Periods and Nori Motives

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Chapter 7: More on diagrams

Chapter 7

More on diagrams

We study additional structures on a diagram and a representation that lead to the construction of a tensor product on the diagram category. The aim is then to turn it into a rigid tensor category with a faithful exact functor to a category of R-modules. The chapter is formal, but the assumptions are tailored to the application to Nori motives.

A particularly puzzling and subtle question is how the question of graded commutativity of the Künneth formula is dealt with.

We continue to work in the setting of Chapter 6.

7.1 Multiplicative structure

Let R a fixed noetherian unital commutative ring.

Recall that $R-\mathrm{Proj}$ is the category of projective R-modules of finite type over R. We only consider representations $T : D \longrightarrow R-\mathrm{Proj}$ where D is a diagram with identities, see Definition 6.1.1.

Definition 7.1.1. Let D_1, D_2 be diagrams with identities. Then $D_1 \times D_2$ is defined as the diagram with vertices of the form (v, w) for v a vertex of D_1, w a vertex of D_2, and with edges of the form (α, id) and (id, β) for α an edge of D_1 and β an edge of D_2 and with $\mathrm{id} = (\mathrm{id}, \mathrm{id})$.

Remark 7.1.2. Levine in [Le] p.466 seems to define $D_1 \times D_2$ by taking the product of the graphs in the ordinary sense. He claims (in the notation of loc. cit.) a map of diagrams

$$H_*\mathrm{Sch}' \times H_*\mathrm{Sch}' \to H_*\mathrm{Sch}'.$$

It is not clear to us how this is defined on general pairs of edges. If α, β are edges corresponding to boundary maps and hence lower the degree by 1, then we would expect $\alpha \times \beta$ to lower the degree by 2. However, there are no such edges in $H_*\mathrm{Sch}'$.

Our restricted version of products of diagrams is enough to get the implications we want.

In order to control signs in the Künneth formula, we need to work in a graded commutative setting.

Definition 7.1.3. A *graded diagram* is a diagram D with identities together with a map

$$|\cdot| : \{\text{vertices of } D\ \} \to \mathbb{Z}/2\mathbb{Z}\ .$$

For an edge $\gamma : v \to v'$ we put $|\gamma| = |v| - |v'|$. If D is a graded diagram, $D \times D$ is equipped with the grading $|(v, w)| = |v| + |w|$.

A *commutative product structure* on a graded D is a map of graded diagrams

$$\times : D \times D \to D$$

together with choices of edges

$$\begin{aligned} \alpha_{v,w} &: v \times w \to w \times v \\ \beta_{v,w,u} &: v \times (w \times u) \to (v \times w) \times u \\ \beta'_{v,w,u} &: (v \times w) \times u \to v \times (w \times u) \end{aligned}$$

for all vertices v, w, h of D.

A *graded multiplicative representation* T of a graded diagram with commutative product structure is a representation of T in $R-\mathrm{Proj}$ together with a choice of isomorphism

$$\tau_{(v,w)} : T(v \times w) \to T(v) \otimes T(w)$$

such that:

1. The composition

$$T(v) \otimes T(w) \xrightarrow{\tau^{-1}_{(v,w)}} T(v \times w) \xrightarrow{T(\alpha_{v,w})} T(w \times v) \xrightarrow{\tau_{(w,v)}} T(w) \otimes T(v)$$

is $(-1)^{|v||w|}$ times the natural map of R-modules.

2. If $\gamma : v \to v'$ is an edge, then the diagram

$$\begin{array}{ccc} T(v \times w) & \xrightarrow{T(\gamma \times \mathrm{id})} & T(v' \times w) \\ \tau \downarrow & & \downarrow \tau \\ T(v) \otimes T(w) & \xrightarrow{(-1)^{|\gamma||w|} T(\gamma) \otimes \mathrm{id}} & T(v') \otimes T(w) \end{array}$$

commutes.

3. If $\gamma : v \to v'$ is an edge, then the diagram
$$\begin{array}{ccc} T(w \times v) & \xrightarrow{T(\mathrm{id}\times\gamma)} & T(w \times v') \\ \tau\downarrow & & \downarrow\tau \\ T(w)\otimes T(v) & \xrightarrow{\mathrm{id}\otimes T(\gamma)} & T(w)\otimes T(v') \end{array}$$
commutes.

4. The diagram
$$\begin{array}{ccc} T(v \times (w \times u)) & \xrightarrow{T(\beta_{v,w,u})} & T((v \times w) \times u) \\ \downarrow & & \downarrow \\ T(v)\otimes T(w \times u) & & T(v\times w)\otimes T(u) \\ \downarrow & & \downarrow \\ T(v)\otimes (T(w)\otimes T(u)) & \longrightarrow & (T(v)\otimes T(w))\otimes T(u) \end{array}$$
commutes under the standard identification
$$T(v)\otimes (T(w)\otimes T(u)) \cong (T(v)\otimes T(w))\otimes T(u).$$
The maps $T(\beta_{v,w,u})$ and $T(\beta'_{v,w,u})$ are inverse to each other.

A *unit* for a graded diagram with commutative product structure D is a vertex $\mathbf{1}$ of degree 0 together with a choice of edges
$$u_v : v \to \mathbf{1} \times v$$
for all vertices of v. A graded multiplicative representation is *unital* if $T(u_v)$ is an isomorphism for all vertices v.

Remark 7.1.4. 1. In particular, $T(\alpha_{v,w})$ and $T(\beta_{v,w,u})$ are isomorphisms. If $v = w$ then $T(\alpha_{v,v}) = (-1)^{|v|}$. If $\mathbf{1}$ is a unit, then $T(\mathbf{1})$ satisfies $T(\mathbf{1}) \cong T(\mathbf{1}) \otimes T(\mathbf{1})$. Hence it is a free R-module of rank 1.

2. Note that the first and the second factor are *not* treated symmetrically. There is a choice of sign convention involved. The convention above is chosen to be conform with the one of Section 1.3. Eventually, we want to view relative singular cohomology as graded multiplicative representation in the above sense.

Let $T : D \longrightarrow R\text{−Proj}$ be a representation of a diagram with identities. Recall that we defined its diagram category $\mathcal{C}(D, T)$ (see Definition 6.1.10). It R is a field or a Dedekind domain, then $\mathcal{C}(D, T)$ can be described as the category of $A(D, T)$-comodules of finite type over R for the coalgebra $A(D, T)$ defined in Theorem 6.1.12.

Proposition 7.1.5. *Let D be a graded diagram with commutative product structure with unit and T a unital graded multiplicative representation of D in $R-$Proj*

$$T : D \longrightarrow R\text{−Proj}.$$

1. *Then $\mathcal{C}(D,T)$ carries the structure of a commutative and associative tensor category with unit and $T : \mathcal{C}(D,T) \to R-$Mod is a tensor functor. On the generators $\tilde{T}(v)$ of $\mathcal{C}(D,T)$ the associativity constraint is induced by the edges $\beta_{v,w,u}$, the commutativity constraint is induced by the edges $\alpha_{v,w}$, the unit object is $\tilde{\mathbf{1}}$ with unital maps induced from the edges u_v.*
2. *If, in addition, R is a field or a Dedekind domain, the coalgebra $A(D,T)$ carries a natural structure of commutative bialgebra (with unit and counit).*

The unit object is going to be denoted $\mathbf{1}$.

Proof. We consider finite diagrams F and F' such that

$$\{v \times w | v, w \in F\} \subset F' .$$

We are going to define natural maps

$$\mu_F^* : \mathrm{End}(T|_{F'}) \to \mathrm{End}(T|_F) \otimes \mathrm{End}(T|_F).$$

Assume this for a moment. Let $X, Y \in \mathcal{C}(D,T)$. We want to define $X \otimes Y$ in $\mathcal{C}(D,T) = 2 - \mathrm{colim}_F \mathcal{C}(F,T)$. Let F such that $X, Y \in \mathcal{C}(F,T)$. This means that X and Y are finitely generated R-modules with an action of $\mathrm{End}(T|_F)$. We equip the R-module $X \otimes Y$ with a structure of $\mathrm{End}(T|_{F'})$-module. It is given by

$$\mathrm{End}(T|_{F'}) \otimes X \otimes Y \to \mathrm{End}(T|_F) \otimes \mathrm{End}(T|_F) \otimes X \otimes Y \to X \otimes Y$$

where we have used the comultiplication map μ_F^* and the module structures of X and Y. This will be independent of the choice of F and F'. Properties of $\otimes$ on $\mathcal{C}(D,T)$ follow from properties of μ_F^*.

If R is a field or a Dedekind domain, let

$$\mu_F : A(F,T) \otimes A(F,T) \to A(D,T)F'T$$

be dual to μ_F^*. Passing to the direct limit defines a multiplication μ on $A(D,T)$.

We now turn to the construction of μ_F^*. Let $a \in \mathrm{End}(T|_{F'})$, i.e., a compatible system of endomorphisms $a_v \in \mathrm{End}(T(v))$ for $v \in F'$. We describe its image $\mu_F^*(a)$. Let $(v, w) \in F \times F$. The isomorphism

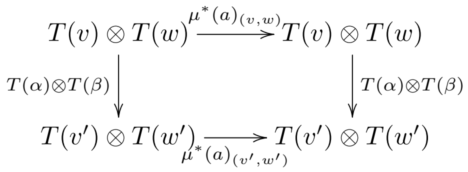
$$\tau : T(v \times w) \to T(v) \otimes T(w)$$

induces an isomorphism

$$\mathrm{End}(T(v \times w)) \cong \mathrm{End}(T(v)) \otimes \mathrm{End}(T(w)).$$

We define the (v, w)-component of $\mu^*(a)$ by the image of $a_{v\times w}$ under this isomorphism.

In order to show that this is a well-defined element of $\mathrm{End}(T|_F) \otimes \mathrm{End}(T|_F)$, we need to check that diagrams of the form

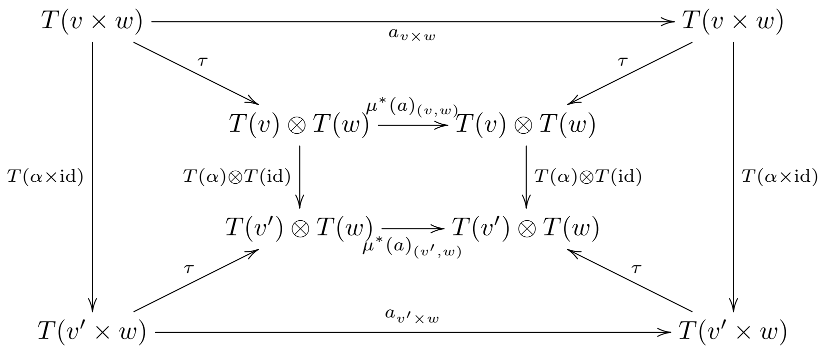


commute for all edges $\alpha : v \to v'$, $\beta : w \to w'$ in F. We factor

$$T(\alpha) \otimes T(\beta) = (T(\mathrm{id}) \otimes T(\beta)) \circ (T(\alpha) \circ T(\mathrm{id}))$$

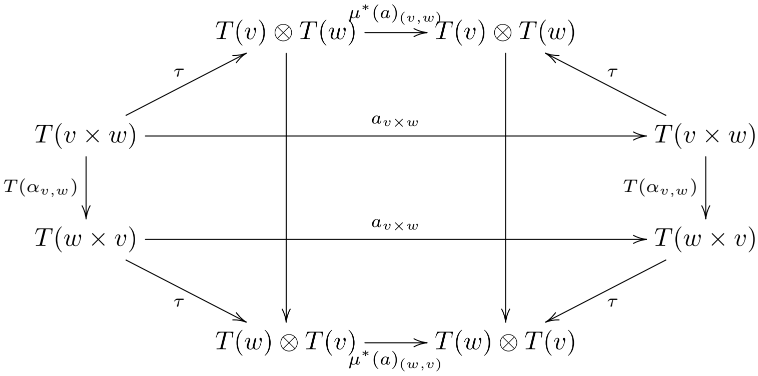
and check the factors separately.

Consider the diagram



The outer square commutes because a is a diagram endomorphism. Top and bottom commute by definition of $\mu^*(a)$. Left and right commute by property (3) up to the same sign $(-1)^{|w||\alpha|}$. Hence the middle square commutes without signs. The analogous diagram for $\mathrm{id} \times \beta$ commutes on the nose. Hence $\mu^*(a)$ is well-defined.

We now want to compare the (v, w)-component to the (w, v)-component. Recall that there is a distinguished edge $\alpha_{v,w} : v \times w \to w \times v$. Consider the diagram



By the construction of $\mu^*(a)_{(v,w)}$ (resp. $\mu^*(a)_{(w,v)}$), the upper (resp. lower) tilted square commutes. By naturality, the middle rectangle with $\alpha_{v,w}$ commutes. By property (1) of a representation of a graded diagram with commutative product, the left and right faces commute where the vertical maps are $(-1)^{|v||w|}$ times the natural commutativity of tensor products of T-modules. Hence the inner square also commutes without the sign factors. This is cocommutativity of μ^*.

The associativity assumption (3) for representations of diagrams with product structure implies the coassociativity of μ^*.

The compatibility of multiplication and comultiplication is built into the definition.

In order to define a unit object in $\mathcal{C}(D,T)$ it suffices to define a counit for $\mathrm{End}(T|_F)$. Assume $\mathbf{1} \in F$. The counit

$$u^* : \mathrm{End}(T|_F) \subset \prod_{v \in F} \mathrm{End}(T(v)) \to \mathrm{End}(T(\mathbf{1})) = R$$

is the natural projection. The assumption on unitality of T allows to check that the required diagrams commute. □

Remark 7.1.6. The proof of Proposition 7.1.5 works without any changes in the arguments when we weaken the assumptions as follows: in Definition 7.1.3 replace $\times$ by a map of diagrams

$$\times : D \times D \to \mathcal{P}(D)$$

where $\mathcal{P}(D)$ is the path category of D: objects are the vertices of D and morphisms the paths. A representation T of D extends canonically to a functor on $\mathcal{P}(D)$.

7.2 Localization

The purpose of this section is to give a diagram version of the localization of a tensor category with respect to one object, i.e., a distinguished object X becomes invertible with respect to tensor product. This is the standard construction used to pass e.g. from effective motives to all motives.

We restrict to the case when R is a field or a Dedekind domain and all representations of diagrams take values in $R-$Proj.

Definition 7.2.1 (Localization of diagrams)**.** Let D^{eff} be a graded diagram with a commutative product structure with unit $\mathbf{1}$. Let $v_0 \in D^{\mathrm{eff}}$ be a vertex. The *localized diagram* D has vertices and edges as follows:

1. for every v a vertex of D^{eff} and $n \in \mathbb{Z}$ a vertex denoted $v(n)$;
2. for every edge $\alpha : v \to w$ in D^{eff} and every $n \in \mathbb{Z}$, an edge denoted $\alpha(n) : v(n) \to w(n)$ in D;
3. for every vertex v in D^{eff} and every $n \in \mathbb{Z}$ an edge denoted $(v \times v_0)(n) \to v(n+1)$.

Put $|v(n)| = |v|$.

We equip D with a weak commutative product structure in the sense of Remark 7.1.6

$$\times : D \times D \to \mathcal{P}(D) \qquad v(n) \times w(m) \mapsto (v \times w)(n+m)$$

together with

$$\begin{aligned} \alpha_{v(n),w(m)} &= \alpha_{v,w}(n+m) \\ \beta_{v(n),w(m),u(r)} &= \beta_{v,w,u}(n+m+r) \\ \beta'_{v(n),w(m),u(r)} &= \beta'_{v,w,u}(n+m+r) \end{aligned}$$

Let $\mathbf{1}(0)$ together with

$$u_{v(n)} = u_v(n)$$

be the unit.

Note that there is a natural inclusion of multiplicative diagrams $D^{\mathrm{eff}} \to D$ which maps a vertex v to $v(0)$.

Remark 7.2.2. The above definition does not spell out $\times$ on edges. It is induced from the product structure on D^{eff} for edges of type (2). For edges of type (3) there is an obvious sequence of edges. We take their composition in $\mathcal{P}(D)$. E.g. for $\gamma_{v,n} : (v \times v_0)(n) \to v(n+1)$ and $\mathrm{id}_{w(m)} = \mathrm{id}_w(m) : w(m) \to w(m)$ we have

$$\gamma_{v,n} \times \mathrm{id}(m) : (v \times v_0)(n) \times w(m) \to v(n+1) \times w(m)$$

via

$$\begin{aligned}(v\times v_0)(n)\times w(m) &= ((v\times v_0)\times w)(n+m)\\ &\xrightarrow{\beta'_{v,v_0,w}(n+m)} (v\times(v_0\times w))(n+m)\\ &\xrightarrow{\mathrm{id}\times\alpha_{v_0,w}(n+m)} (v\times(w\times v_0))(n+m)\\ &\xrightarrow{\beta_{v,w,v_0}(n+m)} ((v\times w)\times v_0)(n+m)\\ &\xrightarrow{\gamma_{v\times w,n+m}} (v\times w)(n+m+1) = v(n+1)\times w(m)\ .\end{aligned}$$

Assumption 7.2.3. Let R be a field or a Dedekind domain. Let T be a multiplicative unital representation of D^{eff} with values in $R-\mathrm{Proj}$ such that $T(v_0)$ is locally free of rank 1 as R-module.

Lemma 7.2.4. *Under Assumption 7.2.3, the representation T extends uniquely to a graded multiplicative representation of D such that $T(v(n)) = T(v)\otimes T(v_0)^{\otimes n}$ for all vertices and $T(\alpha(n)) = T(\alpha)\otimes T(\mathrm{id})^{\otimes n}$ for all edges. It is multiplicative and unital with the choice*

$$\begin{array}{ccc} T(v(n)\times w(m)) & \xrightarrow{\tau_{v(n),w(m)}} & T(v(n))\otimes T(w(m))\\ \Big\downarrow{\scriptstyle \tau_{v,w}(n+m)} & & \Big\downarrow{=}\\ T(v)\otimes T(w)\otimes T(v_0)^{\otimes n+m} & \xrightarrow{\cong} & T(v)\otimes T(v_0)^{\otimes n}\otimes T(w)\otimes T(v_0)^{\otimes m}\end{array}$$

where the last line is the natural isomorphism.

Proof. Define T on the vertices and edges of D via the formula. It is tedious but straightforward to check the conditions. □

Proposition 7.2.5. *Let D^{eff}, D and T be as above. Assume Assumption 7.2.3. Let $A(D,T)$ and $A(D^{\mathrm{eff}},T)$ be the corresponding bialgebras. Then:*

1. *$\mathcal{C}(D,T)$ is the localization of the category $\mathcal{C}(D^{\mathrm{eff}},T)$ with respect to the object $\tilde{T}(v_0)$.*
2. *Let $\chi\in\mathrm{End}(T(v_0))^\vee = A(\{v_0\},T)$ be the dual of $\mathrm{id}\in\mathrm{End}(T(v_0))$. We view it in $A(D^{\mathrm{eff}},T)$. Then $A(D,T) = A(D^{\mathrm{eff}},T)_\chi$ (localization of algebras).*

Proof. Let $D^{\geq n}\subset D$ be the subdiagram with vertices of the form $v(n')$ with $n'\geq n$. Clearly, $D = 2-\mathrm{colim}_n D^{\geq n}$ and hence

$$\mathcal{C}(D,T)\cong 2-\mathrm{colim}_n\mathcal{C}(D^{\geq n},T)\ .$$

Consider the morphism of diagrams

$$D^{\geq n}\to D^{\geq n+1},\ \ v(m)\mapsto v(m+1).$$

It is clearly an isomorphism. We equip $\mathcal{C}(D^{\geq n+1},T)$ with a new fibre functor $f_T\otimes T(v_0)^\vee$. It is faithful exact. The map $v(m)\mapsto \tilde{T}(v(m+1))$ is a representation of $D^{\geq n}$ in the abelian category $\mathcal{C}(D^{\geq n+1},T)$ with fibre functor $f_T\otimes T(v_0)^\vee$. By the universal property, this induces a functor

$$\mathcal{C}(D^{\geq n},T)\to \mathcal{C}(D^{\geq n+1},T)\ .$$

The converse functor is constructed in the same way. Hence

$$\mathcal{C}(D^{\geq n},T)\cong \mathcal{C}(D^{\geq n+1},T),\qquad A(D^{\geq n},T)\cong A(D^{\geq n+1},T).$$

The map of graded diagrams with commutative product and unit

$$D^{\mathrm{eff}}\to D^{\geq 0}$$

induces an equivalence on tensor categories. Indeed, we represent $D^{\geq 0}$ in $\mathcal{C}(D^{\mathrm{eff}},T)$ by mapping $v(m)$ to $\tilde{T}(v)\otimes\tilde{T}(v_0)^m$. By the universal property (see Corollary 6.1.18), this implies that there is a faithful exact functor

$$\mathcal{C}(D^{\geq 0},T)\to \mathcal{C}(D^{\mathrm{eff}},T)$$

inverse to the obvious inclusion. Hence we also have $A(D^{\mathrm{eff}},T)\cong A(D^{\geq 0},T)$ as unital bialgebras.

On the level of coalgebras, this implies

$$A(D,T)=2-\mathrm{colim}_n A(D^{\geq n},T)=2-\mathrm{colim}_n A(D^{\mathrm{eff}},T)$$

because $A(D^{\geq n},T)$ isomorphic to $A(D^{\mathrm{eff}},T)$ as coalgebras. $A(D^{\mathrm{eff}},T)$ also has a multiplication, but the $A(D^{\geq n},T)$ for general $n\in\mathbb{Z}$ do not. However, they carry a weak $A(D^{\mathrm{eff}},T)$-module structure analogous to Remark 7.1.6 corresponding to the map of graded diagrams

$$D^{\mathrm{eff}}\times D^{\geq n}\to \mathcal{P}(D^{\geq n}).$$

We want to describe the transition maps of the direct limit. From the point of view of $D^{\mathrm{eff}}\to D^{\mathrm{eff}}$, it is given by $v\mapsto v\times v_0$.

In order to describe the transition maps $A(D^{\mathrm{eff}},T)\to A(D^{\mathrm{eff}},T)$, it suffices to describe $\mathrm{End}(T|_F)\to\mathrm{End}(T|_{F'})$ where F,F' are finite subdiagrams of D^{eff} such that $v\times v_0\in V(F')$ for all vertices $v\in V(F)$. It is induced by

$$\mathrm{End}(T(v))\to\mathrm{End}(T(v\times v_0))\xrightarrow{\tau}\mathrm{End}(T(v))\otimes\mathrm{End}(T(v_0))\quad a\mapsto a\otimes\mathrm{id}.$$

On the level of coalgebras, this corresponds to the map

$$A(D^{\mathrm{eff}},T)\to A(D^{\mathrm{eff}},T),\quad x\mapsto x\chi.$$

Note finally, that the direct limit $2-\mathrm{colim}A(D^{\mathrm{eff}},T)$ with transition maps given by multiplication by χ agrees with the localization $A(D^{\mathrm{eff}},T)_\chi$. $\square$

7.3 Nori's Rigidity Criterion

Implicit in Nori's construction of motives is a rigidity criterion, which we are now going to formulate and prove explicitly.

Let R be a Dedekind domain or a field and $\mathcal{C}$ an R-linear tensor category. Recall that $R-$Mod is the category of finitely generated R-modules and $R-$Proj the category of finitely generated projective R-modules.

We assume that the tensor product on $\mathcal{C}$ is associative, commutative and unital. Let $\mathbf{1}$ be the unit object. Let $T : \mathcal{C} \to R-$Mod be a faithful tensor functor with values in $R-$Mod. In particular, $T(\mathbf{1}) \cong R$.

We introduce an ad-hoc notion.

Definition 7.3.1. Let V be an object of $\mathcal{C}$. We say that V *admits a perfect duality* if there is morphism

$$q : V \otimes V \to \mathbf{1}$$

or

$$\mathbf{1} \to V \otimes V$$

such that $T(V)$ is projective and $T(q)$ (respectively its dual) is a non-degenerate bilinear form.

Definition 7.3.2. Let V be an object of $\mathcal{C}$. By $\langle V\rangle_\otimes$ we denote the smallest full abelian unital tensor subcategory of $\mathcal{C}$ containing V.

We start with the simplest case of the criterion.

Lemma 7.3.3. *Let V be an object such that $\mathcal{C} = \langle V\rangle_\otimes$ and such that V admits a perfect duality. Then $\mathcal{C}$ is rigid.*

Proof. By standard Tannakian formalism, $\mathcal{C}$ is the category of comodules for a bialgebra A, which is commutative and of finite type as an R-algebra. Indeed: The construction of A as a coalgebra was explained in Proposition 6.1.12. We may view $\mathcal{C}$ as graded diagram (with trivial grading) with a unital commutative product structure in the sense of Definition 7.1.3. The fibre functor T is a unital graded multiplicative representation. The algebra structure on A is the one of Proposition 7.1.5. It is easy to see that A is generated by $A(\{V\}, T,)$ as an algebra. The argument is given in more detail below.

We want to show that A is a Hopf algebra, or equivalently, that the algebraic monoid $M = \mathrm{Spec}A$ is an algebraic group.

By Lemma 7.3.6 it suffices to show that there is a closed immersion $M \to G$ of monoids into an algebraic group G. We are going to construct this group or rather its ring of regular functions. We have

$$A = \lim A_n$$

with $A_n = A(\mathcal{C}_n, T)$ for $\mathcal{C}_n = \langle \mathbf{1}, V, V^{\otimes 2}, \dots, V^{\otimes n}\rangle$, the smallest full abelian subcategory containing $\mathbf{1}, V, \dots, V^{\otimes n}$. By construction, there is a surjective map

$$\bigoplus_{i=0}^{n} \mathrm{End}_R((T(V)^{\otimes i})^\vee \to A_n$$

or, dually, an injective map

$$A_n^\vee \to \bigoplus_{i=0}^{n} \mathrm{End}_R(T(V)^{\otimes i})$$

where $A_n^\vee$ consists of those endomorphisms compatible with all morphisms in $\mathcal{C}_n$. In the limit, there is a surjection of bialgebras

$$\bigoplus_{i=0}^{\infty} \mathrm{End}_R((T(V)^{\otimes i})^\vee) \to A$$

and the kernel is generated by the relation defined by compatibility with morphisms in $\mathcal{C}$. One such relation is the commutativity constraint, hence the map factors via the symmetric algebra

$$S^*(\mathrm{End}(T(V)^\vee) \to A\ .$$

Note that $S^*(\mathrm{End}(T(V)^\vee)$ is canonically the ring of regular functions on the algebraic monoid $\mathrm{End}(T(V))$. Another morphism in $\mathcal{C}$ is the pairing $q : V \otimes V \to \mathbf{1}$. We want to work out the explicit equation induced by q.

We choose a basis $e_1, \dots, e_r$ of $T(V)$. Let

$$a_{i,j} = T(q)(e_i, e_j) \in R$$

By assumption, the matrix is invertible. Let X_{st} be the matrix coefficients on $\mathrm{End}(T(V))$ corresponding to the basis e_i. Compatibility with q gives for every pair (i, j) the equation

$$\begin{aligned} a_{ij} &= q(e_i, e_j) \\ &= q((X_{rs})e_i, (X_{r's'})e_j) \\ &= q\left(\sum_r X_{ri}e_r, \sum_{r'} X_{r'j}e_{r'}\right) \\ &= \sum_{r,r'} X_{ri}X_{r'j}q(e_r, e_{r'}) \\ &= \sum_{r,r'} X_{ri}X_{r'j}a_{rr'} \end{aligned}$$

Note that the latter is the (i, j)-term in the product of matrices

$$(X_{ir})^t(a_{rr'})(X_{r'j})\ .$$

Let $(b_{ij}) = (a_{ij})^{-1}$. With

$$(Y_{ij}) = (b_{ij})(X_{i'r})^t(a_{rr'})$$

we have the coordinates of the inverse matrix. In other words, our set of equations defines the isometry group $G(q) \subset \mathrm{End}(T(V))$. We now have expressed A as quotient of the ring of regular functions of $G(q)$.

The argument works in the same way, if we are given

$$q : \mathbf{1} \to V \otimes V$$

instead. □

Proposition 7.3.4 (Nori)**.** *Let $\mathcal{C}$ and $T : \mathcal{C} \to R-\mathrm{Mod}$ be as defined at the beginning of the section. Let $\{V_i | i \in I\}$ be a set of objects of $\mathcal{C}$ with the properties:*

1. *It generates $\mathcal{C}$ as an abelian tensor category, i.e., the smallest full abelian tensor subcategory of $\mathcal{C}$ containing all V_i is equal to $\mathcal{C}$.*
2. *For every V_i there is an object W_i and a morphism*

   $$q_i : V_i \otimes W_i \to \mathbf{1}$$

   such that $T(q_i) : T(V_i) \otimes T(W_i) \to T(\mathbf{1}) = R$ is a perfect pairing of free R-modules.

Then $\mathcal{C}$ is rigid, i.e., for every object V there is a dual object $V^\vee$ such that

$$\mathrm{Hom}(V \otimes A, B) = \mathrm{Hom}(A, V^\vee \otimes B) , \qquad \mathrm{Hom}(V^\vee \otimes A, B) = \mathrm{Hom}(A, V \otimes B) .$$

This means that the Tannakian dual of $\mathcal{C}$ is not only a monoid but a group.

Remark 7.3.5. The Proposition also holds with the dual assumption, existence of morphisms

$$q_i : \mathbf{1} \to V_i \otimes W_i$$

such that $T(q_i)^\vee : T(V)^\vee \otimes T(W_i)^\vee \to R$ is a perfect pairing.

Proof. Consider $V_i' = V_i \oplus W_i$. The pairing q_i extends to a symmetric map q_i' on $V_i' \otimes V_i'$ such that $T(q_i')$ is non-degenerate. We now replace V_i by V_i'. Without loss of generality, we can assume $V_i = W_i$.

For any finite subset $J \subset I$, let $V_J = \bigoplus_{j \in J} V_j$. Let q_J be the orthogonal sum of the q_j for $j \in J$. It is again a symmetric perfect pairing.

For every object V of $\mathcal{C}$, we write $\langle V \rangle_\otimes$ for the smallest full abelian tensor subcategory of $\mathcal{C}$ containing V. By assumption we have

$$\mathcal{C} = \bigcup_J \langle V_J \rangle_\otimes$$

We apply the standard Tannakian machinery. It attaches to every $\langle V_J \rangle_\otimes$ an R-bialgebra A_J such that $\langle V_J \rangle_\otimes$ is equivalent to the category of A_J-comodules. If we put

$$A = \lim A_J$$

then $\mathcal{C}$ will be equivalent to the category of A-comodules. It suffices to show that A_J is a Hopf-algebra. This is the case by Lemma 7.3.3. □

Finally, the missing lemma on monoids.

Lemma 7.3.6. *Let R be noetherian ring, G be an algebraic group scheme of finite type over R and $M \subset G$ a closed immersion of a submonoid with $1 \in M(R)$. Then M is an algebraic group scheme over R.*

Proof. This seems to be well-known. It is appears as an exercise in [Re] 3.5.1 2. We give the argument:

Let S be any finitely generated R-algebra. We have to show that the value $S \mapsto M(S)$ is a group. We take base change of the situation to S. Hence without loss of generality, it suffices to consider $R = S$. If $g \in G(R)$, we denote the isomorphism $G \to G$ induced by left multiplication with g also by $g : G \to G$. Take any $g \in G(R)$ such that $gM \subset M$ (for example $g \in M(R)$). Then one has

$$M \supseteq gM \supseteq g^2 M \supseteq \cdots$$

As G is Noetherian, this sequence stabilizes, say at $s \in \mathbb{N}$:

$$g^s M = g^{s+1} M$$

as closed subschemes of G. Since every g^s is an isomorphism, we obtain that

$$M = g^{-s} g^s M = g^{-s} g^{s+1} M = gM$$

as closed subschemes of G. So for every $g \in M(R)$ we showed that $gM = M$. Since $1 \in M(R)$, this implies that $M(R)$ is a subgroup. □