T) \xrightarrow{f_T} R$−Mod.*
2. *$\tilde{T}$ satisfies the following universal property: Given*
   - *(a) another R-linear, abelian category $\mathcal{A}$,*
   - *(b) an R-linear, faithful, exact functor, $f : \mathcal{A} \to R$−Mod,*
   - *(c) another representation $F : D \to \mathcal{A}$,*

   such that $f \circ F = T$, then there exists a functor $L(F)$ - unique up to unique isomorphism of functors - such that the following diagram commutes:

$$\begin{tikzcd}
& \mathcal{C}(D,T) \arrow[dd, dashed, "L(F)"] \arrow[dr, "f_T"] & \\
D \arrow[ur, "\tilde{T}"] \arrow[rr, "T"] \arrow[dr, "F"'] & & R\text{−Mod}\ . \\
& \mathcal{A} \arrow[ur, "f"'] &
\end{tikzcd}$$

The category $\mathcal{C}(D,T)$ together with $\tilde{T}$ and f_T is uniquely determined by this property up to unique equivalence of categories. It is explicitly described by the diagram category of Definition 6.1.10. It is functorial in D in the obvious sense.

The proof will be given in Section 6.4. We are going to view f_T as an extension of T from D to $\mathcal{C}(D,T)$ and sometimes write simply T instead of f_T.

The universal property generalizes easily.

Corollary 6.1.14. *Let D, R, T be as in Theorem 6.1.19. Let $\mathcal{A}$ and f, F be as in loc.cit. 2. (a)-(c). Moreover, let S be a faithfully flat commutative unitary R-algebra S and*

$$\phi : T_S \to (f \circ F)_S$$

an isomorphism of representations into $S{-}\mathrm{Mod}$. Then there exists a functor $L(F) : \mathcal{C}(D,T) \to \mathcal{A}$ and an isomorphism of functors

$$\tilde{\phi} : (f_T)_S \to f_S \circ L(F)$$

such that

$$\begin{tikzcd}
 & \mathcal{C}(D,T) \arrow[dd, dashed, "L(F)"] \arrow[dr, "(f_T)_S"] & \\
D \arrow[ur, "\tilde{T}"] \arrow[rr, "T_S"] \arrow[dr, "F"'] & & S{-}\mathrm{Mod} \\
 & \mathcal{A} \arrow[ur, "f_S"'] &
\end{tikzcd}$$

commutes up to ϕ and $\tilde{\phi}$. The pair $(L(F), \tilde{\phi})$ is unique up to unique isomorphism of functors.

The proof will also be given in Section 6.4.

The following properties provide a better understanding of the nature of the category $\mathcal{C}(D,T)$.

Proposition 6.1.15. 1. *As an abelian category $\mathcal{C}(D,T)$ is generated by the $\tilde{T}v$ where v runs through the set of vertices of D, i.e., it agrees with its smallest full subcategory containing all such $\tilde{T}v$.*

2. *Each object of $\mathcal{C}(D,T)$ is a subquotient of a finite direct sum of objects of the form $\tilde{T}v$.*

3. *If $\alpha : v \to v'$ is an edge in D such that $T\alpha$ is an isomorphism, then $\tilde{T}\alpha$ is also an isomorphism.*

Proof. Let $\mathcal{C}' \subset \mathcal{C}(D,T)$ be the subcategory generated by all $\tilde{T}v$. By definition, the representation $\tilde{T}$ factors through $\mathcal{C}'$. By the universal property of $\mathcal{C}(D,T)$, we obtain a functor $\mathcal{C}(D,T) \to \mathcal{C}'$, hence an equivalence of subcategories of $R{-}\mathrm{Mod}$.

The second statement follows from the first criterion since the full subcategory in $\mathcal{C}(D,T)$ of subquotients of finite direct sums is abelian, hence agrees with $\mathcal{C}(D,T)$. The assertion on morphisms follows since the functor $f_T : \mathcal{C}(D,T) \to R-\mathrm{Mod}$ is faithful and exact between abelian categories. Kernel and cokernel of $\tilde{T}\alpha$ vanish if kernel and cokernel of $T\alpha$ vanish. □

Remark 6.1.16. We will later give a direct proof, see Proposition 6.3.20. It will be used in the proof of the universal property.

The diagram category only weakly depends on T.

Corollary 6.1.17. *Let D be a diagram and $T, T' : D \to R-\mathrm{Mod}$ two representations. Let S be a faithfully flat R-algebra and $\phi : T_S \to T'_S$ be an isomorphism of representations in $S-\mathrm{Mod}$. Then it induces an equivalence of categories*

$$\Phi : \mathcal{C}(D,T) \to \mathcal{C}(D,T').$$

Proof. We apply the universal property of Corollary 6.1.14 to the representation T and the abelian category $\mathcal{A} = \mathcal{C}(D,T')$. This yields a functor $\Phi : \mathcal{C}(D,T) \to \mathcal{C}(D,T')$. By interchanging the role of T and T' we also get a functor Φ' in the opposite direction. We claim that they are inverse to each other. The composition $\Phi' \circ \Phi$ can be seen as the universal functor for the representation of D in the abelian category $\mathcal{C}(D,T)$ via T. By the uniqueness part of the universal property, it is the identity. □

Corollary 6.1.18. *Let D be a diagram. Let $T : D \to R-\mathrm{Mod}$ be a representation. Let*

$$D \xrightarrow{\tilde{T}} \mathcal{C}(D,T) \xrightarrow{f_T} R-\mathrm{Mod}$$

be the factorization via the diagram category.

Let $D' \subset D$ be a full subdiagram. It has a representation in $R-\mathrm{Mod}$ by restricting T. Let

$$D' \xrightarrow{T|_{D'}} \mathcal{C}(D',T) \xrightarrow{f_{T|_{D'}}} R-\mathrm{Mod}$$

be the factorization via the diagram category. Let $\iota : \mathcal{C}(D',T) \to \mathcal{C}(D,T)$ be the functor induced from the inclusion of diagrams. Moreover, we assume that there is a representation $T' : D \to \mathcal{C}(D',T)$ compatible with T, i.e., such that

$$T = f_T \circ \iota \circ T' = f_{T|_{D'}} \circ T' \ .$$

Then ι is an equivalence of categories.

Proof. By the universal property of the diagram category, the representation T' induces a functor

$$\pi : \mathcal{C}(D,T) \to \mathcal{C}(D',T) \ .$$

We claim that $\iota \circ \pi$ is the identity functor. As ι is already known to be faithful, this will imply that it is an equivalence.

Let $\mathcal{A} = \mathcal{C}(D,T)$. The representation of D in $\mathcal{A}$ via $f_T \circ \iota \circ T'$ equals T. By functoriality of the universal map, the functor $\iota\pi : \mathcal{C}(D,T) \to \mathcal{A}$ fits into the diagram of the universal property of $\mathcal{C}(D,T)$. By uniqueness, it is equal to the identity. $\square$

The most important ingredient for the proof of the universal property is the following special case.

Theorem 6.1.19. *Let R be a noetherian ring and $\mathcal{A}$ an abelian, R-linear category. Let*

$$T : \mathcal{A} \longrightarrow R\text{-Mod}$$

be a faithful, exact, R-linear functor and

$$\mathcal{A} \xrightarrow{\tilde{T}} \mathcal{C}(\mathcal{A},T) \xrightarrow{f_T} R\text{-Mod}$$

the factorization via its diagram category (see Definition 6.1.10). Then $\tilde{T}$ is an equivalence of categories.

The proof of this theorem will be given in Section 6.3.

6.1.4 Discussion of the Tannakian case

The above may be viewed as a generalization of Tannaka duality. We explain this in more detail. We are not going to use the considerations in the sequel.

Let k be a field, $\mathcal{C}$ a k-linear abelian tensor category, and

$$T : \mathcal{C} \longrightarrow k\text{-Vect}$$

a k-linear faithful tensor functor, all in the sense of [DM]. By standard Tannakian formalism (cf [Sa] and [DM]), there is a k-bialgebra A such that the category is equivalent to the category of A-comodules on finite dimensional k-vector spaces.

On the other hand, if we regard $\mathcal{C}$ as a diagram (with identities) and T as a representation into finite dimensional vector spaces, we can view the diagram category of $\mathcal{C}$ as the category $A(\mathcal{C},T)-$Comod by Theorem 6.1.12. By Theorem 6.1.19 the category $\mathcal{C}$ is equivalent to its diagram category $A(\mathcal{C},T)-$Comod. The construction of the two coalgebras A and $A(\mathcal{C},T)$ coincides. Thus Nori implicitely shows that we can recover the coalgebra structure of A just by looking at the representations of $\mathcal{C}$.

The *algebra* structure on $A(\mathcal{C},T)$ is induced from the tensor product on $\mathcal{C}$ (see also Section 7.1). This defines a pro-algebraic scheme Spec$A(\mathcal{C},T)$. The *coalgebra* structure turns Spec$A(\mathcal{C},T)$ into a monoid scheme. We may interpret $A(\mathcal{C},T)-$Comod as the category of finite-dimensional representations of this monoid scheme.

If the tensor structure is rigid in addition, $\mathcal{C}(D,T)$ becomes what Deligne and Milne call a *neutral Tannakian category* [DM]. The rigidity structure induces an antipodal map, making $A(\mathcal{C},T)$ into a Hopf algebra. This yields the structure of a *group scheme* on $\mathrm{Spec}A(\mathcal{C},T)$, rather than only a monoid scheme.

We record the outcome of the discussion:

Theorem 6.1.20. *Let R be a field and $\mathcal{C}$ be a neutral R-linear Tannakian category with faithful exact fibre functor $T:\mathcal{C}\to R-\mathrm{Mod}$. Then $A(\mathcal{C},T)$ is equal to the Hopf algebra of the Tannakian dual.*

Proof. By construction, see [DM] Theorem 2.11 and its proof. □

A similar result holds in the case that R is a Dedekind domain and

$$T : D \longrightarrow R-\mathrm{Proj}$$

a representation into finitely generated projective R-modules. Again by Theorem 6.1.12, the diagram category $\mathcal{C}(D,T)$ equals $A(\mathcal{C},T)-\mathrm{Comod}$, where $A(\mathcal{C},T)$ is projective over R. Wedhorn shows in [Wed] that if $\mathrm{Spec}A(\mathcal{C},T)$ is a group scheme it is the same to have a representation of $\mathrm{Spec}A(\mathcal{C},T)$ on a finitely generated R-module M and to endow M with an $A(\mathcal{C},T)$-comodule structure.

6.2 First properties of the diagram category

Let R be a unitary commutative noetherian ring, D a diagram and $T:D\to R-\mathrm{Mod}$ a representation. We investigate the category $\mathcal{C}(D,T)$ introduced in Definition 6.1.10.

Lemma 6.2.1. *If D is a finite diagram, then* $\mathrm{End}(T)$ *is an R-algebra which is finitely generated as an R-module.*

Proof. For any $p\in D$ the module Tp is finitely generated. Since R is noetherian, the algebra $\mathrm{End}_R(Tp)$ then is finitely generated as R-module. Thus $\mathrm{End}(T)$ becomes a unitary subalgebra of $\prod_{p\in Ob(D)}\mathrm{End}_R(Tp)$. Since $V(D)$ is finite and R is noetherian,

$$\mathrm{End}(T)\subset\prod_{p\in Ob(D)}\mathrm{End}_R(Tp)$$

is finitely generated as R-module. □

Lemma 6.2.2. *Let D be a finite diagram and $T:D\to R-\mathrm{Mod}$ a representation. Then:*

1. *Let S be a flat R-algebra. Then:*

$$\mathrm{End}_S(T_S)=\mathrm{End}_R(T)\otimes S$$

2. *Let $F : D' \to D$ be morphism of diagrams and $T' = T \circ F$ the induced representation. Then F induces a canonical R-algebra homomorphism*

$$F^* : \mathrm{End}(T) \to \mathrm{End}(T')\ .$$

Proof. The algebra $\mathrm{End}(T)$ is defined as a limit, i.e., a kernel

$$0 \to \mathrm{End}(T) \to \prod_{p \in V(D)} \mathrm{End}_R(Tp) \xrightarrow{\phi} \prod_{m \in D(p,q)} \mathrm{Hom}_R(Tp, Tq)$$

with $\phi(p)(m) = e_q \circ Tm - Tm \circ e_p$. As S is flat over R, this remains exact after tensoring with S. As the R-module Tp is finitely presented and S flat, we have

$$\mathrm{End}_R(Tp) \otimes S = \mathrm{End}_S(T_S p)\ .$$

Hence we get

$$0 \to \mathrm{End}(T|_F) \otimes S \to \prod_{p \in V(D)} \mathrm{End}_S(T_S(p)) \xrightarrow{\phi} \prod_{m \in D(p,q)} \mathrm{Hom}_S(T_S(p), T_S(q))\ .$$

This is the defining sequence for $\mathrm{End}(T_S)$.

The morphism of diagrams $F : D' \to D$ induces a homomorphism

$$\prod_{p \in V(D)} \mathrm{End}_R(Tp) \to \prod_{p' \in V(D')} \mathrm{End}_R(T'p'),$$

by mapping $e = (e_p)_p$ to $F^*(e)$ with $(F^*(e))_{p'} = e_{f(p')}$ in $\mathrm{End}_R(T'p') = \mathrm{End}_R(Tf(p'))$. It is compatible with the induced homomorphism

$$\prod_{m \in D(p,q)} \mathrm{Hom}_R(Tp, Tq) \to \prod_{m' \in D'(p',q')} \mathrm{Hom}_R(T'p', T'q').$$

Hence it induces a homomorphism on the kernels. □

Proposition 6.2.3. *Let R be unitary commutative noetherian ring, D a finite diagram and $T : D \longrightarrow R-\mathrm{Mod}$ be a representation. For any $p \in D$ the object Tp is a natural left $\mathrm{End}(T)$-module. This induces a representation*

$$\tilde{T} : D \longrightarrow \mathrm{End}(T)-\mathrm{Mod},$$

such that T factorises via

$$D \xrightarrow{\tilde{T}} \mathcal{C}(D, T) \xrightarrow{f_T} R-\mathrm{Mod}.$$

Proof. For all $p \in D$ the projection

$$pr : \mathrm{End}(T) \to \mathrm{End}_R(Tp)$$

induces a well-defined action of $\mathrm{End}(T)$ on Tp making Tp into a left $\mathrm{End}(T)$-module. To check that $\tilde{T}$ is a representation of left $\mathrm{End}(T)$-modules, we need $Tm \in \mathrm{Hom}_R(Tp, Tq)$ to be $\mathrm{End}(T)$-linear for all $p, q \in D, m \in D(p,q)$. This corresponds directly to the commutativity of the diagram in Remark 6.1.9. □

Now let D be general. We study the system of finite subdiagrams $F \subset D$. Recall that subdiagrams are full, i.e., they have the same edges.

Corollary 6.2.4. *The finite subdiagrams of D induce a directed system of abelian categories $\big(\mathcal{C}(D, T|_F)\big)_{F\subset D finite}$ with exact, faithful R-linear functors as transition maps.*

Proof. The transition functors are induced from the inclusion via Lemma 6.2.2. $\square$

Recall that we have defined $\mathcal{C}(D,T)$ as 2-colimit of this system, see Definition 6.1.10.

Proposition 6.2.5. *The 2-colimit $\mathcal{C}(D,T)$ exists. It provides an R-linear abelian category such that the composition of the induced representation with the forgetful functor*

$$\begin{array}{ccccc} D & \xrightarrow{\tilde{T}} & \mathcal{C}(D,T) & \xrightarrow{f_T} & R\text{−Mod} \\ p & \mapsto & Tp & \mapsto & Tp. \end{array}$$

yields a factorization of T. The functor f_T is R-linear, faithful and exact.

Proof. It is a straightforward calculation that the limit category inherits all structures of an R-linear abelian category. It has well-defined (co)products and (co)kernels because the transition functors are exact. It has a well-defined R-linear structure as all transition functors are R-linear. Finally, one shows that every kernel resp. cokernel is a monomorphism resp. epimorphism using the fact that all transition functors are faithful and exact.

So for every $p \in D$ the R-module Tp becomes an $\mathrm{End}(T|_F)$-module for all finite $F \subset D$ with $p \in F$. Thus it represents an object in $\mathcal{C}(D,T)$. This induces a representation

$$\begin{array}{ccc} D & \xrightarrow{\tilde{T}} & \mathcal{C}(D,T) \\ p & \mapsto & Tp. \end{array}$$

The forgetful functor is exact, faithful and R-linear. Composition with the forgetful functor f_T obviously yields the initial diagram T. $\square$

We now consider functoriality in D.

Lemma 6.2.6. *Let D_1, D_2 be diagrams and $G : D_1 \to D_2$ a map of diagrams. Let further $T : D_2 \to R$−Mod be a representation and*

$$D_2 \xrightarrow{\tilde{T}} \mathcal{C}(D_2,T) \xrightarrow{f_T} R\text{−Mod}$$

the factorization of T through the diagram category $\mathcal{C}(D_2,T)$ as constructed in Proposition 6.2.5. Let

$$D_1 \xrightarrow{\widetilde{T\circ G}} \mathcal{C}(D_1, T\circ G) \xrightarrow{f_{T\circ G}} R\text{−Mod}$$

be the factorization of $T \circ G$.

Then there exists a faithful R-linear, exact functor $\mathcal{G}$, such that the following diagram commutes.

$$\begin{tikzcd}
D_1 \arrow[rr, "G"] \arrow[d, "\widetilde{T \circ G}"] & & D_2 \arrow[d, "\tilde{T}"] \\
\mathcal{C}(D_1, T \circ G) \arrow[rr, dashed, "\mathcal{G}"] \arrow[dr, "f_{T \circ G}"'] & & \mathcal{C}(D_2, T) \arrow[dl, "f_T"] \\
& R\text{−Mod} &
\end{tikzcd}$$

Proof. Let D_1, D_2 be finite diagrams first. Let $T_1 = T \circ G|_{D_1}$ and $T_2 = T|_{D_2}$. The homomorphism

$$G^* : \mathrm{End}(T_2) \to \mathrm{End}(T_1)$$

of Lemma 6.2.2 induces by restriction of scalars a functor on diagram categories as required.

Let now D_1 be finite and D_2 arbitrary. Let E_2 be finite full subdiagram of D_2 containing $G(D_1)$. We apply the finite case to $G : D_1 \to E_2$ and obtain a functor

$$\mathcal{C}(D_1, T_1) \to \mathcal{C}(E_2, T_2)$$

which we compose with the canonical functor $\mathcal{C}(E_2, T_2) \to \mathcal{C}(D_2, T_2)$. By functoriality, it is independent of the choice of E_2.

Let now D_1 and D_2 be arbitrary. For every finite subdiagram $E_1 \subset E_1$ we have constructed

$$\mathcal{C}(E_1, T_1) \to \mathcal{C}(D_2, T_2) \ .$$

They are compatible and hence define a functor on the limit. □

Isomorphic representations have equivalent diagram categories. More precisely:

Lemma 6.2.7. *Let $T_1, T_2 : D \to R$−Mod be representations and $\phi : T_1 \to T_2$ an isomorphism of representations. Then ϕ induces an equivalence of categories $\Phi : \mathcal{C}(D, T_1) \to \mathcal{C}(D, T_2)$ together with an isomorphism of representations*

$$\tilde{\phi} : \Phi \circ \tilde{T}_1 \to \tilde{T}_2$$

such that $f_{T_2} \circ \tilde{\phi} = \phi$.

Proof. We only sketch the argument which is analogous to the proof of Lemma 6.2.6.

It suffices to consider the case $D = F$ finite. The functor

$$\Phi : \mathrm{End}(T_1){-}\mathrm{Mod} \to \mathrm{End}(T_2){-}\mathrm{Mod}$$

is the extension of scalars for the R-algebra isomorphism $\mathrm{End}(T_1) \to \mathrm{End}(T_2)$ induced by conjugating by ϕ. $\square$

6.3 The diagram category of an abelian category

In this section we give the proof of Theorem 6.1.19: the diagram category of the diagram category of an abelian category with respect to a representation given by an exact faithful functor is the abelian category itself.

We fix a commutative noetherian ring R with unit and an R-linear abelian category $\mathcal{A}$. By R-algebra we mean a unital R-algebra, not necessarily commutative.

6.3.1 A calculus of tensors

We start with some general constructions of functors. We fix a unital R-algebra E, finitely generated as R-module, not necessarily commutative. The most important case is $E = R$, but this is not enough for our application.

In the simpler case where R is a field, the constructions in this sections can already be found in [DMOS].

Definition 6.3.1. Let E be an R-algebra which is finitely generated as R-module. We denote $E{-}\mathrm{Mod}$ the category of finitely generated left E-modules.

The algebra E and the objects of $E{-}\mathrm{Mod}$ are noetherian because R is.

Definition 6.3.2. Let $\mathcal{A}$ be an R-linear abelian category and p be an object of $\mathcal{A}$. Let E be a not necessarily commutative R algebra and

$$E^{op} \xrightarrow{f} \mathrm{End}_{\mathcal{A}}(p)$$

be a morphism of R-algebras. We say that p is a *right E-module in $\mathcal{A}$*.

Example 6.3.3. Let $\mathcal{A}$ be the category of left R'-modules for some R-algebra R'. Then a right E-module in $\mathcal{A}$ is the same thing as an (R', E)-bimodule, i.e., a left R'-module with the structure of a right E-module.

Lemma 6.3.4. *Let $\mathcal{A}$ be an R-linear abelian category and p be an object of $\mathcal{A}$. Let E be a not necessarily commutative R-algebra and p a right E-module in $\mathcal{A}$. Then*

$$\mathrm{Hom}_{\mathcal{A}}(p, \_) : \mathcal{A} \to R{-}\mathrm{Mod}$$

can naturally be viewed as a functor to $E{-}\mathrm{Mod}$.

Proof. For every $q \in \mathcal{A}$, the algebra E operates on $\mathrm{Hom}_{\mathcal{A}}(p, q)$ via functoriality. □

Proposition 6.3.5. *Let $\mathcal{A}$ be an R-linear abelian category and p be an object of $\mathcal{A}$. Let E be a not necessarily commutative R algebra and p a right E-module in $\mathcal{A}$. Then the functor*

$$\mathrm{Hom}_{\mathcal{A}}(p, \_) : \mathcal{A} \longrightarrow E\mathrm{-Mod}$$

has an R-linear left adjoint

$$p \otimes_E \_ : E\mathrm{-Mod} \longrightarrow \mathcal{A}.$$

It is right exact. It satisfies

$$p \otimes_E E = p,$$

and on endomorphisms of E we have (using $\mathrm{End}_E(E) \cong E^{op}$)

$$p \otimes_E \_ : \quad \begin{array}{ccc} \mathrm{End}_E(E) & \longrightarrow & \mathrm{End}_{\mathcal{A}}(p) \\ a & \longmapsto & f(a). \end{array}$$

Proof. Right exactness of $p \otimes_E \_$ follows from the universal property. For every $M \in E\mathrm{-Mod}$, the value of $p \otimes_E M$ is uniquely determined by the universal property. In the case of $M = E$, it is satisfied by p itself because we have for all $q \in \mathcal{A}$

$$\mathrm{Hom}_{\mathcal{A}}(p, q) = \mathrm{Hom}_E(E, \mathrm{Hom}_{\mathcal{A}}(p, q)).$$

This identification also implies the formula on endomorphisms of $M = E$.

By compatibility with direct sums, this implies that $p \otimes_E E^n = \bigoplus_{i=1}^n p$ for free E-modules. For the same reason, morphisms $E^m \xrightarrow{(a_{ij})_{ij}} E^n$ between free E-modules must be mapped to $\bigoplus_{i=1}^m p \xrightarrow{f(a_{ij})_{ij}} \bigoplus_{i=1}^n p$.

Let M be a finitely presented left E-module. We fix a finite presentation

$$E^{m_1} \xrightarrow{(a_{ij})_{ij}} E^{m_0} \overset{\pi_a}{\twoheadrightarrow} M \to 0.$$

Since $p \otimes_E \_$ preserves cokernels (if it exists), we need to define

$$p \otimes_E M := \mathrm{Coker}(p^{m_1} \xrightarrow{\tilde{A} := f(a_{ij})_{ij}} p^{m_0}).$$

We check that it satisfies the universal property. Indeed, for all $q \in \mathcal{A}$, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{A}}(p \otimes E^{m_1}, q) \arrow[d, "\cong"] & \mathrm{Hom}_{\mathcal{A}}(p \otimes E^{m_0}, q) \arrow[l] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathcal{A}}(p \otimes M, q) \arrow[l] \arrow[d, dashed] & 0 \arrow[l] \\
\mathrm{Hom}_E(E^{m_1}, \mathrm{Hom}_{\mathcal{A}}(p, q)) & \mathrm{Hom}_E(E^{m_0}, \mathrm{Hom}_{\mathcal{A}}(p, q)) \arrow[l] & \mathrm{Hom}_E(M, \mathrm{Hom}_{\mathcal{A}}(p, q)) \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

Hence the dashed arrow exists and is an isomorphism.

The universal property implies that $p \otimes_E M$ is independent of the choice of presentation and functorial. We can also make this explicit. For a morphism between arbitrary modules $\varphi : M \to N$ we choose lifts

$$\begin{tikzcd}
E^{m_1} \arrow[r, "A"] \arrow[d, "\varphi^1"] & E^{m_0} \arrow[r, "\pi_A"] \arrow[d, "\varphi^0"] & M \arrow[r] \arrow[d, "\varphi"] & 0 \\
E^{n_1} \arrow[r, "B"] & E^{n_0} \arrow[r, "\pi_B"] & N \arrow[r] & 0.
\end{tikzcd}$$

The respective diagram in $\mathcal{A}$,

$$\begin{tikzcd}
p^{m_1} \arrow[r, "\tilde{A}"] \arrow[d, "\tilde{\varphi^1}"] & p^{m_0} \arrow[r, "\pi_{\tilde{A}}"] \arrow[d, "\tilde{\varphi^0}"] & \mathrm{Coker}(\tilde{A}) \arrow[r] \arrow[d, dashed, "\exists!"] & 0 \\
p^{n_1} \arrow[r, "\tilde{B}"] & p^{n_0} \arrow[r, "\pi_{\tilde{B}}"] & \mathrm{Coker}(\tilde{B}) \arrow[r] & 0.
\end{tikzcd}$$

induces a unique morphism $p \otimes_E \text{\_}(\varphi) : p \otimes_E M \to p \otimes_E N$ that keeps the diagram commutative. It is independent of the choice of lifts as different lifts of projective resolutions are homotopic. This finishes the construction. $\square$

Corollary 6.3.6. *Let E be an R-algebra finitely generated as R-module and $\mathcal{A}$ an R-linear abelian category. Let*

$$T : \mathcal{A} \longrightarrow E\text{−Mod}$$

be an exact, R-linear functor into the category of finitely generated E-modules. Further, let p be a right E-module $\mathcal{A}$ with structure given by

$$E^{op} \xrightarrow{f} \mathrm{End}_{\mathcal{A}}(p)$$

a morphism of R-algebras. Then the composition

$$E^{op} \xrightarrow{f} \mathrm{End}_{\mathcal{A}}(p) \xrightarrow{T} \mathrm{End}_E(Tp).$$

induces a right action on Tp, making it into an E-bimodule. The composition

$$\begin{array}{ccccc} E\text{−Mod} & \xrightarrow{p \otimes_E \text{\_}} & \mathcal{A} & \xrightarrow{T} & E\text{−Mod} \\ M & \mapsto & p \otimes_E M & \mapsto & Tp \otimes_E M \end{array}$$

becomes the usual tensor functor of E-modules.

Proof. It is obvious that the composition

$$\begin{array}{ccccc} E-\mathrm{Mod} & \xrightarrow{p\otimes_E -} & \mathcal{A} & \xrightarrow{T} & E-\mathrm{Mod} \\ E^n & \mapsto & p\otimes_E E^n & \mapsto & Tp\otimes_E E^n \end{array}$$

induces the usual tensor functor

$$Tp\otimes_E \_ : E-\mathrm{Mod} \longrightarrow E-\mathrm{Mod}$$

on free E-modules. For arbitrary finitely generated E-modules this follows from the fact that $Tp\otimes_E \_$ is right exact and T is exact. □

Remark 6.3.7. Let E be an R-algebra, let M be a right E-module and N be a left E-module. We obtain the tensor product $M\otimes_E N$ by dividing out the equivalence relation $m\cdot e\otimes n \sim m\otimes e\cdot n$ for all $m\in M, n\in N, e\in E$ of the tensor product $M\otimes_R N$ of R-modules. We will now see that a similar approach holds for the abstract tensor products $p\otimes_R M$ and $p\otimes_E M$ in $\mathcal{A}$ as defined in Proposition 6.3.5. For the easier case that R is a field, this approach has been used in [DM].

Lemma 6.3.8. *Let $\mathcal{A}$ be an R-linear, abelian category, E a not necessarily commutative R-algebra which is finitely generated as R-module and $p\in\mathcal{A}$ a right E-module in $\mathcal{A}$. Let $M\in E-\mathrm{Mod}$ and $E'\in E-\mathrm{Mod}$ be in addition a right E-module. Then $p\otimes_E E'$ is a right E-module in $\mathcal{A}$ and we have*

$$p\otimes_E (E'\otimes_E M) = (p\otimes_E E')\otimes_E M.$$

Moreover,

$$(p\otimes_E E)\otimes_R M = p\otimes_R M.$$

Proof. The right E-module structure on $p\otimes_E E'$ is defined by functoriality. The equalities are immediate from the universal property. □

Proposition 6.3.9. *Let $\mathcal{A}$ be an R-linear, abelian category. Let further E be a unital R-algebra with finite generating family $e_1,\dots,e_m$. Let p a right E-module in $\mathcal{A}$ with structure given by*

$$E^{op} \xrightarrow{f} \mathrm{End}_{\mathcal{A}}(p).$$

Let M be a left E-module.

Then $p\otimes_E M$ is isomorphic to the cokernel of the map

$$\Sigma : \bigoplus_{i=1}^{m}(p\otimes_R M) \longrightarrow p\otimes_R M$$

given by

$$\sum_{i=1}^{m}(f(e_i)\otimes \mathrm{id}_M - \mathrm{id}_p\otimes e_i\mathrm{id}_M)\,\pi_i$$

with π_i the projection to the i-summand.

More suggestively (even if not quite correct), we write

$$\Sigma : (x_i \otimes v_i)_{i=1}^m \mapsto \sum_{i=1}^m (f(e_i)(x_i) \otimes v_i - x_i \otimes (e_i \cdot v_i))$$

for $x_i \in p$ and $v_i \in M$.

Proof. Consider the sequence

$$\bigoplus_{i=1}^m E \otimes_R E \longrightarrow E \otimes E \longrightarrow E \longrightarrow 0$$

where the first map is given by

$$(x_i \otimes y_i)_{i=1}^m \mapsto \sum_{i=1}^m x_i e_i \otimes y_i - x_i \otimes e_i y_i$$

and the second is multiplication. We claim that it is exact. The sequence is exact in E because E is unital. The composition of the two maps is zero, hence the cokernel maps to E. The elements in the cokernel satisfy the relation $\bar{x}e_i \otimes \bar{y} = \bar{x} \otimes e_i\bar{y}$ for all $\bar{x}, \bar{y}$ and $i = 1, \dots, m$. The e_i generate E, hence $\bar{x}e \otimes \bar{y} = \bar{x} \otimes e\bar{y}$ for all $\bar{x}, \bar{y}$ and all $e \in E$. Hence the cokernel equals $E \otimes_E E$ which is E via the multiplication map.

Now we tensor the sequence from the left by p and from the right by M and obtain an exact sequence

$$\bigoplus_{i=1}^m p \otimes_E (E \otimes_R E) \otimes_E M \longrightarrow p \otimes_E (E \otimes_R E) \otimes_E M \longrightarrow p \otimes_E E \otimes_E M \to 0.$$

Applying the computation rules of Lemma 6.3.8, we get the sequence in the proposition. □

Similarly to Proposition 6.3.5 and Corollary 6.3.6, but less general, we construct a contravariant functor $\mathrm{Hom}_R(p, \_)$:

Proposition 6.3.10. *Let $\mathcal{A}$ be an R-linear abelian category, and p be an object of $\mathcal{A}$. Then the functor*

$$\mathrm{Hom}_{\mathcal{A}}(\_, p) : \mathcal{A}^\circ \longrightarrow R\mathrm{-Mod}$$

has a left adjoint

$$\mathrm{Hom}_R(\_, p) : R\mathrm{-Mod} \longrightarrow \mathcal{A}^\circ.$$

This means that for all $M \in R\mathrm{-Mod}$ and $q \in \mathcal{A}$, we have

$$\mathrm{Hom}_{\mathcal{A}}(q, \mathrm{Hom}_R(M, p)) = \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathcal{A}}(q, p)).$$

It is left exact. It satisfies

$$\mathrm{Hom}_R(R, p) = p.$$

If

$$T : \mathcal{A} \longrightarrow R\text{–Mod}$$

is an exact, R-linear functor into the category of finitely generated R-modules then the composition

$$\begin{array}{ccccc} R\text{–Mod} & \xrightarrow{\mathrm{Hom}(\_,p)} & \mathcal{A} & \xrightarrow{T} & R\text{–Mod} \\ M & \mapsto & \mathrm{Hom}_R(M,p) & \mapsto & \mathrm{Hom}_R(M,Tp) \end{array}$$

is the usual $\mathrm{Hom}(\_, Tp)$*-functor in* R–Mod.

Proof. The arguments are the same as in the proof of Proposition 6.3.5 and Corollary 6.3.6. □

Remark 6.3.11. Let $\mathcal{A}$ be an R-linear, abelian category. The functors $\mathrm{Hom}_R(\_, p)$ as defined in Proposition 6.3.10 and $p \otimes_R \_$ as defined in Proposition 6.3.6 are also functorial in p, i.e., we have even functors

$$\mathrm{Hom}_R(\_, \_) : (R\text{–Mod})^\circ \times \mathcal{A} \longrightarrow \mathcal{A}$$

and

$$\_ \otimes_R \_ : \mathcal{A} \times R\text{–Mod} \longrightarrow \mathcal{A}.$$

We will denote the image of a morphism $p \xrightarrow{\alpha} q$ under the functor $\mathrm{Hom}_R(M, \_)$ by

$$\mathrm{Hom}_R(M,p) \xrightarrow{\alpha\circ} \mathrm{Hom}_R(M,q)$$

This notation $\alpha\circ$ is natural since by composition

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{\mathrm{Hom}(M,\_)} & \mathcal{A} & \xrightarrow{T} & R\text{–Mod} \\ p & \mapsto & \mathrm{Hom}_R(M,p) & \mapsto & \mathrm{Hom}_R(M,Tp) \end{array}$$

$T(\alpha\circ)$ becomes the usual left action of $T\alpha$ on $\mathrm{Hom}_R(M, Tp)$.

Proof. This follows from the universal property. □

We will now check that the above functors have very similar properties to usual tensor and Hom-functors in R–Mod.

Lemma 6.3.12. *Let $\mathcal{A}$ be an R-linear, abelian category and M a finitely generated R-module. Then the functor* $\mathrm{Hom}_R(M, \_)$ *is right-adjoint to the functor* $\_ \otimes_R M$.

If

$$T : \mathcal{A} \longrightarrow R\text{–Mod}$$

is an R-linear, exact functor into finitely generated R-modules, the composed functors $T \circ \mathrm{Hom}_R(M, \_)$ *and* $T \circ (\_ \otimes_R M)$ *yield the usual hom-tensor adjunction in* R–Mod.

Proof. The assertion follows from the universal property and the identification $T \circ \mathrm{Hom}_R(M, \_) = \mathrm{Hom}_R(M, T\_)$ in Proposition 6.3.10 and $T \circ \_ \otimes_R M = (T\_) \otimes_R M$ in Proposition 6.3.6. □

6.3.2 Construction of the equivalence

Definition 6.3.13. Let $\mathcal{A}$ be an abelian category and $\mathcal{S}$ a not necessarily abelian subcategory. With $\langle \mathcal{S} \rangle$ we denote the smallest full abelian subcategory of $\mathcal{A}$ containing $\mathcal{S}$, i.e., the intersection of all full subcategories of $\mathcal{A}$ that are abelian, contain $\mathcal{S}$, and for which the inclusion functor is exact.

Lemma 6.3.14. *Let $\mathcal{A} = \langle F \rangle$ for a finite set of objects. Let $T : \langle F \rangle \to R\mathrm{-Mod}$ be a faithful exact functor. Then the inclusion $F \to \langle F \rangle$ induces an equivalence*

$$\mathrm{End}(T|_F)\mathrm{-Mod} \longrightarrow \mathcal{C}(\langle F \rangle, T).$$

Proof. Let $E = \mathrm{End}(T|_F)$. Its elements are tuples of endomorphisms of Tp for $p \in F$ commuting with all morphisms $p \to q$ in F.

We have to show that $E = \mathrm{End}(T)$. In other words, that any element of E defines a unique endomorphism of Tq for all objects q of $\langle F \rangle$ and commutes with all morphisms in $\langle F \rangle$.

Any object q is a subquotient of a finite direct sum of copies of objects $p \in F$. The operation of E on Tp with $p \in F$ extends uniquely to an operation on direct sums, kernels and cokernels of morphisms. It is also easy to see that the action commutes with Tf for all morphisms f between these objects. This means that it extends to all objects $\langle F \rangle$, compatible with all morphisms. □

We first concentrate on the case $\mathcal{A} = \langle p \rangle$. From now on, we abbreviate $\mathrm{End}(T|_{\{p\}})$ by $E(p)$.

Lemma 6.3.15. *Let $\langle p \rangle = \mathcal{A}$ be an abelian category. Let $\langle p \rangle \xrightarrow{T} R\mathrm{-Mod}$ a faithful exact R-linear functor into the category of finitely generated R-modules. Let*

$$\langle p \rangle \xrightarrow{\tilde{T}} E(p)\mathrm{-Mod} \xrightarrow{f_T} R\mathrm{-Mod}$$

be the factorization via the diagram category of T constructed in Proposition 6.2.5. Then:

1. *There exists an object $X(p) \in Ob(\langle p \rangle)$ such that*

$$\tilde{T}(X(p)) = E(p).$$

2. *The object $X(p)$ has a right $E(p)$-module structure in $\mathcal{A}$*

$$E(p)^{op} \to \mathrm{End}_{\mathcal{A}}(X(p))$$

such that the induced $E(p)$-module structure on $E(p)$ is the product.

3. *There is an isomorphism*

$$\tau : X(p) \otimes_{E(p)} \tilde{T}p \to p$$

which is natural in $f \in \mathrm{End}_{\mathcal{A}}(p)$*, i.e.,*

$$\begin{tikzcd} p \arrow[r, "f"] \arrow[d, "\tau"] & p \arrow[d, "\tau"] \\ X(p) \otimes_{E(p)} \tilde{T}p \arrow[r, "id \otimes \tilde{T}f"] & X(p) \otimes_{E(p)} \tilde{T}p \end{tikzcd}$$

An easier construction of $X(p)$ in the field case can be found in [DM], the construction for R being a noetherian ring is due to Nori [N].

Proof. We consider the object $\mathrm{Hom}_R(Tp, p) \in \mathcal{A}$. Via the contravariant functor

$$\begin{array}{ccc} R\text{−Mod} & \overset{\mathrm{Hom}(\_,p)}{\longrightarrow} & \mathcal{A} \\ Tp & \mapsto & \mathrm{Hom}_R(Tp, p) \end{array}$$

of Proposition 6.3.10 it is a right $\mathrm{End}_R(Tp)$-module in $\mathcal{A}$ which, after applying T just becomes the usual right $\mathrm{End}(Tp)$-module $\mathrm{Hom}_R(Tp, Tp)$. For each $\varphi \in \mathrm{End}(Tp)$, k we will write $\circ\varphi$ for the action on $\mathrm{Hom}(Tp, p)$ as well. By Lemma 6.3.12 the functors $\mathrm{Hom}_R(Tp, \_)$ and $\_ \otimes_R Tp$ are adjoint, so we obtain an evaluation map

$$\tilde{ev} : \mathrm{Hom}_R(Tp, p) \otimes_R Tp \longrightarrow p$$

that becomes the usual evaluation in R−Mod after applying T. Our aim is now to define $X(p)$ as a suitable subobject of $\mathrm{Hom}_R(Tp, p) \in \mathcal{A}$. The structures on $X(p)$ will be induced from the structures on $\mathrm{Hom}_R(Tp, p)$.

Let $M \in R$−Mod. We consider the functor

$$\begin{array}{ccc} \mathcal{A} & \overset{\mathrm{Hom}_R(M,\_)}{\longrightarrow} & \mathcal{A} \\ p & \mapsto & \mathrm{Hom}_R(M, p) \end{array}$$

of Remark 6.3.11. The endomorphism ring $\mathrm{End}_{\mathcal{A}}(p)) \subset \mathrm{End}_R(Tp)$ is finitely generated as R-module, since T is faithful and R is noetherian. Let $\alpha_1, ..., \alpha_n$ be a generating family. Since

$$E(p) = \{\varphi \in \mathrm{End}(Tp) | T\alpha \circ \varphi = \varphi \circ T\alpha \ \forall \alpha : p \to p\},$$

we can write $E(p)$ as the kernel of

$$\begin{array}{ccc} \mathrm{Hom}(Tp, Tp) & \longrightarrow & \bigoplus_{i=1}^n \mathrm{Hom}(Tp, Tp) \\ u & \mapsto & u \circ T\alpha_i - T\alpha_i \circ u \end{array}$$

By the exactness of T, the kernel $X(p)$ of

$$\begin{array}{ccc} \mathrm{Hom}(Tp,p) & \longrightarrow & \bigoplus_{i=1}^n \mathrm{Hom}(Tp,p) \\ u & \mapsto & u \circ T\alpha_i - \alpha_i \circ u \end{array}$$

is a preimage of $E(p)$ under T in $\mathcal{A}$.

By construction, the right $\mathrm{End}_R(Tp)$-module structure on $\mathrm{Hom}_R(Tp,p)$ restricts to a right $E(p)$-module structure on $X(p)$ whose image under $\tilde{T}$ yields the natural $E(p)$ right-module structure on $E(p)$.

Now consider the evaluation map

$$\tilde{ev} : \mathrm{Hom}_R(Tp,p) \otimes_R Tp \longrightarrow p$$

mentioned at the beginning of the proof. By Proposition 6.3.9 we know that the cokernel of the map Σ defined there is isomorphic to $X(p) \otimes_{E(p)} \tilde{T}p$. The diagram

$$\begin{tikzcd} \bigoplus_{i=1}^k (X(p) \otimes_R Tp) \arrow[r, "\Sigma"] & X(p) \otimes_R Tp \arrow[r, "\mathrm{incl} \otimes id"] \arrow[dr, "\mathrm{Coker}(\Sigma)"] & \mathrm{Hom}_R(Tp,p) \otimes_R Tp \arrow[r, "\tilde{\mathrm{ev}}"] & p \\ & & X(p) \otimes_{E(p)} \tilde{T}p & \end{tikzcd}$$

in $\mathcal{A}$ maps via T to the diagram

$$\begin{tikzcd} \bigoplus_{i=1}^k (E(p) \otimes_R Tp) \arrow[r, "\Sigma"] & E(p) \otimes_R Tp \arrow[r, "\mathrm{incl} \otimes id"] \arrow[dr, "\mathrm{Coker}(\Sigma)"] & \mathrm{Hom}_R(Tp,Tp) \otimes_R Tp \arrow[r, "\mathrm{ev}"] & Tp \\ & & E(p) \otimes_{E(p)} \tilde{T}p & \end{tikzcd}$$

in $R-$Mod, where the composition of the horizontal maps becomes zero. Since T is faithful, the respective horizontal maps in $\mathcal{A}$ are zero as well and induce a map

$$\tau : X(p) \otimes_{E(p)} Tp \longrightarrow p$$

that keeps the diagram commutative. By definition of Σ in Proposition 6.3.9, the respective map

$$\tilde{T}\tau : E(p) \otimes_{E(p)} \tilde{T}p \longrightarrow \tilde{T}p$$

becomes the natural evaluation isomorphism of E-modules. Since $\tilde{T}$ is faithful, τ is an isomorphism as well.

Naturality in f holds since $\tilde{T}$ is faithful and

$$\begin{array}{ccc} \tilde{T}p & \xrightarrow{\tilde{T}f} & \tilde{T}p \\ \downarrow{\scriptstyle \tilde{T}\tau} & & \downarrow{\scriptstyle \tilde{T}\tau} \\ E(p)\otimes_{E(p)}\tilde{T}p & \xrightarrow{id\otimes\tilde{T}f} & E(p)\otimes_{E(p)}\tilde{T}p \end{array}$$

commutes in $E(p)-$Mod. $\square$

Proposition 6.3.16. *Let $\langle p\rangle = \mathcal{A}$ be an R-linear, abelian category and*

$$\mathcal{A} \xrightarrow{T} R-\mathrm{Mod}$$

be as in Theorem 6.1.19. Let

$$\mathcal{A} \xrightarrow{\tilde{T}} \mathcal{C}(\mathcal{A},T) \xrightarrow{f_T} R-\mathrm{Mod}$$

be the factorization of T via its diagram category. Then $\tilde{T}$ is an equivalence of categories with inverse given by $X(p)\otimes_{E(p)}$ _ with $X(p)$ the object constructed in Lemma 6.3.15.

Proof. We have $\mathcal{A} = \langle p\rangle$, thus $\mathcal{C}(\mathcal{A},T) = E(p)-$Mod. Consider the object $X(p)$ of Lemma 6.3.15. It is a right $E(p)$-module in $\mathcal{A}$, in other words

$$\begin{array}{cccc} f: & \big(E(p)\big)^{op} & \longrightarrow & \mathrm{End}_{\mathcal{A}}(X(p)) \\ & \varphi & \longmapsto & \circ\varphi \end{array}$$

We apply Corollary 6.3.6 to $E = E(p)$, the object $X(p)$, the above f and the functor

$$\tilde{T} : \langle p\rangle \longrightarrow E(p)-\mathrm{Mod}\ .$$

It yields the functor

$$X(p)\otimes_{E(p)} \_ : E(p)-\mathrm{Mod} \longrightarrow \langle p\rangle$$

such that the composition

$$\begin{array}{ccccc} E(p)-\mathrm{Mod} & \xrightarrow{X(p)\otimes_{E(p)}\_} & \langle p\rangle & \xrightarrow{\tilde{T}} & E(p)-\mathrm{Mod} \\ M & \longmapsto & X(p)\otimes_{E(p)} M & \mapsto & \tilde{T}(X(p))\otimes_{E(p)} M = E(p)\otimes_{E(p)} M \end{array}$$

becomes the usual tensor product of $E(p)$-modules, and therefore yields the identity functor.

We want to check that $X(p) \otimes_{E(p)}$ _ is a left-inverse functor of $\tilde{T}$ as well. Thus we need to find a natural isomorphism τ, i.e., for all objects $p_1, p_2 \in \mathcal{A}$ we need isomorphisms τ_{p_1}, τ_{p_2} such that for morphisms $f : p_1 \to p_2$ the following diagram commutes:

$$\begin{tikzcd}
X(p) \otimes_{E(p)} \tilde{T}p_1 \arrow[r, "id \otimes \tilde{T}f"] \arrow[d, "\tau_{p_1}"] & X(p) \otimes_{E(p)} \tilde{T}p_2 \arrow[d, "\tau_{p_2}"] \\
p_1 \arrow[r, "f"] & p_2
\end{tikzcd}$$

Since the functors T and f_T are faithful and exact, and we have $T = f_t \circ \tilde{T}$, we know that $\tilde{T}$ is faithful and exact as well. We have already shown that $\tilde{T} \circ X(p) \otimes_{E(p)}$ _ is the identity functor. So $X(p) \otimes_{E(p)}$ _ is faithful exact as well. Since $\mathcal{A}$ is generated by p, it suffices to find a natural isomorphism for p and its endomorphisms. This is provided by the isomorphism τ of Lemma 6.3.15. $\square$

Proof of Theorem 6.1.19. If $\mathcal{A}$ is generated by one object p, then the functor $\tilde{T}$ is an equivalence of categories by Proposition 6.3.16. It remains to reduce to this case.

The diagram category $\mathcal{C}(\mathcal{A}, T)$ arises as a direct limit, hence we have

$$2\text{--}\mathrm{colim}_{F \subset Ob(\mathcal{A})} \mathrm{End}(T|_F)\text{--Mod}$$

and in the same way we have

$$\mathcal{A} = 2\text{--}\mathrm{colim}_{F \subset Ob(\mathcal{A})} \langle F \rangle$$

with F ranging over the system of full subcategories of $\mathcal{A}$ that contain only a finite number of objects. Moreover, by Lemma 6.3.14, we have $\mathrm{End}(T|_F) = \mathrm{End}(T|_{\langle F \rangle})$. Hence it suffices to check equivalence of categories

$$\langle F \rangle \xrightarrow{\tilde{T}|_{\langle F \rangle}} \mathrm{End}(T|_F)\text{--Mod}$$

for all abelian categories that are generated by a finite number of objects.

We now claim that $\langle F \rangle \cong \langle \bigoplus_{p \in F} p \rangle$ are equivalent: indeed, since any endomorphism of $\bigoplus_{p \in F} p$ is of the form $(a_{pq})_{p,q \in F}$ with $a_{pq} : p \to q$, and since F has all finite direct sums, we know that $\langle \bigoplus_{p \in F} p \rangle$ is a full subcategory of $\langle F \rangle$. On the other hand, for any $q, q' \in F$ the inclusion $q \hookrightarrow \bigoplus_{p \in F} p$ is a kernel and the projection $\bigoplus_{p \in F} p \twoheadrightarrow q'$ is a cokernel, so for any set of morphisms $(a_{qq'})_{q,q' \in F}$, the morphism $a_{qq'} : q \to q'$ by composition

$$q \hookrightarrow \bigoplus_{p \in F} \xrightarrow{(a_{pp'})_{p,p' \in F}} \bigoplus_{p' \in F} p' \twoheadrightarrow q'$$

is contained in $\langle\bigoplus_{p\in F} p\rangle$. Thus $\langle F\rangle$ is a full subcategory of $\langle\bigoplus_{p\in F} p\rangle$.

Similarly one sees that $\mathrm{End}(T|_{\{p\}})-\mathrm{Mod}$ is equivalent to $\mathrm{End}(T|_F)-\mathrm{Mod}$. So we can even assume that our abelian category is generated by just one object $q = \bigoplus_{p\in F} p$. □

6.3.3 Examples and applications

We work out a couple of explicit examples in order to demonstrate the strength of Theorem 6.1.19. We also use the arguments of the proof to deduce an additional property of the diagram property as a first step towards its universal property.

Throughout let R be a noetherian unital ring.

Example 6.3.17. Let $T : R-\mathrm{Mod} \to R-\mathrm{Mod}$ be the identity functor viewed as a representation. Note that $R-\mathrm{Mod}$ is generated by the object R^n. By Theorem 6.1.19 and Lemma 6.3.14, we have equivalences of categories

$$\mathrm{End}(T|_{\{R^n\}})-\mathrm{Mod} \longrightarrow \mathcal{C}(R-\mathrm{Mod}, T) \longrightarrow R-\mathrm{Mod}.$$

By definition, $E = \mathrm{End}(T|_{\{R^n\}})$ consists of those elements of $\mathrm{End}_R(R^n)$ which commute with all elements of $\mathrm{End}_{\mathcal{A}}(R^n)$, i.e., the center of the matrix algebra, which is R.

This can be made more interesting by playing with the representation.

Example 6.3.18 (Morita equivalence)**.** Let R be a noetherian commutative unital ring, $\mathcal{A} = R-\mathrm{Mod}$. Let P be a flat finitely generated R-module and

$$T : R-\mathrm{Mod} \longrightarrow R-\mathrm{Mod}, \quad M \mapsto M \otimes_R P.$$

It is faithful and exact, hence the assumptions of Theorem 6.1.19 are satisfied and we get an equivalence

$$\mathcal{C}(R-\mathrm{Mod}, T) \longrightarrow R-\mathrm{Mod}\ .$$

Note that $\mathcal{A} = \langle R\rangle$ and hence by Lemma 6.3.14, $\mathcal{C}(R-\mathrm{Mod}, T) = E-\mathrm{Mod}$ with $E = \mathrm{End}_R(T|_{\{R\}}) = \mathrm{End}_R(P)$. Hence we have shown that

$$\mathrm{End}_R(P)-\mathrm{Mod} \to R-\mathrm{Mod}$$

is an equivalence of categories. This is a case of Morita equivalence of categories of modules.

Example 6.3.19. Let R be a noetherian commutative unital ring and E an R-algebra finitely generated as an R-module. Let

$$T : E-\mathrm{Mod} \to R-\mathrm{Mod}$$

be the forgetful functor. The category $E-\mathrm{Mod}$ is generated by the module E. Hence by Theorem 6.1.19 and Lemma 6.3.14, we have again equivalences of categories

$$E'-\mathrm{Mod} \longrightarrow \mathcal{C}(E-\mathrm{Mod}, T) \longrightarrow E-\mathrm{Mod},$$

where $E' = \mathrm{End}(T|_{\{E\}})$ is the subalgebra of $\mathrm{End}_R(E)$ of endomorphisms compatible with all E-morphisms $E \to E$. Note that $\mathrm{End}_E(E) = E^{op}$ and hence E' is the centralizer of E^{op} in $\mathrm{End}_R(E)$

$$E' = C_{\mathrm{End}_R(E)}(E^{op}) = E \ .$$

Hence in this case the functor $\mathcal{A} \to \mathcal{C}(\mathcal{A}, T)$ is the identity.

We deduce another consequence of the explicit description of $\mathcal{C}(D, T)$.

Proposition 6.3.20. *Let D be a diagram and $T : D \to R-\mathrm{Mod}$ a representation. Let*

$$D \xrightarrow{\tilde{T}} \mathcal{C}(D, T) \xrightarrow{f_T} R-\mathrm{Mod}$$

its factorization. Then the category $\mathcal{C}(D, T)$ is generated by the image of $\tilde{T}$:

$$\mathcal{C}(D, T) = \langle \tilde{T}(D) \rangle \ .$$

Proof. It suffices to consider the case when D is finite. Let $X = \bigoplus_{p\in D} Tp$ and $\mathbb{E} = \mathrm{End}_R(X)$. Let $S \subset \mathbb{E}$ be the R-subalgebra generated by Te for $e \in E(D)$ and the projectors $p_p : X \to T(p)$. Then

$$E = \mathrm{End}(T) = C_{\mathbb{E}}(S)$$

is the commutator of S in $\mathbb{E}$. (The endomorphisms commuting with the projectors are those respecting the decomposition. By definition, $\mathrm{End}(T)$ consists of those endomorphisms of the summands commuting with all Te.)

By construction $\mathcal{C}(D, T) = E-\mathrm{Mod}$. We claim that it is equal to

$$\mathcal{A} := \langle \{\tilde{T}p | p \in D\} \rangle = \langle \tilde{X} \rangle$$

with $\tilde{X} = \bigoplus_{p\in D} \tilde{T}p$. The category has a faithful exact representation by $f_T|_{\mathcal{A}}$. Note that $f_T(\tilde{X}) = X$. By Theorem 6.1.19, the category $\mathcal{A}$ is equivalent to its diagram category $\mathcal{C}(\langle \tilde{X} \rangle, f_T) = E'-\mathrm{Mod}$ with $E' = \mathrm{End}(f_T|_{\mathcal{A}})$. By Lemma 6.3.14, E' consists of elements of $\mathbb{E}$ commuting with all elements of $\mathrm{End}_{\mathcal{A}}(\tilde{X})$. Note that

$$\mathrm{End}_{\mathcal{A}}(\tilde{X}) = \mathrm{End}_E(X) = C_{\mathbb{E}}(E)$$

and hence

$$E' = C_{\mathbb{E}}(C_{\mathbb{E}}(E)) = C_{\mathbb{E}}(C_{\mathbb{E}}(C_{\mathbb{E}}(S)) = C_{\mathbb{E}}(S)$$

because a triple commutator equals the simple commutator. We have shown $E = E'$ and the two categories are equivalent.

$\square$

Remark 6.3.21. This is a direct proof of Proposition 6.1.15.

6.4 Universal property of the diagram category

At the end of this section we will be able to establish the universal property of the diagram category.

Let $T : D \longrightarrow R-\mathrm{Mod}$ be a diagram and

$$D \xrightarrow{\tilde{T}} \mathcal{C}(D,T) \xrightarrow{f_T} R-\mathrm{Mod}$$

the factorization of T via its diagram category. Let $\mathcal{A}$ be another R-linear abelian category, $F : D \to \mathcal{A}$ a representation, and $f : \mathcal{A} \to R-\mathrm{Mod}$ a faithful, exact, R-linear functor into the categories of finitely generated R-modules such that $f \circ F = T$.

Our aim is to deduce that there exists - uniquely up to isomorphism - an R-linear exact faithful functor

$$L(F) : \mathcal{C}(D,T) \to \mathcal{A},$$

making the following diagram commute:

$$\begin{tikzcd}
& D \arrow[dl, "\tilde{T}"'] \arrow[dr, "F"] & \\
\mathcal{C}(D,T) \arrow[rr, dashed, "\exists! L(F)"] \arrow[dr, "f_T"'] & & \mathcal{A} \arrow[dl, "T_{\mathcal{A}}"] \\
& R-\mathrm{Mod} &
\end{tikzcd}$$

Proposition 6.4.1. *There is a functor $L(F)$ making the diagram commute.*

Proof. We can regard $\mathcal{A}$ as a diagram and obtain a representation

$$\mathcal{A} \xrightarrow{T_{\mathcal{A}}} R-\mathrm{Mod},$$

that factorizes via its diagram category

$$\mathcal{A} \xrightarrow{\tilde{T}_{\mathcal{A}}} \mathcal{C}(\mathcal{A}, T_{\mathcal{A}}) \xrightarrow{f_{T_{\mathcal{A}}}} R-\mathrm{Mod}.$$

We obtain the following commutative diagram

$$\begin{tikzcd}
D \arrow[rr, "F"] \arrow[d, "\tilde{T}_D"] \arrow[dr, "T"] & & \mathcal{A} \arrow[d, "\tilde{T_{\mathcal{A}}}"] \arrow[dl, "T_{\mathcal{A}}"'] \\
\mathcal{C}(D,T) \arrow[r, "f_T"'] & R\mathrm{-Mod} & \mathcal{C}(\mathcal{A},T_{\mathcal{A}}) \arrow[l, "f_{T_{\mathcal{A}}}"]
\end{tikzcd}$$

By functoriality of the diagram category (see Proposition 6.2.6) there exists an R-linear faithful exact functor $\mathcal{F}$ such that the following diagram commutes:

$$\begin{tikzcd}
D \arrow[rr, "F"] \arrow[d, "\tilde{T}_D"] & & \mathcal{A} \arrow[d, "\tilde{T_{\mathcal{A}}}"] \\
\mathcal{C}(D,T) \arrow[rr, dashed, "\mathcal{F}"] \arrow[dr, "f_T"'] & & \mathcal{C}(\mathcal{A},T_{\mathcal{A}}) \arrow[dl, "f_{T_{\mathcal{A}}}"] \\
& R\mathrm{-Mod} &
\end{tikzcd}$$

Since $\mathcal{A}$ is R-linear, abelian, and T is faithful, exact, R-linear, we know by Proposition 6.1.19, that $\tilde{T}_{\mathcal{A}}$ is an equivalence of categories. The functor

$$L(F) : \mathcal{C}(D,T) \to \mathcal{A},$$

is given by the composition of $\mathcal{F}$ with the inverse of $\tilde{T}_{\mathcal{A}}$. Since an equivalence of R-linear categories is exact, faithful and R-linear, $L(F)$ is so as well, as it is the composition of such functors. $\square$

Proposition 6.4.2. *The functor $L(F)$ is unique up to unique isomorphism.*

Proof. Let L' be another functor satisfying the condition in the diagram. Let $\mathcal{C}'$ be the subcategory of $\mathcal{C}(D,T)$ on which $L' = L(F)$. We claim that the inclusion is an equivalence of categories. Without loss of generality, we may assume D is finite.

Note that the subcategory is full because $f : \mathcal{A} \to R\mathrm{-Mod}$ is faithful. It contains all objects of the form $\tilde{T}p$ for $p \in D$. As the functors are additive, this implies that they also have to agree (up to canonical isomorphism) on finite direct sums of objects. As the functors are exact, they also have to agree on and all kernels and cokernels. Hence $\mathcal{C}'$ is the full abelian subcategory of $\mathcal{C}(D,T)$ generated by $\tilde{T}(D)$. By Proposition 6.3.20 this is all of $\mathcal{C}(D,T)$. $\square$

Proof of Theorem 6.1.13. Let $T : D \to R{-}\mathrm{Mod}$ be a representation and $f : \mathcal{A} \to R{-}\mathrm{Mod}$, $F : D \to \mathcal{A}$ as in the statement. By Proposition 6.4.1 the functor $L(F)$ exists. It is unique by Proposition 6.4.2. Hence $\mathcal{C}(D,T)$ satisfies the universal property of Theorem 6.1.13.

Let $\mathcal{C}$ be another category satisfying the universal property. By the universal property for $\mathcal{C}(D,T)$ and the representation of D in $\mathcal{C}$, we get a functor $\Psi : \mathcal{C}(D,T) \to \mathcal{C}$. By interchanging their roles, we obtain a functor Ψ' in the opposite direction. Their composition $\Psi' \circ \Psi$ satisfies the universal property for $\mathcal{C}(D,T)$ and the representation $\tilde{T}$. By the uniqueness part, it is isomorphic to the identity functor. The same argument also applies to $\Psi \circ \Psi'$. Hence Ψ is an equivalence of categories.

Functoriality of $\mathcal{C}(D,T)$ in D is Lemma 6.2.6. □

The generalized universal property follows by a trick.

Proof of Corollary 6.1.14. Let $T : D \to R{-}\mathrm{Mod}$, $f : \mathcal{A} \to R{-}\mathrm{Mod}$ und $F : D \to \mathcal{A}$ be as in the corollary. Let S be a faithfully flat R-algebra and

$$\phi : T_S \to (f \circ F)_S$$

an isomorphism of representations into $S{-}\mathrm{Mod}$. We first show the existence of $L(F)$.

Let $\mathcal{A}'$ be the category with objects of the form (V_1, V_2, ψ) where $V_1 \in R{-}\mathrm{Mod}$, $V_2 \in \mathcal{A}$ and $\psi : V_1 \otimes_R S \to f(V_2) \otimes_R S$ an isomorphism. Morphisms are defined as pairs of morphisms in $R{-}\mathrm{Mod}$ and $\mathcal{A}$ such the obvious diagram commutes. This category is abelian because S is flat over R. Kernels and cokernels are taken componentwise. Let $f' : \mathcal{A}' \to R{-}\mathrm{Mod}$ be the projection to the first component. It is faithful and exact because S is faithfully flat over R.

The data T, F and ϕ define a representation $F' : D \to \mathcal{A}'$ compatible with T. By the universal property of Theorem 6.1.13, we obtain a factorization

$$F' : D \xrightarrow{\tilde{T}} \mathcal{C}(D,T) \xrightarrow{L(F')} \mathcal{A}' .$$

We define $L(F)$ as the composition of $L(F')$ with the projection to the second component. The transformation

$$\tilde{\phi} : (f_T)_S \to f_S \circ L(F)$$

is defined on $X \in \mathcal{C}(D,T)$ using the isomorphism ψ part of the object $L(F')(X) \in \mathcal{A}'$.

Conversely, the triple $(f, L(F), \tilde{\phi})$ satisfies the universal property of $L(F')$. By the uniqueness part of the universal property, this means that it agrees with $L(F')$. This makes $L(F)$ unique. □

6.5 The diagram category as a category of comodules

Under more restrictive assumptions on R and T, we can give a description of the diagram category of comodules, see Theorem 6.1.12.

6.5.1 Preliminary discussion

In [DM] Deligne and Milne note that if R is a field, E a finite-dimensional R-algebra, and V an E-module that is finite-dimensional as R-vector space then V has a natural structure as comodule over the coalgebra $E^\vee := \mathrm{Hom}_R(E,R)$. For an algebra E finitely generated as an R-module over an arbitrary noetherian ring R, the R-dual $E^\vee$ does not even necessarily carry a natural structure of an R-coalgebra. The problem is that the dual map to the algebra multiplication

$$E^\vee \xrightarrow{\mu^*} (E \otimes_R E)^\vee$$

does not generally define a comultiplication because the canonical map

$$\rho : E^\vee \otimes_R E^\vee \to \mathrm{Hom}(E, E^\vee) \cong (E \otimes_R E)^\vee$$

fails to be an isomorphism in general. In this chapter we will see that this isomorphism holds true for the R-algebras $\mathrm{End}(T_F)$ if we assume that R is a Dedekind domain or field. We will then show that by

$$\begin{aligned}\mathcal{C}(D,T) &= 2-\mathrm{colim}_{F\subset D}(\mathrm{End}(T_F)-\mathrm{Mod})\\ &= 2-\mathrm{colim}_{F\subset D}(\mathrm{End}(T_F)^\vee-\mathrm{Comod}) = (2-\mathrm{colim}_{F\subset D}\mathrm{End}(T_F)^\vee)-\mathrm{Comod}\end{aligned}$$

we can view the diagram category $\mathcal{C}(D,T)$ as the category of finitely generated comodules over the coalgebra $2-\mathrm{colim}_{F\subset D}\mathrm{End}(T_F)^\vee$.

Remark 6.5.1. Note that the category of comodules over an arbitrary coalgebra C is not abelian in general, since the tensor product $X \otimes_R -$ is right exact, but in general not left exact. If C is flat as R-algebra (e.g. free), then the category of C-comodules is abelian [MM, pg. 219].

6.5.2 Coalgebras and comodules

Let R be a noetherian ring with unit.

Proposition 6.5.2. *Let E be an R-algebra which is finitely generated as R-module. Then the canonical map*

$$\begin{array}{rccc}\rho: & E^\vee \otimes_R M & \to & \mathrm{Hom}(E,M)\\ & \varphi\otimes m & \mapsto & (n\mapsto \varphi(n)\cdot m)\end{array}$$

becomes an isomorphism for all R-modules M if and only if E is projective.

Proof. [Str, Proposition 5.2] □

Lemma 6.5.3. *Let E be an R-algebra which is finitely generated and projective as an R-module.*

1. *The R-dual module $E^\vee$ carries a natural structure of a counital coalgebra.*
2. *Any left E-module that is finitely generated as R-module carries a natural structure as left $E^\vee$-comodule.*
3. *We obtain an equivalence of categories between the category of finitely generated left E-modules and the category of finitely generated left $E^\vee$-comodules.*

Proof. By the repeated application of Proposition 6.5.2, this becomes a straightforward calculation. We will sketch the main steps of the proof.

1. If we dualize the associativity constraint of E we obtain a commutative diagram of the form

$$\begin{tikzcd}
(E \otimes_R E \otimes_R E)^\vee & (E \otimes_R E)^\vee \arrow[l, "(\mu \otimes id)^*"'] \\
(E \otimes_R E)^\vee \arrow[u, "(id \otimes \mu)^*"] & E^\vee . \arrow[l, "\mu^*"] \arrow[u, "\mu^*"']
\end{tikzcd}$$

   By the use of the isomorphism in Propostion 6.5.2 and Hom-Tensor adjunction we obtain the commutative diagram

$$\begin{tikzcd}
E^\vee \otimes_R E^\vee \otimes_R E^\vee & E^\vee \otimes_R E^\vee \arrow[l, "\mu^* \otimes id^*"'] \\
E^\vee \otimes_R E^\vee \arrow[u, "id^* \otimes \mu^*"] & E^\vee , \arrow[l, "\mu^*"] \arrow[u, "\mu^*"']
\end{tikzcd}$$

   which induces a cocommutative comultiplication on $E^\vee$. Similarly we obtain the counit diagram, so $E^\vee$ naturally gets a coalgebra structure.

2. For an E-module M we analogously dualize the respective diagram

$$\begin{tikzcd}
M & E \otimes_R M \arrow[l, "m"'] \\
E \otimes_R M \arrow[u, "m"] & E \otimes_R E \otimes_R M \arrow[l, "\mu \otimes id"] \arrow[u, "id \otimes m"']
\end{tikzcd}$$

and use Proposition 6.5.2 and Hom-Tensor adjunction to see that the E-multiplication induces a well-defined $E^\vee$-comultiplication

$$\begin{tikzcd} M \arrow[r, "\hat{m}"] \arrow[d, "\hat{m}"] & E^\vee \otimes_R M \arrow[d, "id \otimes \hat{m}"] \\ E^\vee \otimes_R M \arrow[r, "\mu^* \otimes id"] & E^\vee \otimes_R E^\vee \otimes_R M \end{tikzcd}$$

on M.

3. For any homomorphism $f : M \longrightarrow N$ of left E-modules, the commutative diagram

$$\begin{tikzcd} M \arrow[r, "f"] & N \\ E \otimes_R M \arrow[u, "\mu"] \arrow[r, "id \otimes f"] & E \otimes_R N \arrow[u, "\mu"] \end{tikzcd}$$

induces a commutative diagram

$$\begin{tikzcd} E^\vee \otimes_R M \arrow[r, "id \otimes f"] & E^\vee \otimes_R N, \\ M \arrow[u, "\hat{\mu}"] \arrow[r, "f"] & N \arrow[u, "\hat{\mu}"] \end{tikzcd}$$

thus f is a homomorphism of left $E^\vee$-comodules.

4. Conversely, we can dualize the $E^\vee$-comodule structure to obtain a $(E^\vee)^\vee = E$-module structure. The two constructions are inverse to each other.

□

Definition 6.5.4. Let A be a coalgebra over R. Then we denote by $A-$Comod the category of comodules over A that are finitely generated as a R-modules.

Recall that $R-$Proj denotes the category of finitely generated projective R-modules.

Corollary 6.5.5. *Let R be a field or Dedekind domain, D a diagram and*

$$T : D \longrightarrow R\text{−Proj}$$

a representation. Set $A(D,T) := \varinjlim_{F\subset D \text{ finite}} \mathrm{End}(T_F)^\vee$. Then $A(D,T)$ has the structure of a coalgebra and the diagram category of T is the abelian category $A(D,T)$−Comod.

Proof. For any finite subset $F \subset D$ the algebra $\mathrm{End}(T_F)$ is a submodule of the finitely generated projective R-module $\prod_{p\in F} \mathrm{End}(T_p)$. Since R is a field or Dedekind domain, for a finitely generated module to be projective is equivalent to being torsion free. Hence the submodule $\mathrm{End}(T_F)$ is also finitely generated and torsion-free, or equivalently, projective. By the previous lemma, $\mathrm{End}(T_F)^\vee$ is an R-coalgebra and $\mathrm{End}(T_F)\text{−Mod} \cong \mathrm{End}(T_F)^\vee\text{−Comod}$. From now on, we denote $\mathrm{End}(T_F)^\vee$ with $A(F,T)$. Taking limits over the direct system of finite subdiagrams as in Definition 6.1.10, we obtain

$$\begin{aligned}\mathcal{C}(D,T) &:= 2\text{−colim}_{F\subset D \text{ finite}}\mathrm{End}(T_F)\text{−Mod}\\ &= 2\text{−colim}_{F\subset D \text{ finite}}A(F,T)\text{−Comod}.\end{aligned}$$

Since the category of coalgebras is cocomplete, $A(D,T) = \varinjlim_{F\subset D} A(F,T)$ is a coalgebra as well.

We now need to show that the categories $2\text{−colim}_{F\subset D \text{ finite}}(A(F,T)\text{−Comod})$ and $A(D,T)$−Comod are equivalent. For any finite F the canonical map $A(F,T) \longrightarrow A(D,T)$ via restriction of scalars induces a functor

$$\phi_F : A(F,T)\text{−Comod} \longrightarrow A(D,T)\text{−Comod}$$

and therefore by the universal property a unique functor

$$u : \varinjlim A(F,T)\text{−Comod} \longrightarrow A(D,T)\text{−Comod}.$$

such that for all finite $F', F'' \subset D$ with $F' \subset F''$ and the canonical functors

$$\psi_F : A(F',T)\text{−Comod} \longrightarrow \varinjlim_{F\subset D} A(F,T)\text{−Comod}$$

the following diagram commutes:

$$\begin{tikzcd}
A(F',T)-\mathrm{Comod} \arrow[rr, "\phi_{F'F''}"] \arrow[dr, "\psi_{F'}"] \arrow[ddr, "\phi_{F'}"'] & & A(F'',T)-\mathrm{Comod} \arrow[dl, "\psi_{F''}"'] \arrow[ddl, "\phi_{F''}"] \\
& \varinjlim_{F\subset D}(A(F,T)-\mathrm{Comod}) \arrow[d, "\exists! u"] & \\
& A(D,T)-\mathrm{Comod} &
\end{tikzcd}$$

We construct an inverse map to u: Let M be an $A(D,T)$-comodule and

$$m : M \to M \otimes_R A(D,T)$$

be the comultiplication. Let $M = \langle x_1, .., x_n \rangle_R$. Then $m(x_i) = \sum_{k=1}^n a_{ki} \otimes x_k$ for certain $a_{ki} \in A(D,T)$. Every a_{ki} is already contained in an $A(F,T)$ for sufficiently large F. By taking the union of these finitely many F, we can assume that all a_{ki} are contained in one coalgebra $A(F,T)$. Since $x_1, .., x_n$ generate M as R-module, m defines a comultiplication

$$\tilde{m} : M \to M \otimes_R A(F,T).$$

So M is an $A(F,T)$-comodule in a natural way, thus via ψ_F an object of $2-\mathrm{colim}_I(A_i-\mathrm{Comod})$. □

We also need to understand the behavior of $A(D,T)$ under base-change.

Lemma 6.5.6 (Base change)**.** *Let R be a field or a Dedekind domain and $T : D \to R-\mathrm{Proj}$ a representation. Then*

$$A(D, T_S) = A(D,T) \otimes_R S\ .$$

Proof. Let $F \subset D$ be a finite subdiagram. Recall that

$$A(F,T) = \mathrm{Hom}_R(\mathrm{End}(T|_F), R)\ .$$

Both R and $\mathrm{End}_R(T|_F)$ are projective because R is a field or a Dedekind domain. Hence by Lemma 6.2.2

$$\mathrm{Hom}_R(\mathrm{End}_R(T|_F), R) \otimes S \cong \mathrm{Hom}_S(\mathrm{End}_R(T|_F) \otimes S, S) \cong \mathrm{Hom}_S(\mathrm{End}_S((T_S)|_F), S).$$

This is nothing but $A(F, T_S)$. Tensor products commute with direct limits, hence the statement for $A(D,T)$ follows immediately. □