Periods and Nori Motives

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Part I

Part I

Background material

Chapter 1

General Set-up

In this chapter we collect some standard notation used throughout the book.

1.1 Varieties

Let k be field. It will almost always be of characteristic zero or even a subfield of $\mathbb{C}$.

By a *scheme* over k we mean a separated scheme of finite type over k. Let Sch be the category of k-schemes. By a *variety* over k we mean a quasi-projective reduced scheme of finite type over k. Let Var be the category of varieties over k. Let Sm and Aff be the full subcategories of smooth varieties and affine varieties, respectively.

1.1.1 Linearizing the category of varieties

We also need the additive categories generated by these categories of varieties. More precisely:

Definition 1.1.1. Let $\mathbb{Z}[\mathrm{Var}]$ be the category with objects the objects of Var. If $X = X_1 \cup \cdots \cup X_n$, $Y = Y_1 \cup \cdots \cup Y_m$ are varieties with connected components X_i, Y_j, we put

$$\mathrm{Mor}_{\mathbb{Z}[\mathrm{Var}]}(X, Y) = \left\{ \sum_{i,j} a_{ij} f_{ij} | a_{ij} \in \mathbb{Z},\ f_{ij} \in \mathrm{Mor}_{\mathrm{Var}}(X_i, Y_j) \right\}$$

with the addition of formal linear combinations. Composition of morphisms is defined by extending composition of morphisms of varieties $\mathbb{Z}$-linearly.

Analogously, we define $\mathbb{Z}[\mathrm{Sm}]$, $\mathbb{Z}[\mathrm{Aff}]$ from Sm and Aff. Moreover, let $\mathbb{Q}[\mathrm{Var}]$,

$\mathbb{Q}[\mathrm{Sm}]$ and $\mathbb{Q}[\mathrm{Aff}]$ be the analogous $\mathbb{Q}$-linear additive categories where morphisms are formal $\mathbb{Q}$-linear combinations of morphisms of varieties.

Let $F = \sum a_i f_i : X \to Y$ be a morphism in $\mathbb{Z}[\mathrm{Var}]$. The *support* of F is the set of f_i with $a_i \neq 0$.

$\mathbb{Z}[\mathrm{Var}]$ is an additive category with direct sum given by the disjoint union of varieties. The zero object corresponds to the empty variety, or, if you prefer, has to be added formally.

We are also going to need the category of *smooth correspondences* SmCor. It has the same objects as Sm and as morphisms *finite correspondences*

$$\mathrm{Mor}_{\mathrm{SmCor}}(X, Y) = \mathrm{Cor}(X, Y),$$

where $\mathrm{Cor}(X, Y)$ is the free $\mathbb{Z}$-module with generators integral subschemes $\Gamma \subset X \times Y$ such that $\Gamma \to X$ is finite and dominant over a component of X.

1.1.2 Divisors with normal crossings

Definition 1.1.2. Let X be a smooth variety of dimension n and $D \subset X$ a closed subvariety. D is called *divisor with normal crossings* if for every point of D there is an affine neighbourhood U of x in X which is étale over $\mathbb{A}^n$ via coordinates $t_1, \dots, t_n$ and such that $D|_U$ has the form

$$D|_U = V(t_1 t_2 \cdots t_r)$$

for some $1 \leq r \leq n$.

D is called a *simple divisor with normal crossings* if in addition the irreducible components of D are smooth.

In other words, D looks étale locally like an intersection of coordinate hyperplanes.

Example 1.1.3. Let $D \subset \mathbb{A}^2$ be the nodal curve, given by the equation $y^2 = x^2(x-1)$. It is smooth in all points different from $(0, 0)$ and looks étale locally like $xy = 0$ in the origin. Hence it is a divisor with normal crossings but not a simple normal crossings divisor.

1.2 Complex analytic spaces

A classical reference for complex analytic spaces is the book of Grauert and Remmert [GR].

Definition 1.2.1. A *complex analytic space* is a locally ringed space $(X, \mathcal{O}_X^{\mathrm{hol}})$ with X paracompact and Hausdorff, and such that $(X, \mathcal{O}_X^{\mathrm{hol}})$ is locally isomorphic to the vanishing locus Z of a set S of holomorphic functions in some open

$U \subset \mathbb{C}^n$ and $\mathcal{O}_Z^{\text{hol}} = \mathcal{O}_U^{\text{hol}}/\langle S \rangle$, where $\mathcal{O}_U^{\text{hol}}$ is the sheaf of holomorphic functions on U.

A *morphism* of complex analytic spaces is a morphism $f : (X, \mathcal{O}_X^{\text{hol}}) \to (Y, \mathcal{O}_Y^{\text{hol}})$ of locally ringed spaces, which is given by a morphism of sheaves $\tilde{f} : \mathcal{O}_Y^{\text{hol}} \to f_*\mathcal{O}_X^{\text{hol}}$ that sends a germ $h \in \mathcal{O}_{Y,y}^{\text{hol}}$ of a holomorphic function h near y to the germs $h \circ f$, which are holomorphic near x for all x with $f(x) = y$. A morphism will sometimes simply be called a holomorphic map, and will be denoted in short form as $f : X \to Y$.

Let An be the category of complex analytic spaces.

Example 1.2.2. Let X be a complex manifold. Then it can be viewed as a complex analytic space. The structure sheaf is defined via the charts.

Definition 1.2.3. A morphism $X \to Y$ between complex analytic spaces is called *proper* if the preimage of any compact subset in Y is compact.

1.2.1 Analytification

Polynomials over $\mathbb{C}$ can be viewed as holomorphic functions. Hence an affine variety immediately defines a complex analytic space. If X is smooth, it is even a complex submanifold. This assignment is well-behaved under gluing and hence it globalizes. A general reference for this is [SGA1], exposé XII by M. Raynaud.

Proposition 1.2.4. *There is a functor*

$$\cdot^{\text{an}} : \text{Sch}_{\mathbb{C}} \to \text{An}$$

which assigns to a scheme of finite type over $\mathbb{C}$ its analytification. There is a natural morphism of locally ringed spaces

$$\alpha : (X^{\text{an}}, \mathcal{O}_{X^{\text{an}}}^{\text{hol}}) \to (X, \mathcal{O}_X)$$

and $\cdot^{\text{an}}$ is universal with this property. Moreover, α is the identity on points.

If X is smooth, then X^{an} is a complex manifold. If $f : X \to Y$ is proper, the so ist f^{an}.

Proof. By the universal property it suffices to consider the affine case where the obvious construction works. Note that X^{an} is Hausdorff because X is separated, and it is paracompact because it has a finite cover by closed subsets of some $\mathbb{C}^n$. If X is smooth then X^{an} is smooth by [SGA1], Prop. 2.1 in exposé XII, or simply by the Jacobi criterion. The fact that f^{an} is proper if f is proper is shown in [SGA1], Prop. 3.2 in exposé XII. $\square$

1.3 Complexes

1.3.1 Basic definitions

Let $\mathcal{A}$ be an additive category. If not specified otherwise, a complex will always mean a cohomological complex, i.e., a sequence A^i for $i \in \mathbb{Z}$ of objects of $\mathcal{A}$ with *ascending* differential $d^i : A^i \to A^{i+1}$ such that $d^{i+1}d^i = 0$ for all $i \in \mathbb{Z}$. The category of complexes is denoted by $C(\mathcal{A})$. We denote $C^+(\mathcal{A})$, $C^-(\mathcal{A})$ and $C^b(\mathcal{A})$ the full subcategories of complexes bounded below, bounded above and bounded, respectively.

If $K^\bullet \in C(\mathcal{A})$ is a complex, we define the *shifted* complex $K^\bullet[1]$ with

$$(K^\bullet[1])^i = K^{i+1} \ , \quad d^i_{K^\bullet[1]} = -d^{i+1}_{K^\bullet} \ .$$

If $f : K^\bullet \to L^\bullet$ is a morphism of complexes, its *cone* is the complex $\mathrm{Cone}(f)^\bullet$ with

$$\mathrm{Cone}(f)^i = K^{i+1} \oplus L^i, d^i_{\mathrm{Cone}(f)} = (-d^{i+1}_K, f^{i+1} + d^i_L) \ .$$

By construction there are morphisms

$$L^\bullet \to \mathrm{Cone}(f) \to K^\bullet[1] \ ,$$

Let $K(\mathcal{A})$, $K^+(\mathcal{A})$, $K^-(\mathcal{A})$ and $K^b(\mathcal{A})$ be the corresponding homotopy categories where the objects are the same and morphisms are homotopy classes of morphisms of complexes. Note that these categories are always triangulated with the above shift functor and the class of distinguished triangles are those homotopy equivalent to

$$K^\bullet \xrightarrow{f} L^\bullet \to \mathrm{Cone}(f) \to K^\bullet[1]$$

for some morphism of complexes f.

Recall:

Definition 1.3.1. Let $\mathcal{A}$ be an abelian category. A morphism $f^\bullet : K^\bullet \to L^\bullet$ of complexes in $\mathcal{A}$ is called *quasi-isomorphism* if

$$H^i(f) : H^i(K^\bullet) \to H^i(L^\bullet)$$

is an isomorphism for all $i \in \mathbb{Z}$.

We will always assume that an abelian category has enough injectives, or is essentially small, in order to avoid set-theoretic problems. If $\mathcal{A}$ is abelian, let $D(\mathcal{A})$, $D^+(\mathcal{A})$, $D^-(\mathcal{A})$ and $D^b(\mathcal{A})$ the induced derived categories where the objects are the same as in $K^?(\mathcal{A})$ and morphisms are obtained by localizing $K^?(\mathcal{A})$ with respect to the class of quasi-isomorphisms. A triangle is distinguished if it is isomorphic in $D^?(\mathcal{A})$ to a distinguished triangle in $K^?(\mathcal{A})$.

Remark 1.3.2. Let $\mathcal{A}$ be abelian. If $f : K^\bullet \to L^\bullet$ is a morphism of complexes, then

$$0 \to L^\bullet \to \mathrm{Cone}(f) \to K^\bullet[1] \to 0$$

is an exact sequence of complexes. Indeed, it is degreewise split-exact.

1.3.2 Filtrations

Filtrations on complexes are used in order to construct spectral sequences. We mostly need two standard cases.

Definition 1.3.3. 1. Let $\mathcal{A}$ be an additive category, $K^\bullet$ a complex in $\mathcal{A}$. The *stupid filtration* $F^{\geq p}K^\bullet$ on $K^\bullet$ is given by

$$F^{\geq p}K^\bullet = \begin{cases} K^i & i \geq p, \\ 0 & i < p. \end{cases}$$

The quotient $K^\bullet/F^{\geq p}K^\bullet$ is given by

$$F^{<p}K^\bullet = \begin{cases} 0 & i \geq p, \\ K^i & i < p. \end{cases}$$

2. Let $\mathcal{A}$ be an abelian category, $K^\bullet$ a complex in $\mathcal{A}$. The *canonical filtration* $\tau_{\leq p}K^\bullet$ on $K^\bullet$ is given by

$$F^{\leq p}K^\bullet = \begin{cases} K^i & i < p, \\ \mathrm{Ker}(d^p) & i = p, \\ 0 & i > p. \end{cases}$$

The quotient $K^\bullet/F^{\leq p}K^\bullet$ is given by

$$\tau_{>p}K^\bullet = \begin{cases} 0 & i < p, \\ K^p/\mathrm{Ker}(d^p) & i = p, \\ K^i & i > p. \end{cases}$$

The associated graded pieces of the stupid filtration are given by

$$F^{\geq p}K^\bullet/F^{\geq p+1}K^\bullet = K^p \ .$$

The associated graded pieces of the canonical filtration are given by

$$\tau_{\leq p}K^\bullet/\tau_{\leq p-1}K^\bullet = H^p(K^\bullet) \ .$$

1.3.3 Total complexes and signs

We return to the more general case of an additive category $\mathcal{A}$. We consider complexes in $K^{\bullet,\bullet} \in C(\mathcal{A})$, i.e., double complexes consisting of a set of objects $K^{i,j} \in \mathcal{A}$ for $i,j \in \mathbb{Z}$ with differentials

$$d_1^{i,j} : K^{i,j} \to K^{i,j+1} \ , \quad d_2^{i,j} : K^{i,j} \to K^{i+1,j}$$

such that $(K^{i,\bullet}, d_2^{i,\bullet})$ and $(K^{\bullet,j}, d_1^{\bullet,j})$ are complexes and the diagrams

$$\begin{array}{ccc} K^{i,j+1} & \xrightarrow{d_2^{i,j+1}} & K^{i+1,j+1} \\ d_1^{i,j}\Big\uparrow & & \Big\uparrow d_1^{i+1,j} \\ K^{i,j} & \xrightarrow{d_2^{i,j}} & K^{i+1,j} \end{array}$$

commute for all $i, j \in \mathbb{Z}$. The *associated simple complex* or *total complex* $\mathrm{Tot}(K^{\bullet,\bullet})$ is defined as

$$\mathrm{Tot}(K^{\bullet,\bullet})^n = \bigoplus_{i+j=n} K^{i,j} \ , \quad d^n_{\mathrm{Tot}(K^{\bullet,\bullet})} = \sum_{i+j=n} (d_1^{i,j} + (-1)^j d_2^{i,j}) \ .$$

In order to take the direct sum, either the category has to allow infinite direct sums or we have to assume boundedness conditions to make sure that only finite direct sums occur. This is the case if $K^{i,j} = 0$ unless $i, j \geq 0$.

Examples 1.3.4. 1. Our definition of the cone is a special case: for $f : K^\bullet \to L^\bullet$

$$\mathrm{Cone}(f) = \mathrm{Tot}(\tilde{K}^{\bullet,\bullet}) \ , \qquad \tilde{K}^{\bullet,-1} = K^\bullet, \tilde{K}^{\bullet,0} = L^\bullet \ .$$

2. Another example is given by the tensor product. Given two complexes $(F^\bullet, d_F)$ and $(G^\bullet, d_G)$, the tensor product

$$(F^\bullet \otimes G^\bullet)^n = \bigoplus_{i+j=n} F^i \otimes G^j$$

has a natural structure of a double complex with $K^{i,j} = F^i \otimes G^j$, and the differential is given by $d = \mathrm{id}_F \otimes d_G + (-1)^i d_F \otimes \mathrm{id}_F$.

Remark 1.3.5. There is a choice of signs in the definition of the total complex. See for example [Hu1] §2.2 for a discussion. We use the convention opposite to the one of loc. cit. For most formulae it does matter which choice is used, as long as it is used consistently. However, it does have an asymmetric effect on the formula for the compatibility of cup-products with boundary maps. We spell out the source of this assymmetry.

Lemma 1.3.6. *Let $F^\bullet$, $G^\bullet$ be complexes in an additive tensor category. Then:*

1. $F^\bullet \otimes (G^\bullet[1]) = (F^\bullet \otimes G^\bullet)[1]$.
2. $\epsilon : (F^\bullet[1] \otimes G^\bullet) \to (F^\bullet \otimes G^\bullet)[1]$ *with* $\epsilon = (-1)^j$ *on* $F^i \otimes G^j$ *(in degree* $i + j - 1$*) is an isomorphism of complexes.*

Proof. We compute the differential on $F^i \otimes G^i$ in all three complexes. Note that

$$F^i \otimes G^j = (F[1])^{i-1} \otimes G^j = F^i \otimes (G[1])^{j-1}.$$

For better readability, we drop $\otimes\mathrm{id}$ and $\mathrm{id}\otimes$ and $|_{F^i\otimes G^j}$ everywhere. Hence we have

$$\begin{aligned}
d^{i+j-1}_{(F^\bullet\otimes G^\bullet)[1]} &= -d^{i+j}_{F^\bullet\otimes G^\bullet}\\
&= -\left(d^j_{G^\bullet} + (-1)^j d^i_{F^\bullet}\right)\\
&= -d^j_{G^\bullet} + (-1)^{j-1} d^i_{F^\bullet}\\
d^{i+j-1}_{F^\bullet\otimes(G^\bullet[1])} &= d^{j-1}_{G^\bullet[1]} + (-1)^{j-1} d^i_{F^\bullet}\\
&= -d^j_{G^\bullet} + (-1)^{j-1} d^i_{F^\bullet}\\
d^{i+j-1}_{(F^\bullet[1])\otimes G^\bullet} &= d^j_{G^\bullet} + (-1)^j d^{i-1}_{F^\bullet[1]}\\
&= d^j_{G^\bullet} + (-1)^{j-1} d^i_{F^\bullet}
\end{aligned}$$

We see that the first two complexes agree, whereas the differential of the third is different. Multiplication by $(-1)^j$ on the summand $F^i\otimes G^j$ is a morphism of complexes. □

1.4 Hypercohomology

Let X be a topological space and $\mathrm{Sh}(X)$ the abelian category of sheaves of abelian groups on X.

We want to extend the definition of sheaf cohomology on X, as explained in [Ha2], Chap. III, to complexes of sheaves.

1.4.1 Definition

Definition 1.4.1. Let $\mathcal{F}^\bullet$ be a bounded below complex of sheaves of abelian groups on X. An *injective resolution* of $\mathcal{F}^\bullet$ is a quasi-isomorphism

$$\mathcal{F}^\bullet \to \mathcal{I}^\bullet$$

where $\mathcal{I}^\bullet$ is a bounded below complex with $\mathcal{I}^n$ *injective* for all n, i.e., $\mathrm{Hom}(-,\mathcal{I}^n)$ is exact.

Sheaf cohomology of X with coefficients in $\mathcal{F}^\bullet$ is defined as

$$H^i(X,\mathcal{F}^\bullet) = H^i\left(\Gamma(X,\mathcal{I}^\bullet)\right) \qquad i\in\mathbb{Z}$$

where $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ is an injective resolution.

Remark 1.4.2. In the older literature, it is customary to write $\mathbb{H}^i(X,\mathcal{F}^\bullet)$ instead of $H^i(X,\mathcal{F}^\bullet)$ and call it *hypercohomology.* We do not see any need to distinguish. Note that in the special case $\mathcal{F}^\bullet = \mathcal{F}[0]$ a sheaf viewed as a complex concentrated in degree 0, the notion of an injective resolution in the above sense agrees with the ordinary one in homological algebra.

Remark 1.4.3. In the language of derived categories, we have

$$H^i(X, \mathcal{F}^\bullet) = \mathrm{Hom}_{D^+(\mathrm{Sh}(X))}(\mathbb{Z}, \mathcal{F}^\bullet[i])$$

because $\Gamma(X, \cdot) = \mathrm{Hom}_{\mathrm{Sh}(X)}(\mathbb{Z}, \cdot)$.

Lemma 1.4.4. *$H^i(X, \mathcal{F}^\bullet)$ is well-defined and functorial in $\mathcal{F}^\bullet$.*

Proof. We first need existence of injective resolutions. Recall that the category $\mathrm{Sh}(X)$ has enough injectives. Hence every sheaf has an injective resolution. This extends to bounded below complexes in $\mathcal{A}$ by [We] Lemma 5.7.2 (or rather, its analogue for injective rather than projective objects).

Let $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ and $\mathcal{G}^\bullet \to \mathcal{J}^\bullet$ be injective resolutions. By loc.cit. Theorem 10.4.8

$$\mathrm{Hom}_{D^+(\mathrm{Sh}(X))}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \mathrm{Hom}_{K^+(\mathrm{Sh}(X))}(\mathcal{I}^\bullet, \mathcal{J}^\bullet).$$

This means in particular that every morphism of complexes lifts to a morphism of injective resolutions and that the lift is unique up to homotopy of complexes. Hence the induced maps

$$H^i(\Gamma(X, \mathcal{I}^\bullet)) \to H^i(\Gamma(X, \mathcal{J}^\bullet))$$

agree. This implies that $H^i(X, \mathcal{F}^\bullet)$ is well-defined and a functor. $\square$

Remark 1.4.5. Injective sheaves are abundant (by our general assumption that there are enough injectives), but not suitable for computations. Every injective sheaf $\mathcal{F}$ is *flasque* [Ha1, III. Lemma 2.4], i.e., the restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$ between non-empty open sets $V \subset U$ are always surjective. Below we will introduce the canonical flasque Godemont resolution for any sheaf $\mathcal{F}$. More generally, every flasque sheaf $\mathcal{F}$ is *acyclic*, i.e., $H^i(X, \mathcal{F}) = 0$ for $i > 0$. One may compute sheaf cohomology of $\mathcal{F}$ using any acyclic resolution $F^\bullet$. This follows from the hypercohomology spectral sequence

$$E_2^{p,q} = H^p(H^q(F^\bullet)) \Rightarrow H^{p+q}(X, \mathcal{F})$$

which is supported entirely on the $q = 0$-line.

Special acylic resolutions on X are the so-called *fine* resolutions. See [Wa, pg. 170] for a definition of fine sheaves involving partitions of unity. Their importance comes from the fact that sheaves of $\mathcal{C}^\infty$-functions and sheaves of $\mathcal{C}^\infty$-differential forms on X are fine sheaves.

1.4.2 Godement resolutions

For many purposes, it is useful to have functorial resolutions of sheaves. One such is given by the Godement resolution introduced in [God] chapter II §3.

Let X be a topological space. Recall that a sheaf on X vanishes if and only the stalks at all $x \in X$ vanish. For all $x \in X$ we denote $i_x : x \to X$ the natural inclusion.

Definition 1.4.6. Let $\mathcal{F} \in \mathrm{Sh}(X)$. Put

$$I(\mathcal{F}) = \prod_{x \in X} i_{x*}\mathcal{F}_x \ .$$

Inductively, we define the *Godement resolution* $Gd^\bullet(\mathcal{F})$ of $\mathcal{F}$ by

$$\begin{aligned} Gd^0(\mathcal{F}) &= I(\mathcal{F}) \ , \\ Gd^1(\mathcal{F}) &= I(\mathrm{Coker}(\mathcal{F} \to Gd^0(\mathcal{F}))) \ , \\ Gd^{n+1}(\mathcal{F}) &= I(\mathrm{Coker}(Gd^{n-1}(\mathcal{F}) \to Gd^n(\mathcal{F}))) \quad n > 0. \end{aligned}$$

Lemma 1.4.7. 1. *Gd is an exact functor with values in $C^+(\mathrm{Sh}(X))$.*

2. *The natural map $\mathcal{F} \to Gd^\bullet(\mathcal{F})$ is a flasque resolution.*

Proof. Functoriality is obvious. The sheaf $I(\mathcal{F})$ is given by

$$U \mapsto \prod_{x \in U} i_{x*}\mathcal{F}_x \ .$$

All the sheaves involved are flasque, hence acyclic. In particular, taking the direct products is exact (it is not in general), turning $I(\mathcal{F})$ into an exact functor. $\mathcal{F} \to I(\mathcal{F})$ is injective, and hence by construction, $Gd^\bullet(\mathcal{F})$ is then a flasque resolution. □

Definition 1.4.8. Let $\mathcal{F}^\bullet \in C^+(\mathrm{Sh}(X))$ be a complex of sheaves. We call

$$Gd(\mathcal{F}^\bullet) := \mathrm{Tot}(Gd^\bullet(\mathcal{F}^\bullet))$$

the *Godement resolution* of $\mathcal{F}^\bullet$.

Corollary 1.4.9. *The natural map*

$$\mathcal{F} \to Gd(\mathcal{F}^\bullet)$$

is a quasi-isomorphism and

$$H^i(X, \mathcal{F}^\bullet) = H^i\left(\Gamma(X, Gd(\mathcal{F}^\bullet))\right) \ .$$

Proof. By Lemma 1.4.7, the first assertion holds if $\mathcal{F}^\bullet$ is concentrated in a single degree. The general case follows by the hypercohomology spectral sequence or by induction on the length of the complex using the stupid filtration.

All terms in $Gd(\mathcal{F}^\bullet)$ are flasque, hence acyclic for $\Gamma(X, \cdot)$. □

We now study functoriality of the Godement resolution. For a continuous map $f : X \to Y$ be denote f^{-1} the pull-back functor on sheaves of abelian groups. Recall that it is exact.

Lemma 1.4.10. *Let $f : X \to Y$ be a continuous map between topological spaces, $\mathcal{F}^\bullet \in C^+(\mathrm{Sh}(Y))$. Then there is a natural quasi-isomorphism*

$$f^{-1}Gd_Y(\mathcal{F}^\bullet) \to Gd_X(f^{-1}\mathcal{F}^\bullet) \ .$$

Proof. Consider a sheaf $\mathcal{F}$ on Y. We want to construct

$$f^{-1}I(\mathcal{F}) \to I(f^{-1}\mathcal{F}) = \prod_{x\in X} i_{x*}(f^{-1}\mathcal{F})_x = \prod_{x\in X} i_{x*}\mathcal{F}_{f(x)} \ .$$

By the universal property of the direct product and adjunction for f^{-1}, this is equivalent to specifying for all $x \in X$

$$\prod_{y\in Y} i_{y*}\mathcal{F}_y = I(\mathcal{F}) \to f_* i_{x*}\mathcal{F}_{f(x)} = i_{f(x)*}\mathcal{F}_{f(x)} \ .$$

We use the natural projection map. By construction, we have a natural commutative diagram

$$\begin{array}{ccccc} f^{-1}\mathcal{F} & \longrightarrow & f^{-1}I(\mathcal{F}) & \longrightarrow & \mathrm{Coker}\left(f^{-1}\mathcal{F} \to f^{-1}I(\mathcal{F})\right) \\ {\scriptstyle =}\downarrow & & \downarrow & & \\ f^{-1}\mathcal{F} & \longrightarrow & I(f^{-1}\mathcal{F}) & \longrightarrow & \mathrm{Coker}\left(f^{-1}\mathcal{F} \to I(f^{-1}\mathcal{F})\right) \end{array}$$

It induces a map between the cokernels. Proceeding inductively, we obtain a morphism of complexes

$$f^{-1}Gd^\bullet_Y(\mathcal{F}) \to Gd^\bullet_X(f^{-1}\mathcal{F}) \ .$$

It is a quasi-isomorphism because both are resolutions of $f^{-1}\mathcal{F}$. This transformation of functors extends to double complexes and hence defines a transformation of functors on $C^+(\mathrm{Sh}(Y))$. $\square$

Remark 1.4.11. We are going to apply the theory of Godement resolutions in the case where X is a variety over a field k, a complex manifold or more generally a complex analytic space. The continuous maps that we need to consider are those in these categories, but also the maps of schemes $X_K \to X_k$ for the change of base field K/k and a variety over k; and the continuous map $X^{\mathrm{an}} \to X$ for an algebraic variety over $\mathbb{C}$ and its analytification.

1.4.3 Čech cohomology

Neither the definition of sheaf cohomology via injective resolutions nor Godement resolutions are convenient for concrete computations. We introduce Čech cohomology for this task. We follow [Ha2], Chap. III §4, but extend to hypercohomology.

Let k be a field. We work in the category of varieties over k. Let $I = \{1, \dots, n\}$ as ordered set and $\mathfrak{U} = \{U_i | i \in I\}$ an affine open cover of X. For any subset $J \subset \{1, \dots, n\}$ we denote

$$U_J = \bigcap_{j \in J} U_j \ .$$

As X is separated, they are all affine.

Definition 1.4.12. Let X and $\mathfrak{U}$ be as above. Let $\mathcal{F} \in \mathrm{Sh}(X)$. We define the *Čech complex* of $\mathcal{F}$ as

$$C^p(\mathfrak{U}, \mathcal{F}) = \prod_{J \subset I, |J| = p+1} \mathcal{F}(U_J) \qquad p \geq 0$$

with differential $\delta^p : C^p(\mathfrak{U}, \mathcal{F}) \to C^{p+1}(\mathfrak{U}, \mathcal{F})$

$$(\delta^p \alpha)_{i_0 < i_1 < \cdots < i_p} = \sum_{j=0}^{p+1} (-1)^p \alpha_{i_0 \cdots < \hat{i}_j < \cdots < i_{p+1}} |_{U_{i_0 \cdots i_{p+1}}} \ ,$$

where, as usual, $i_0 \cdots < \hat{i}_j < \cdots < i_{p+1}$ means the tuple with $\hat{i}_j$ removed.

We define the p-th *Čech cohomology* of X with coefficients in $\mathcal{F}$ as

$$\check{H}^p(\mathfrak{U}, \mathcal{F}) = H^p(C^\bullet(\mathfrak{U}, \mathcal{F}), \delta) \ .$$

Proposition 1.4.13 ([Ha2], chap. III Theorem 4.5)**.** *Let X be a variety, $\mathfrak{U}$ an affine open cover as before. Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules on X. Then there is a natural isomorphism*

$$H^p(X, \mathcal{F}) = \check{H}^p(\mathfrak{U}, \mathcal{F}) \ .$$

We now extend to complexes. We can apply the functor $C^\bullet(\mathfrak{U}, \cdot)$ to all terms in a complex $\mathcal{F}^\bullet$ and obtain a double complex $C^\bullet(\mathfrak{U}, \mathcal{F}^\bullet)$.

Definition 1.4.14. Let X and $\mathfrak{U}$ as before. Let $\mathcal{F}^\bullet \in C^+(\mathrm{Sh}(X))$. We define the *Čech complex* of $\mathfrak{U}$ with coefficients in $\mathcal{F}^\bullet$ as

$$C^\bullet(\mathfrak{U}, \mathcal{F}^\bullet) = \mathrm{Tot}\,(C^\bullet(\mathfrak{U}, \mathcal{F}^\bullet))$$

and *Čech cohomology* as

$$\check{H}^p(\mathfrak{U}, \mathcal{F}) = H^p(C^\bullet(\mathfrak{U}, \mathcal{F}^\bullet)) \ .$$

Proposition 1.4.15. *Let X be a variety, $\mathfrak{U}$ as before an open affine cover of X. Let $\mathcal{F}^\bullet \in C^+(\mathrm{Sh}(X))$ be complex such that all $\mathcal{F}^n$ are coherent sheaves of $\mathcal{O}_X$-modules. Then there is a natural isomorphism*

$$H^p(X, \mathcal{F}) \to \check{H}^p(\mathfrak{U}, \mathcal{F}^\bullet) \ .$$

Proof. The essential step is to define the map. We first consider a single sheaf $\mathcal{G}$. Let $\mathcal{C}^\bullet(\mathfrak{U},\mathcal{G})$ be a sheafified version of the Čech complex for a sheaf $\mathcal{G}$. By [Ha2], chap. III Lemma 4.2, it is a resolution of $\mathcal{G}$. We apply the Godement resolution and obtain a flasque resolution of $\mathcal{G}$ by

$$\mathcal{G} \to \mathcal{C}^\bullet(\mathfrak{U},\mathcal{G}) \to Gd\,(\mathcal{C}^\bullet(\mathfrak{U},\mathcal{G}))\ .$$

By Proposition 1.4.13, the induced map

$$C^\bullet(\mathfrak{U},\mathcal{G}) \to \Gamma(X, Gd\,(\mathcal{C}^\bullet(\mathfrak{U},\mathcal{G}))$$

is a quasi-isomorphism as both compute $H^i(X,\mathcal{G})$.

The construction is functorial in $\mathcal{G}$, hence we can apply it to all components of a complex $\mathcal{F}^\bullet$ and obtain double complexes. We use the previous results for all $\mathcal{F}^n$ and take total complexes. Hence

$$\mathcal{F}^\bullet \to \mathrm{Tot}\mathcal{C}^\bullet(\mathfrak{U},\mathcal{F}^\bullet) \to Gd\,(\mathcal{C}^\bullet(\mathfrak{U},\mathcal{F}^\bullet))$$

are quasi-isomorphisms. Taking global sections we get a quasi-isomorphism

$$\mathrm{Tot}C^\bullet(\mathfrak{U},\mathcal{F}^\bullet) \to \mathrm{Tot}\Gamma(X, Gd\,(\mathcal{C}^\bullet(\mathfrak{U},\mathcal{F}^\bullet)))\ .$$

By definition, the complex on the left computes Čech cohomology of $\mathcal{F}^\bullet$ and the complex on right computes hypercohomology of $\mathcal{F}^\bullet$. □

Corollary 1.4.16. *Let X be an affine variety and $\mathcal{F}^\bullet \in C^+(\mathrm{Sh}(X))$ such that all $\mathcal{F}^n$ are coherent sheaves of $\mathcal{O}_X$-modules. Then*

$$H^i(\Gamma(X,\mathcal{F}^\bullet)) = H^i(X,\mathcal{F}^\bullet)\ .$$

Proof. We use the affine covering $\mathfrak{U} = \{X\}$ and apply Proposition 1.4.15. □

1.5 Simplicial objects

We introduce simplicial varieties in order to carry out some of the constructions in [D5]. Good general references on the topic are [May] or [We] Ch. 8.

Definition 1.5.1. Let Δ be the category whose objects are finite ordered sets

$$[n] = \{0, 1, \ldots, n\} \quad n \in \mathbb{N}_0$$

with morphisms nondecreasing monotone maps. Let Δ_N be the full subcategory with objects the $[n]$ with $n \le N$.

If $\mathcal{C}$ is a category, we denote by $\mathcal{C}^\Delta$ the *category of simplicial objects* in $\mathcal{C}$ defined as contravariant functors

$$X_\bullet : \Delta \to \mathcal{C}$$

with transformation of functors as morphisms. We denote by $\mathcal{C}^{\Delta^\circ}$ the *category of cosimplicial objects* in $\mathcal{C}$ defined as covariant functors

$$X^\bullet : \Delta \to \mathcal{C} \ .$$

Similarly, we defined the categories $\mathcal{C}^{\Delta_N}$ and $\mathcal{C}^{\Delta_N^\circ}$ of *N-truncated simplicial and cosimplicial objects.*

Example 1.5.2. Let X be an object of $\mathcal{C}$. The constant functor

$$\Delta^\circ \to \mathcal{C}$$

which maps all objects to X and all morphism to the identity morphism is a simplicial object. It is called the *constant simplicial object* associated to X.

In Δ, we have in particular the *face maps*

$$\epsilon_i : [n-1] \to [n] \quad i = 0, \dots, n,$$

the unique injective map leaving out the index i, and the *degeneracy maps*

$$\eta_i : [n+1] \to [n] \quad i = 0, \dots, n,$$

the unique surjective map with two elements mapping to i. More generally, a map in Δ is called *face* or *degeneracy* if it is a composition of ϵ_i or η_i, respectively. Any morphism in Δ can be decomposed into a degeneracy followed by a face ([We] Lemma 8.12).

For all $m \geq n$, we denote $S_{m,n}$ the set of all degeneracy maps $[m] \to [n]$.

A simplicial object $X_\bullet$ is determined by a sequence of objects

$$X_0, X_1, \dots$$

and face and degeneracy morphisms between them. In particular, we write

$$\partial_i : X_n \to X_{n-1}$$

for the image of ϵ_i and

$$s_i : X_n \to X_{n+1}$$

for the image of η_i.

Example 1.5.3. For every n, there is a simplicial set $\Delta[n]$ with

$$\Delta[n]_m = \mathrm{Mor}_\Delta([m], [n])$$

and the natural face and degeneracy maps. It is called the *simplicial n-simplex.* For $n = 0$, this is the *simplicial point*, and for $n = 1$ the *simplicial interval.* Functoriality in the first argument induces maps of simplicial sets. In particular, there are

$$\delta_0 = \epsilon_0^*, \delta_1 = \epsilon_1^* : \Delta[1] \to \Delta[0] \ .$$

Definition 1.5.4. Let $\mathcal{C}$ be a category with finite products and coproducts. Let $\star$ be the final object. Let $X_\bullet$, $Y_\bullet$ simplicial objects in $\mathcal{C}$ and $S_\bullet$ a simplicial set

1. $X_\bullet \times Y_\bullet$ is the simplicial object with
$$(X_\bullet \times Y_\bullet)_n = X_n \times Y_n$$
with face and degeneracy maps induced from $X_\bullet$ and $Y_\bullet$.
2. $X_\bullet \times S_\bullet$ is the simplicial object with
$$(X_\bullet \times S_\bullet)_n = \coprod_{s\in S_n} X_n$$
with face and degeneracy maps induced from $X_\bullet$ and $S_\bullet$.
3. Let $f, g : X_\bullet \to Y_\bullet$ be morphisms of simplicial objects. Then f is called *homotopic* to g if there is a morphism
$$h : X_\bullet \times \Delta[1] \to Y_\bullet$$
such that $h \circ \delta_0 = f$ and $h \circ \delta_1 = g$.

The inclusion $\Delta_N \to \Delta$ induces a natural restriction functor
$$\mathrm{sq}_N : \mathcal{C}^\Delta \to \mathcal{C}^{\Delta_N} .$$
For a simplicial object $X_\bullet$, we call $\mathrm{sq}_N X_\bullet$ its *N-skeleton.* If $Y_\bullet$ is a fixed simplicial objects, we also denote sq_N the restriction functor from simplicial objects over $Y_\bullet$ to simplicial objects over $\mathrm{sq}_N Y_\bullet$.

Remark 1.5.5. The skeleta $\mathrm{sq}_k X_\bullet$ define the *skeleton filtration*, i.e., a chain of maps
$$\mathrm{sq}_0 X_\bullet \to \mathrm{sq}_1 X_\bullet \to \cdots \to \mathrm{sq}_N X_\bullet = X_\bullet.$$
Later, in section 2.3, we will define the topological realization $|X_\bullet|$ of a simplicial set $X_\bullet$. The skeleton filtration then defines a filtration of $|X_\bullet|$ by closed subspaces.

An important example in this book is the case when the simplicial set $X_\bullet$ is a finite set, i.e., all X_n are finite sets, and empty for $n > N$ sufficiently large. See section 2.3.

Lemma 1.5.6. *Let $\mathcal{C}$ be a category with finite limits. Then the functor sq_N has a right adjoint*
$$\mathrm{cosq}_N : \mathcal{C}^{\Delta_N} \to \mathcal{C}^\Delta .$$
If $Y_\bullet$ is a fixed simplicial object, then
$$\mathrm{cosq}_N^{Y_\bullet}(X_\bullet) = \mathrm{cosq}_N X_\times \times_{\mathrm{cosq}_N \mathrm{sq}_N Y_\bullet} Y_\bullet$$
is the right adjoint of the relative version of sq_N.

Proof. The existence of cosq_N is an instance of a Kan extension. We refer to [ML, chap. X] or [AM, chap. 2] for its existence. The relative case follows from the universal properties of fibre products. □

If $X_\bullet$ is an N-truncated simplicial object, we call $\mathrm{cosq}_N X_\bullet$ its *coskeleton*.

Remark 1.5.7. We apply this in particular to the case where $\mathcal{C}$ is one of the categories Var, Sm or Aff over a fixed field k. The disjoint union of varieties is a coproduct in these categories and the direct product a product.

Definition 1.5.8. Let S be a class of covering maps of varieties containing all identity morphisms. A morphism $f : X_\bullet \to Y_\bullet$ of simplicial varieties (or the simplicial variety $X_\bullet$ itself) is called an *S-hypercovering* if the adjunction morphisms

$$X_n \to (\mathrm{cosq}_{n-1}^{Y_\bullet} \mathrm{sq}_{n-1} X_\bullet)_n$$

are in S.

If S is the class of proper, surjective morphisms, we call $X_\bullet$ a *proper hypercover* of $Y_\bullet$.

Definition 1.5.9. Let $X_\bullet$ be a simplical variety. It is called *split* if for all $n \in \mathbb{N}_0$

$$N(X_n) = X_n \smallsetminus \bigcup_{i=0}^{n-1} s_i(X_{n-1})$$

is an open and closed subvariety of X_n.

We call $N(X_n)$ the non-degenerate part of X_n. If $X_\bullet$ is a split simplicial variety, we have a decomposition as varieties

$$X_n = N(X_n) \amalg \coprod_{m<n} \coprod_{s \in S_{m,n}} sN(X_m)$$

where $S_{m,n}$ is the set of degeneracy maps from X_m to X_n.

Theorem 1.5.10 (Deligne)**.** *Let k be a field and Y a variety over k. Then there is a split simplicial variety $X_\bullet$ with all X_n smooth and a proper hypercover $X_\bullet \to Y$.*

Proof. The construction is given in [D5] Section (6.2.5). It depends on the existence of resolutions of singularities. In positive characteristic, we may use de Jong's result on alterations instead. □

The other case we are going to need is the case of additive categories.

Definition 1.5.11. Let $\mathcal{A}$ be an additive category. We define a functor

$$C : \mathcal{A}^\Delta \to C^-(\mathcal{A})$$

by mapping a simplicial object $X_\bullet$ to the cohomological complex

$$\dots X_{-n} \xrightarrow{d^{-n}} X_{-(n-1)} \to \dots \to X_0 \to 0$$

with differential

$$d^{-n} = \sum_{i=0}^{n} (-1)^i \partial_i \ .$$

We define a functor

$$C : \mathcal{A}^{\Delta^\circ} \to C^+(\mathcal{A})$$

by mapping a cosimplicial object $X^\bullet$ to the cohomological complex

$$0 \to X^0 \to \dots X^n \xrightarrow{d^n} X_{n+1} \to \dots$$

with differential

$$d^n = \sum_{i=0}^{n} (-1)^i \partial_i \ .$$

Let $\mathcal{A}$ be an abelian category. We define a functor

$$N : \mathcal{A}^{\Delta^\circ} \to C^+(\mathcal{A})$$

by mapping a cosimplicial object $X^\bullet$ to the *normalized complex* $N(X^\bullet)$ with

$$N(X^\bullet)_n = \bigcap_{i=0}^{n-1} \mathrm{Ker}(s_i : X^n \to X^{n-1})$$

and differential $d^n|_{N(X^\bullet)}$.

Proposition 1.5.12 (Dold-Kan correspondence)**.** *Let $\mathcal{A}$ be an abelian category, $X^\bullet \in \mathcal{A}^{\Delta^\circ}$ a cosimplicial object. Then the natural map*

$$N(X^\bullet) \to C(X^\bullet)$$

is a quasi-isomorphism.

Proof. This is the dual result of [We], Theorem 8.3.8. □

Remark 1.5.13. We are going to apply this in the case of cosimplicial complexes, i.e., to $C(\mathcal{A})^{\Delta^\circ}$, where $\mathcal{A}$ is abelian, e.g., a category of vector spaces.

1.6 Grothendieck topologies

Grothendieck topologies generalize the notion of open subsets in topological spaces. Using them one can define cohomology theories in more abstract settings. To define a Grothendieck topology, we need the notion of a *site, or situs.* Let $\mathcal{C}$ be a category. A basis for a Grothendieck topology on $\mathcal{C}$ is given by *covering families* in the category $\mathcal{C}$ satisfying the following definition.

Definition 1.6.1. A *site/situs* is a category $\mathcal{C}$ together with a collection of morphism in $\mathcal{C}$

$$(\varphi_i : V_i \longrightarrow U)_{i \in I},$$

the *covering families.*

The covering families satisfy the following axioms:

- An isomorphism $\varphi : V \to U$ is a covering family with an index set containing only one element.

- If $(\varphi_i : V_i \longrightarrow U)_{i \in I}$ is a covering family, and $f : V \to U$ a morphism in $\mathcal{C}$, then for each $i \in I$ there exists the pullback diagram

$$\begin{array}{ccc} V \times_U V_i & \xrightarrow{F_i} & V_i \\ {\scriptstyle \Phi_i}\big\downarrow & & \big\downarrow{\scriptstyle \varphi_i} \\ V & \xrightarrow{f} & U \end{array}$$

  in $\mathcal{C}$, and $(\Phi_i : V \times_U V_i \to V)_{i \in I}$ is a covering family of V.

- If $(\varphi_i : V_i \longrightarrow U)_{i \in I}$ is a covering family of U, and for each V_i there is given a covering family $\left(\varphi^i_j : V^i_j \to V_i\right)_{j \in J(i)}$, then

$$\left(\varphi_i \circ \varphi^i_j : V^i_j \to U\right)_{i \in I, j \in J(i)}$$

  is a covering family of U.

Example 1.6.2. Let X be a topological space. Then the category of open sets in X together with inclusions as morphisms form a site, where the covering maps are the families $(U_i)_{i \in I}$ of open subsets of U such that $\cup_{i \in I} U_i = U$. Thus each topological space defines a canonical site. For the Zariski open subsets of a scheme X this is called the *(small) Zariski site* of X.

Definition 1.6.3. A *presheaf* $\mathcal{F}$ of abelian groups on a situs $\mathcal{C}$ is a contravariant functor

$$\mathcal{F} : \mathcal{C} \to \mathrm{Ab}, U \mapsto \mathcal{F}(U).$$

A presheaf $\mathcal{F}$ is a *sheaf*, if for each covering family $(\varphi_i : V_i \longrightarrow U)_{i \in I}$, the difference kernel sequence

$$0 \to \mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(V_i) \rightrightarrows \prod_{(i,j) \in I \times I} \mathcal{F}(V_i \times_U V_j)$$

is exact. This means that a section $s \in \mathcal{F}(U)$ is determined by its restrictions to each V_i, and a tuple $(s_i)_{i \in I}$ of sections comes from a section on U, if one has $s_i = s_j$ on pullbacks to the fiber product $V_i \times_U V_j$.

Once we have a notion of sheaves in a certain Grothendieck topology, then we may define cohomology groups $H^*(X, \mathcal{F})$ by using injective resolutions as in section 1.4 as the right derived functor of the left-exact global section functor $X \mapsto \mathcal{F}(X) = H^0(X, \mathcal{F})$ in the presence of enough injectives.

Example 1.6.4. The *(small) étale site* over a smooth variety X consists of the category of all étale morphisms $\varphi : U \to X$ from a smooth variety U to X. See [Ha2, Chap. III] for the notion of étale maps. We just note here that étale maps are quasi-finite, i.e., have finite fibers, are unramified, and the image $\varphi(U) \subset X$ is a Zariski open subset.

A morphism in this site is given by a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & U \\ \downarrow & & \downarrow \\ X & \xrightarrow{\text{id}} & X. \end{array}$$

Let U be étale over X. A family $(\varphi_i : V_i \longrightarrow U)_{i \in I}$ of étale maps over X is called a covering family of U, if $\bigcup_{i \in I} \varphi_i(V_i) = U$, i.e., the images form a Zariski open covering of U.

This category has enough injectives by Grothendieck [SGA4.2], and thus we can define étale cohomology $H^*_{\text{et}}(X, \mathcal{F})$ for étale sheaves $\mathcal{F}$.

Example 1.6.5. In Section 2.7 we are going to introduce the h′-topology on the category of analytic spaces.

Definition 1.6.6. Let $\mathcal{C}$ and $\mathcal{C}'$ be sites. A *morphism of sites* $f : \mathcal{C} \to \mathcal{C}'$ consists of a functor $F : \mathcal{C}' \to \mathcal{C}$ (sic) which preserves fibre products and such that the F applied to a covering family of $\mathcal{C}'$ yields a covering family of $\mathcal{C}$.

A morphism of sites induces an adjoint pair of functors (f^*, f_*) between sheaves of sets on $\mathcal{C}$ and $\mathcal{C}'$.

Example 1.6.7.

1. Let $f : X \to Y$ be continuous map of topological spaces. As in Example 1.6.2 we view them as sites. Then the functor F mapping an open subset of Y to its preimage $f^{-1}(U)$.
2. Let X be a scheme. Then there is morphism of sites from the small étale site of X to the Zariski-site of X. The functor views an open subscheme $U \subset X$ as an étale X-scheme. Open covers are in particular étale covers.

Definition 1.6.8. Let $\mathcal{C}$ be a site. A $\mathcal{C}$-hypercover is an S-hypercover in the sense of Definition 1.5.8 with S the class of morphism

$$\coprod_{i \in I} U_i \to U$$

for all covering families $\{\phi_i : U_i \to U\}_{i \in I}$ in the site.

If $X_\bullet$ is a simplical object and $\mathcal{F}$ is a presheaf of abelian groups, then $\mathcal{F}(X_\bullet)$ is a cosimplicial abelian group. By applying the total complex functor C of Definition 1.5.11, we get a complex of abelian groups.

Proposition 1.6.9. *Let $\mathcal{C}$ be a site closed under finite products and fibre products, $\mathcal{F}$ a sheaf of abelian groups on $\mathcal{C}$, $X \in \mathcal{C}$. Then*

$$H^i(X, \mathcal{F}) = \lim_{X_\bullet \to X} H^i\left(C(\mathcal{F}(X_\bullet))\right)$$

where the direct limit runs through the system of all $\mathcal{C}$-hypercovers of X.

Proof. This is [SGA4V, Théorème 7.4.1] □

1.7 Torsors

Informally, a torsor is a group without a unit. The standard notion in algebraic geometry is sheaf theoretic: A torsor under a sheaf of groups G is a sheaf of sets X with an operation $G \times X \to X$ such that there is a cover over which X becomes isomorphic to G and the action becomes the group operation. This makes sense in any site.

In this chapter, we are going to discuss a variant of this idea which does not involve sites or topologies but rather schemes. This approach fits well with the Tannaka formalism that we have discussed in previous chapters.

It is used by Kontsevich in [K]. This notion at least goes back to a paper of R. Baer [Ba] from 1929, see the footnote on page 202 of loc. cit. where Baer explains how the notion of a torsor comes up in the context of earlier work of H. Prüfer [Pr]. In yet another context, ternary operations satisfying these axioms are called associative Malcev operations, see [Joh] for a short account.

1.7.1 Sheaf theoretic definition

Definition 1.7.1. Let $\mathcal{C}$ be a category equipped with a Grothendieck topology t. Assume S is a final object of $\mathcal{C}$. Let G be a group object in $\mathcal{C}$. A *(left) G-torsor* is an object X with a (left) operation

$$\mu : G \times X \to X$$

such that there is a t-covering $U \to S$ such that restriction of G and X to U is a trivial, i.e., $X(U)$ is non-empty, and the choice $x \in X(U)$ induces a natural isomorphism

$$\begin{aligned} \cdot x : G(U') &\to X(U') \\ g &\mapsto \mu(g, x). \end{aligned}$$

for all $U' \to U$.

The condition can also be formulated as an isomorphism

$$G \times U \to X \times U$$
$$(g, u) \mapsto g(u), u)$$

Remark 1.7.2. 1. As μ is an operation, the isomorphism of the definition is compatible with the operation as well, i.e., the diagram

$$\begin{tikzcd} G(U') \times X(U') \arrow[r, "\mu"] & X(U') \\ G(U') \times G(U') \arrow[u, "(\mathrm{id}, \cdot x)"] \arrow[r] & G(U') \arrow[u, "\cdot x"'] \end{tikzcd}$$

commutes.

2. If, moreover, $X \to S$ is a t-cover, then $X(X)$ is always non-empty and we recover a version of the definition that often appears in the literature, namely that
$$G \times X \to X \times X$$
has to be an isomorphism.

We are interested in the topology that is in use in Tannaka theory. It is basically the flat topology, but we have to be careful what we mean by this because the schemes involved are not of finite type over the base.

Definition 1.7.3. Let S be an affine scheme and $\mathcal{C}$ the category of affine S-schemes. The *fpqc*-topology on $\mathcal{C}$ is generated by covers of the form $X \to Y$ with $\mathcal{O}(X)$ faithfully flat over $\mathcal{O}(Y)$.

The letters *fpqc* stand for fidèlement plat quasi-compact. Recall that Spec$A$ is quasi-compact for all rings A.

We do not discuss the non-affine case at all, but see the survey [Vis] by Vistoli for the general case. The topology is discussed in loc. cit. Section 2.3.2. The above formulation follows from loc. cit. Lemma 2.60.

Remark 1.7.4. If, moreover, $S = \mathrm{Spec}(k)$ is the spectrum of a field, then any non-trivial Spec$A \to Spec(k)$ is an *fpqc*-cover. Hence, we are in the situation of Remark 1.7.2. Note that X still has to be non-empty!

The importance of the *fpqc*-topology is that all representable presheaves are *fpqc*-sheaves, see [Vis, Theorem 2.55].

1.7.2 Torsors in the category of sets

Definition 1.7.5 ([Ba] p. 202, [K] p. 61, [Fr] Definition 7.2.1)**.** A *torsor* is a set X together with a map

$$(\cdot, \cdot, \cdot) : X \times X \times X \to X$$

satisfying:

1. $(x,y,y) = (y,y,x) = x$ for all $x, y \in X$
2. $((x,y,z),u,v) = (x,(u,z,y),v) = (x,y,(z,u,v))$ for all $x,y,z,u,v \in X$.

Morphisms are defined in the obvious way, i.e., maps $X \to X'$ of sets commuting with the torsor structure.

Lemma 1.7.6. *Let G be a group. Then $(g,h,k) = gh^{-1}k$ defines a torsor structure on G.*

Proof. This is a direct computation:

$$\begin{aligned}
(x,y,y) &= xy^{-1}y = x = yy^{-1}x = (y,y,x),\\
((x,y,z),u,v) &= (xy^{-1}z,u,v) = xy^{-1}zu^{-1}v = (x,y,zu^{-1}v) = (x,y,(z,u,v)),\\
(x,(u,z,y),v) &= (x,uz^{-1}y,v) = x(uz^{-1}y)^{-1}v) = xy^{-1}zu^{-1}v.
\end{aligned}$$

□

Lemma 1.7.7 ([Ba] page 202)**.** *Let X be a torsor, $e \in X$ an element. Then $G_e := X$ carries a group structure via*

$$gh := (g,e,h), \quad g^{-1} := (e,g,e).$$

Moreover, the torsor structure on X is given by the formula $(g,h,k) = gh^{-1}k$ in G_e.

Proof. First we show associativity:

$$(gh)k = (g,e,h)k = ((g,e,h),e,k) = (g,e,(h,e,k)) = g(h,e,k) = g(hk).$$

e becomes the neutral element:

$$eg = (e,e,g) = g; ge = (g,e,e) = g.$$

We also have to show that g^{-1} is indeed the inverse element:

$$gg^{-1} = g(e,g,e) = (g,e,(e,g,e)) = ((g,e,e),g,e) = (g,g,e) = e.$$

Similarly one shows that $g^{-1}g = e$. One gets the torsor structure back, since

$$\begin{aligned}
gh^{-1}k &= g(e,h,e)k = (g,e,(e,h,e))k = ((g,e,(e,h,e)),e,k)\\
&= (g,(e,(e,h,e),e),k) = (g,((e,e,h),e,e),k)\\
&= (g,(h,e,e),k) = (g,h,k).
\end{aligned}$$

□

Proposition 1.7.8. *Let $\mu_l : X^2 \times X^2 \to X^2$ be given by*

$$\mu_l\left((a,b),(c,d)\right) = ((a,b,c),d).$$

Then μ_l is associative and has (x,x) for $x \in X$ as left-neutral elements. Let $G^l = X^2/\sim_l$ where $(a,b) \sim_l (a,b)(x,x)$ for all $x \in X$ is an equivalence relation. Then μ_l is well-defined on G^l and turns G^l into a group. Moreover, the torsor structure map factors via a simply transitive left G^l-operation on X which is defined by

$$(a,b)x := (a,b,x).$$

Let $e \in X$. Then

$$i_e : G_e \to G^l, \qquad x \mapsto (x,e)$$

is group isomorphism inverse to $(a,b) \mapsto (a,b,e)$.
In a similar way, using $\mu_r\left((a,b),(c,d)\right) := (a,(b,c,d))$ we obtain a group G^r with analogous properties acting transitively on the right on X and such that μ_r factors through the action $X \times G^r \to X$.

Proof. First we check associativity of μ_l:

$$(a,b)[(c,d)(e,f)] = (a,b)((c,d,e),f) = ((a,b,(c,d,e)),f) = (((a,b,c),d,e),f)$$
$$[(a,b)(c,d)](e,f) = ((a,b,c),d)(e,f) = (((a,b,c),d,e),f)$$

(x,x) is a left neutral element for every $x \in X$:

$$(x,x)(a,b) = ((x,x,a),b) = (a,b)$$

We also need to check that $\sim_l$ is an equivalence relation: $\sim_l$ is reflexive, since one has $(a,b) = ((a,b,b),b) = (a,b)(b,b)$ by the first torsor axiom and the definition of μ. For symmetry, assume $(c,d) = (a,b)(x,x)$. Then

$$\begin{aligned}(a,b) &= ((a,b,b),b) = ((a,b,(x,x,b)),b) = (((a,b,x),x,b),b)\\ &= ((a,b,x),x)(b,b) = (a,b)(x,x)(b,b) = (c,d)(b,b)\end{aligned}$$

again by the torsor axioms and the definition of μ_l. For transitivity observe that

$$(a,b)(x,x)(y,y) = (a,b)((x,x,y),y) = (a,b)(y,y).$$

Now we show that μ_l is well-defined on G^l:

$$[(a,b)(x,x)][(c,d)(y,y)] = (a,b)[(x,x)(c,d)](y,y) = (a,b)(c,d)(y,y).$$

The inverse element to (a,b) in G^l is given by (b,a), since

$$(a,b)(b,a) = ((a,b,b),a) = (a,a).$$

Define the left G^l-operation on X by $(a,b)x := (a,b,x)$. This is compatible with μ_l, since

$$\begin{aligned}
[(a,b)(c,d)]x &= ((a,b,c),d)x = ((a,b,c),d,x),\\
(a,b)[(c,d)x] &= (a,b)(c,d,x) = ((a,b,(c,d,x))
\end{aligned}$$

are equal by the second torsor axiom. The left G^l-operation is well-defined with respect to $\sim_l$:

$$[(a,b)(x,x)]y = ((a,b,x),x)y = ((a,b,x),x,y) = (a,(x,x,b),y) = (a,b,y) = (a,b)y.$$

Now we show that i_e is a group homomorphism:

$$ab = (a,e,b) \mapsto ((a,e,b),e) = (a,e)(b,e)$$

The inverse group homomorphism is given by

$$(a,b)(c,d) = ((a,b,c),d) \mapsto ((a,b,c),d,e).$$

On the other hand in G_e one has:

$$(a,b,e)(c,d,e) = ((a,b,e),e,(c,d,e)) = (a,b,(e,e,(c,d,e))) = (a,b,(c,d,e)).$$

This shows that i_e is an isomorphism. The fact that G_e is a group implies that the operation of G^l on X is simply transitive. Indeed the group structure on $G_e = X$ is the one induced by the operation of G^l. The analogous group G^r is constructed using μ_r and an equivalence relation $\sim_r$ with opposite order, i.e., $(a,b) \sim_r (x,x)(a,b)$ for all $x \in X$. The properties of G^r can be verified in the same way as for G^l and are left to the reader. □

1.7.3 Torsors in the category of schemes (without groups)

Definition 1.7.9. Let S be a scheme. A *torsor* in the category of S-schemes is a non-empty scheme X and a morphism

$$X \times X \times X \to X$$

which on T-valued points is a torsor in the sense of Definition 1.7.5 for all T over S.

This simply means that the diagrams of the previous definition commute as morphisms of schemes. The following is the scheme theoretic version of Lemma 1.7.8.

Recall the *fpqc*-topology of Definition 1.7.3.

Proposition 1.7.10. *Let S be affine. Let X be a torsor in the category of affine schemes. Assume that X/S is faithfully flat. Then there are affine group* *schemes G^l and G^r operating from the left and right on X, respectively, such that the natural maps*

$$\begin{aligned} G^l \times X \to X \times X \quad & (g,x) \mapsto (gx,x) \\ X \times G^r \to X \times X \quad & (x,g') \mapsto (x,xg') \end{aligned}$$

are isomorphisms.

Moreover, X is a left G^l- and right G^r-torsor with respect to the fpqc-topology on the category of affine schemes.

Proof. We consider G^l. The arguments for G^r are the same. We define G^l as the *fpqc*-sheafification of the presheaf

$$T \mapsto X^2(T)/\sim_l$$

We are going to see below that it is representable by an affine scheme. The map of presheaves μ_l defines a multiplication on G^l. It is associative as it is associative on the presheaf level.

We construct the neutral element. Recall that $X \to S$ is an *fpqc*-cover. The diagonal $\Delta : X \to X^2/\sim_l$ induces a section $e \in G(X)$. It satisfies descent for the cover X/S by the definition of the equivalence relation $\sim_l$. Hence it defines an element $e \in G(S)$. We claim that it is the neutral element of G. This can be tested *fpqc*-locally, e.g., after base change to X. For T/X the set $X(T)$ is non-empty, hence $X^2/\sim_l(T)$ is a group with neutral element e by Proposition 1.7.8.

The inversion map ι exists on $X^2(T)/\sim_l$ for T/X, hence it also exists and is the inverse on $G(T)$ for T/X. By the sheaf condition this gives a well-defined map with the correct properties on G.

By the same arguments, the action homomorphism $X^2(T)/\sim_l \times X(T) \to X(T)$ defines a left action $G^l \times X \to X$. The induced map $G^l \times X \to X \times X$ is an isomorphism because it as an isomorphism on the presheaf level for T/X. In particular, X is a left G^l-torsor.

We now turn to representability.

We are going to construct G^l by flat descent with respect to the faithfully flat cover $X \to S$ following [BLR, Section 6.1]. In order to avoid confusion, put $T = X$ and $Y = X \times X$ viewed as T-scheme over the second factor. A descent datum on $Y \to T$ consists of the choice of an isomorphism

$$\phi : p_1^*Y \to p_2^*Y$$

subject to the coycle condition

$$p_{13}^*\phi = p_{23}^*\phi \circ p_{12}^*\phi$$

with the obvious notation. We have $p_1^*Y = Y \times T = X^2 \times X$ and $p_2^*Y = T \times Y = X \times X^2$ and use

$$\phi(x_1,x_2,x_3) = (x_2, \rho(x_1,x_2,x_3), x_3)$$

where $\rho : X^2 \to X$ is the structural morphism of X. We have $p_{12}^* p_1^* Y = X^2 \times X \times X$ etc. and

$$\begin{aligned} p_{12}^*\phi(x_1,x_2,x_3,x_4) &= (x_2,\rho(x_1,x_2,x_3),x_3,x_4) \\ p_{23}^*\phi(x_1,x_2,x_3,x_4) &= (x_1,x_3,\rho(x_2,x_3,x_4),x_4) \\ p_{13}^*\phi(x_1,x_2,x_3,x_4) &= (x_2,x_3\rho(x_1,x_3,x_4),x_4) \end{aligned}$$

and the cocyle condition is equivalent to

$$\rho(\rho(x_1,x_2,x_3),x_3,x_4) = \rho(x_1,x_2,x_4),$$

which is an immediate consequence of the properties of a torsor. In the affine case (that we are in) any descent datum is effective, i.e., induced from a uniquely determined S-scheme $\tilde{G}^l$. In other words, it represents the *fpqc*-sheaf defined as the coequalizer of

$$X^2 \times X \rightrightarrows X^2$$

with respect to the projection p_1 mapping (x_1,x_2,x_3) to (x_1,x_2) and $p_2 \circ \phi : X^2 \times X \to X \times X^2 \to X^2$ mapping

$$(x_1,x_2,x_3) \mapsto (x_2,\rho(x_1,x_2,x_3),x_3) \mapsto (\rho(x_1,x_2,x_3),x_3)$$

This is precisely the equivalence relation $\sim_l$. Hence

$$\tilde{G}^l = X^2/\sim_l$$

as *fpqc*-sheaves. □

Remark 1.7.11. If S is the spectrum of a field, then the flatness assumption is always satisfied. In general, some kind of assumption is needed, as the following example shows. Let S be the spectrum of a discrete valuation ring with closed point s. Let G be an algebraic group over s and $X = G$ the trivial torsor defined by G. In particular, we have the structure map

$$X \times_s X \times_s X \to X.$$

We now view X as an S-scheme. Note that

$$X \times_S X \times_S X = X \times_s X \times_s X$$

hence X is also a torsor over S in the sense of Definition 1.7.9. However, it is not a torsor with respect to the *fpqc* (or any other reasonable Grothendieck topology) as $X(T)$ is empty for all $T \to S$ surjective.

Chapter 2

Singular Cohomology

In this chapter we give a short introduction to singular cohomology. Many properties are only sketched, as this theory is considerably easier than de Rham cohomology for example.

2.1 Sheaf cohomology

Let X be a topological space. Sometimes, if indicated, X will be the underlying topological space of an analytic or algebraic variety also denoted by X. To avoid technicalities, X will always be assumed to be a paracompact space, i.e., locally compact, Hausdorff, and satisfying the second countability axiom.

From now on, let $\mathcal{F}$ be a sheaf of abelian groups on X and consider sheaf cohomology $H^i(X, \mathcal{F})$ from Section 1.4. Mostly, we will consider the case of the constant sheaf $\mathcal{F} = \mathbb{Z}$. Later we will also consider other constant coefficients $R \supset \mathbb{Z}$, but this will not change the following topological statements.

Definition 2.1.1 (Relative Cohomology)**.** Let $A \subset X$ be a closed subset, $U = X \setminus A$ the open complement, $i : A \hookrightarrow X$ and $j : U \hookrightarrow X$ be the inclusion maps. We define *relative cohomology* as

$$H^i(X, A; \mathbb{Z}) := H^i(X, j_!\mathbb{Z}),$$

where $j_!$ is the extension by zero, i.e., the sheafification of the presheaf $V \mapsto \mathbb{Z}$ for $V \subset U$ and $V \mapsto 0$ else.

Remark 2.1.2 (Functoriality and homotopy invariance)**.** The association

$$(X, A) \mapsto H^i(X, A; \mathbb{Z})$$

is a contravariant functor from pairs of topological spaces to abelian groups. In particular, for every continuous map $f : (X, A) \to (X', A')$ of pairs, i.e., satisfying $f(A) \subset A'$, one has a homomorphism $f^* : H^i(X', A'; \mathbb{Z}) \to H^i(X, A; \mathbb{Z})$.

Given two homotopic maps f and g, then the homomorphisms f^*, g^* are equal. As a consequence, if two pairs (X, A) and (X', A') are homotopy equivalent, then the cohomology groups $H^i(X', A'; \mathbb{Z})$ and $H^i(X, A; \mathbb{Z})$ are isomorphic.

Proposition 2.1.3. *There is a long exact sequence*

$$\cdots \to H^i(X, A; \mathbb{Z}) \to H^i(X, \mathbb{Z}) \to H^i(A, \mathbb{Z}) \xrightarrow{\delta} H^{i+1}(X, A; \mathbb{Z}) \to \cdots$$

Proof. This follows from the exact sequence of sheaves

$$0 \to j_!\mathbb{Z} \to \mathbb{Z} \to i_*\mathbb{Z} \to 0.$$

□

Note that by our definition of cones, see section 1.3, one has a quasi-isomorphism $j_!\mathbb{Z} = \mathrm{Cone}(\mathbb{Z} \to i_*\mathbb{Z})[-1]$. For Nori motives we need a version for triples, which can be proved using iterated cones by the same method:

Corollary 2.1.4. *Let $X \supset A \supset B$ be a triple of relative closed subsets. Then there is a long exact sequence*

$$\cdots \to H^i(X, A; \mathbb{Z}) \to H^i(X, B; \mathbb{Z}) \to H^i(A, B; \mathbb{Z}) \xrightarrow{\delta} H^{i+1}(X, A; \mathbb{Z}) \to \cdots$$

Here, δ is the connecting homomorphism, which in the cone picture is explained in Section 1.3.

Proposition 2.1.5 (Mayer-Vietoris)**.** *Let $\{U, V\}$ be an open cover of X. Let $A \subset X$ be closed. Then there is a natural long exact sequence*

$$\begin{aligned} \cdots \to H^i(X, A; \mathbb{Z}) \to H^i_{\mathrm{dR}}(U, U \cap A; \mathbb{Z}) \oplus H^i(V, V \cap A; \mathbb{Z}) \\ \to H^i(U \cap V, U \cap V \cap A; \mathbb{Z}) \to H^{i+1}(X, A; \mathbb{Z}) \to \cdots \end{aligned}$$

Proof. Pairs (U, V) of open subsets form an excisive couple in the sense of [Sp, pg. 188], and therefore the Mayer-Vietoris property holds by [Sp, pg. 189-190]. □

Theorem 2.1.6 (Proper base change)**.** *Let $\pi : X \to Y$ be proper (i.e., the preimage of a compact subset is compact). Let $\mathcal{F}$ be a sheaf on X. Then the stalk in $y \in Y$ is computed as*

$$(R^i\pi_*\mathcal{F})_y = H^i(X_y, \mathcal{F}|_{X_y}).$$

Proof. See [KS] Proposition 2.6.7. As π is proper, we have $R\pi_* = R\pi_!$. □

Now we list some properties of the sheaf cohomology of algebraic varieties over a field $k \hookrightarrow \mathbb{C}$. As usual, we will not distinguish in notation between a variety X and the topological space $X(\mathbb{C})$. The first property is:

Proposition 2.1.7 (Excision, or abstract blow-up)**.** *Let $f : (X', D') \to (X, D)$ be a proper, surjective morphism of algebraic varieties over $\mathbb{C}$, which induces an isomorphism $F : X' \setminus D' \to X \setminus D$. Then*

$$f^* : H^*(X, D; \mathbb{Z}) \cong H^*(X', D'; \mathbb{Z}).$$

Proof. This fact goes back to A. Aeppli [Ae]. It is a special case of proper-base change: Let $j : U \to X$ be the complement of D and $j' : U \to X'$ its inclusion into X'. For all $x \in X$, we have

$$R^i \pi_* j'_! \mathbb{Z} = H^i(X_x, j'_! \mathbb{Z}|_{X'_x}).$$

For $x \in U$, the fibre is one point. It has no higher cohomology. For $x \in D$, the restriction of $j'_! \mathbb{Z}$ to X'_x is zero. Together this means

$$R\pi_* j'_! \mathbb{Z} = j_! \mathbb{Z}.$$

The statement follows from the Leray spectral sequence. □

We will later prove a slightly more general proper base change theorem for singular cohomology, see Theorem 2.5.11.

The second property is:

Proposition 2.1.8 (Gysin isomorphism, localization, weak purity)**.** *Let X be an irreducible variety of dimension n over k, and Z a closed subvariety of pure codimension r. Then there is an exact sequence*

$$\cdots \to H^i_Z(X, \mathbb{Z}) \to H^i(X, \mathbb{Z}) \to H^i(X \setminus Z, \mathbb{Z}) \to H^{i+1}_Z(X, \mathbb{Z}) \to \cdots$$

where $H^i_Z(X, \mathbb{Z})$ is cohomology with supports in Z, defined as the hypercohomology of $\mathrm{Cone}(\mathbb{Z}_X \to \mathbb{Z}_{X \setminus U})[-1]$.

If, moreover, X and Z are both smooth, then one has the Gysin isomorphism

$$H^i_Z(X, \mathbb{Z}) \cong H^{i-2r}(Z, \mathbb{Z}).$$

In particular, one has weak purity:

$$H^i_Z(X, \mathbb{Z}) = 0 \text{ for } i < 2r,$$

and $H^{2r}_Z(X, \mathbb{Z}) = H^0(Z, \mathbb{Z})$ is free of rank the number of components of Z.

Proof. See [Pa, Sect. 2] for this statement and an axiomatic treatment with more general properties and examples of cohomology theories. □

2.2 Singular (co)homology

Let X be a topological space (same general assumptions as in section 2.1). The definition of singular homology and cohomology uses topological simplexes.

Definition 2.2.1. The topological *n-simplex* Δ_n is defined as

$$\Delta_n := \{(t_0, ..., t_n) \mid \sum_{i=0}^{n} t_i = 1,\ t_i \geq 0\}\ .$$

Δ_n has natural codimension one faces defined by $t_i = 0$.

Singular (co)homology is defined by looking at all possible continuous maps from simplices to X.

Definition 2.2.2. A singular *n-simplex* σ is a continuous map

$$f\colon \Delta_n \to X.$$

In the case where X is a differentiable manifold, a singular simplex σ is called *differentiable*, if the map f can be extended to a $\mathcal{C}^\infty$-map from a neighbourhood of $\Delta_n \subset \mathbb{R}^{n+1}$ to X. The *group of singular n-chains* is the free abelian group

$$S_n(X) := \mathbb{Z}[f\colon \Delta_n \to X \mid f \text{ singular chain }].$$

In a similar way, we denote by $S_n^\infty(X)$ the free abelian *group of differentiable chains*. The boundary map $\partial_n : S_n(X) \to S_{n-1}(X)$ is defined as

$$\partial_n(f) := \sum_{i=0}^{n}(-1)^i f|_{t_i=0}.$$

The *group of singular n-cochains* is the free abelian group

$$S^n(X) := \mathrm{Hom}_{\mathbb{Z}}(S_n(X), \mathbb{Z}).$$

The *group of differentiable singular n-cochains* is the free abelian group

$$S^n(X) := \mathrm{Hom}_{\mathbb{Z}}(S_n^\infty(X), \mathbb{Z}).$$

The adjoint of ∂_{n+1} defines the boundary map

$$d_n : S^n(X) \to S^{n+1}(X).$$

Lemma 2.2.3. *One has $\partial_{n-1}\partial_n = 0$ and $d_{n+1}d_n = 0$, i.e., the groups $S_\bullet(X)$ and $S^\bullet(X)$ define complexes of abelian groups.*

The proof is left to the reader as an exercise.

Definition 2.2.4. Singular homology and cohomology with values in $\mathbb{Z}$ is defined as

$$H^i_{\text{sing}}(X,\mathbb{Z}) := H^i(S^\bullet(X), d_\bullet),\ H_i^{\text{sing}}(X,\mathbb{Z}) := H_i(S_\bullet(X), \partial_\bullet)\ .$$

In a similar way, we define (for X a manifold) the differentiable singular (co)homology as

$$H^i_{\text{sing},\infty}(X,\mathbb{Z}) := H^i(S^\bullet_\infty(X), d_\bullet),\ H_i^{\text{sing},\infty}(X,\mathbb{Z}) := H_i(S^\infty_\bullet(X), \partial_\bullet)\ .$$

Theorem 2.2.5. *Assume that X is a locally contractible topological space, i.e., every point has an open contractible neighborhood. In this case, singular cohomology $H^i_{\text{sing}}(X,\mathbb{Z})$ agrees with sheaf cohomology $H^i(X,\mathbb{Z})$ with coefficients in $\mathbb{Z}$. If Y is a differentiable manifold, differentiable singular (co)homology agrees with singular (co)homology.*

Proof. Let $\mathcal{S}^n$ be the sheaf associated to the presheaf $U \mapsto S^n(U)$. One shows that $\mathbb{Z} \to \mathcal{S}^\bullet$ is a fine resolution of the constant sheaf $\mathbb{Z}$ [Wa, pg. 196]. In the proof it is used that X is locally contractible, see [Wa, pg. 194]. If X is a manifold, differentiable cochains also define a fine resolution [Wa, pg. 196]. Therefore, the inclusion of complexes $S^\infty_\bullet(X) \hookrightarrow S_\bullet(X)$ induces isomorphisms

$$H^i_{\text{sing},\infty}(X,\mathbb{Z}) \cong H^i_{\text{sing}}(X,\mathbb{Z}) \text{ and } H_i^{\text{sing},\infty}(X,\mathbb{Z}) \cong H_i^{\text{sing}}(X,\mathbb{Z})\,.$$

□

Of course, topological manifolds satisfy the assumption of the theorem.

2.3 Simplicial cohomology

In this section we want to introduce simplicial (co)homology and its relation to singular (co)homology. Simplicial (co)homology can be defined for topological spaces with an underlying combinatorial structure.

In the literature there are various notions of such spaces. In increasing order of generality, these are: (geometric) simplicial complexes and topological realizations of abstract simplicial complexes, of Δ-complexes (sometimes also called semi-simplicial complexes), and of simplicial sets. A good reference with a discussion of various definitions is the book by Hatcher [Hat], or the introductory paper [Fri] by Friedman.

By construction, such spaces are built from topological simplices Δ_n in various dimensions n, and the faces of each simplex are of the same type. Particularly nice examples are polyhedra, for example a tetrahedron, where the simplicial structure is obvious.

Geometric simplicial complexes come up more generally in geometric situations in the triangulations of manifolds with certain conditions. An example is the case of an analytic space X^{an} where X is an algebraic variety defined over $\mathbb{R}$. There one can always find a semi-algebraic triangulation by a result of Lojasiewicz, cf. Hironaka [Hi2, p. 170] and Prop. 2.6.8.

In this section, we will think of a simplicial space as the topological realization of a finite simplicial set:

Definition 2.3.1. Let $X_\bullet$ be a finite simplicial set in the sense of Remark 1.5.5. One has the face maps

$$\partial_i : X_n \to X_{n-1} \quad i = 0, \ldots, n,$$

and the degeneracy maps

$$s_i : X_n \to X_{n+1} \quad i = 0, \ldots, n.$$

The topological realization $|X_\bullet|$ of $X_\bullet$ is defined as

$$|X_\bullet| := \coprod_{n=0}^{\infty} X_n \times \Delta_n / \sim,$$

where each X_n carries the discrete topology, Δ_n is the topological n-simplex, and the equivalence relation is given by the two relations

$$(x, \partial_i(y)) \sim (\partial_i(x), y), \quad (x, s_i(y)) \sim (s_i(x), y), \quad x \in X_{n-1},\ y \in \Delta_n.$$

(Note that we denote the face and degeneracy maps for the n-simplex by the same letters ∂_i, s_i.)

In this way, every finite simplicial set gives rise to a topological space $|X_\bullet|$. It is known that $|X_\bullet|$ is a compactly generated CW-complex [Hat, Appendix]. In fact, every finite CW-complex is homotopy equivalent to a finite simplical complex of the same dimension by [Hat, Thm. 2C.5]. Thus, our restriction to realizations of finite simplicial sets is not a severe restriction.

The skeleton filtration from Remark 1.5.5 defines a filtration of $|X_\bullet|$

$$|\mathrm{sq}_0 X_\bullet| \subseteq |\mathrm{sq}_1 X_\bullet| \subseteq \cdots \subseteq |\mathrm{sq}_N X_\bullet| = |X_\bullet|$$

by closed subspaces, if X_n is empty for $n > N$.

There is finite number of simplices in each degree n. Associated to each of them is a continuous map $\sigma : \Delta_n \to |X_\bullet|$. We denote the free abelian group of all such σ of degree n by $C_n^{\Delta}(X_\bullet)$

$$\partial_n : C_n^{\Delta}(X_\bullet) \to C_{n-1}^{\Delta}(X_\bullet)$$

are given by alternating restriction maps to faces, as in the case of singular homology. Note that the vertices of each simplex are ordered, so that this is well-defined.

Definition 2.3.2. Simplicial homology of the topological space $X = |X_\bullet|$ is defined as

$$H_n^{\mathrm{simpl}}(X, \mathbb{Z}) := H_n(C_*^{\Delta}(X_\bullet), \partial_*),$$

and simplicial cohomology as

$$H^n_{\mathrm{simpl}}(X; \mathbb{Z}) := H^n(C^*_{\Delta}(X_\bullet), d_*),$$

where $C^n_{\Delta}(X_\bullet) = \mathrm{Hom}(C_n^{\Delta}(X_\bullet), \mathbb{Z})$ and d_n is adjoint to ∂_n.

Example 2.3.3. A tetrahedron arises from a simplicial set with four vertices (0-simplices), six edges (1-simplices), and four faces (2-simplices). A computation shows that $H_n = \mathbb{Z}$ for $i = 0, 2$ and zero otherwise (this was a priori clear, since it is topologically a sphere).

A torus T^2 can be obtained from a square by identifying opposite sides, called a and b. If we look at the diagonal of the square, we see that there is a simplicial complex with one vertex (!), three edges, and two faces. A computation shows that $H_1(T^2, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$ as expected, and $H_0(T^2, \mathbb{Z}) = H_2(T^2, \mathbb{Z}) = \mathbb{Z}$.

This definition does not depend on the representation of a topological space X as the topological realization of a simplicial set, since one can prove that simplicial (co)homology coincides with singular (co)homology:

Theorem 2.3.4. *Singular and simplicial (co)homology of X are equal.*

Proof. (For homology only.) The chain of closed subsets

$$|\mathrm{sq}_0 X_\bullet| \subseteq |\mathrm{sq}_1 X_\bullet| \subseteq \cdots \subseteq |\mathrm{sq}_N X_\bullet| = |X_\bullet|$$

gives rise to long exact sequences of simplicial homology groups

$$\cdots \to H_n^{\mathrm{simpl}}(|\mathrm{sq}_{n-1} X_\bullet|, \mathbb{Z}) \to H_n^{\mathrm{simpl}}(|\mathrm{sq}_n X_\bullet|, \mathbb{Z}) \to H_n^{\mathrm{simpl}}(|\mathrm{sq}_{n-1} X_\bullet|, |\mathrm{sq}_n X_\bullet|; \mathbb{Z}) \to \cdots$$

A similar sequence holds for singular homology, and there is a canonical map $C_n^{\Delta}(X) \to C_n(X)$ from simplicial to singular chains. The result is then proved by induction on n. We use that the relative complex $C_n(|\mathrm{sq}_{n-1} X_\bullet|, |\mathrm{sq}_n X_\bullet|)$ has zero differential and is a free abelian group of rank equal to the cardinality of X_n. Therefore, one concludes by observing a computation of the singular (co)homology of Δ_n, i.e., $H^i(\Delta_n, \mathbb{Z}) = \mathbb{Z}$ for $i = 0$ and zero otherwise. □

In a similar way, one can define the simplicial (co)homology of a pair $(X, D) = (|X_\bullet|, |D_\bullet|)$, where $D_\bullet \subset X_\bullet$ is a simplicial subobject. The associated chain complex is given by the quotient complex $C_*(X_\bullet)/C_*(D_\bullet)$. The same proof will then show that the singular and simplicial (co)homology of pairs coincide.

From the definition of the topological realization, we see that X is a CW-complex. In the special case, when X is the topological space underlying an affine algebraic variety X over $\mathbb{C}$, or more generally a Stein manifold, then one can show:

Theorem 2.3.5 (Artin vanishing)**.** *Let X be an affine variety over $\mathbb{C}$ of dimension n. Then $H^q(X^{\mathrm{an}}, \mathbb{Z}) = 0$ for $q > n$. In fact, X^{an} is homotopy equivalent to a finite simplicial complex where all cells are of dimension $\leq n$.*

Proof. The proof was first given by Andreotti and Fraenkel [AF] for Stein spaces. An algebraic proof was given by M. Artin [A, Cor. 3.5, tome 3]. □

Corollary 2.3.6 (Good topological filtration)**.** *Let X be an affine variety over $\mathbb{C}$ of dimension n. Then the skeleton filtration of X^{an} is given by*

$$X^{\mathrm{an}} = X_n \supset X_{n-1} \supset \cdots \supset X_0$$

where the pairs (X_i, X_{i-1}) have only cohomology in degree i.

Remark 2.3.7. The Basic Lemma of Nori and Beilinson, see Thm. 2.5.6, shows that there is even an algebraic variant of this topological skeleton filtration.

Corollary 2.3.8 (Artin vanishing for relative cohomology)**.** *Let X be an affine variety of dimension n and $Z \subset X$ a closed subvariety. Then*

$$H^i(X^{\mathrm{an}}, Z^{\mathrm{an}}, \mathbb{Z}) = 0 \text{ for } i > n.$$

Proof. Consider the long exact sequence for relative cohomology and use Artin vanishing for X and Z from Thm.2.3.5. □

The following theorem is strongly related to the Artin vanishing theorem.

Theorem 2.3.9 (Lefschetz hyperplane theorem)**.** *Let $X \subset \mathbb{P}^N_{\mathbb{C}}$ be an integral projective variety of dimension n, and $H \subset \mathbb{P}^N_{\mathbb{C}}$ a hyperplane section such that $H \cap X$ contains the singularity set X_{sing} of X. Then the inclusion $H \cap X \subset X$ is $(n-1)$-connected. In particular, one has $H^q(X, \mathbb{Z}) = H^q(X \cap H, \mathbb{Z})$ for $q \leq n$.*

Proof. See for example [AF]. □

2.4 Künneth formula and Poincaré duality

Assume that we have given two topological spaces X and Y, and two closed subsets $j : A \hookrightarrow X$, and $j' : C \hookrightarrow Y$. By the above, we have

$$H^*(X, A; \mathbb{Z}) = H^*(X, j_!\mathbb{Z})$$

and

$$H^*(Y, C; \mathbb{Z}) = H^*(Y, j'_!\mathbb{Z}) \ .$$

The relative cohomology group

$$H^*(X \times Y, X \times C \cup A \times Y; \mathbb{Z})$$

can be computed as $H^*(X \times Y, \tilde{j}_! \mathbb{Z})$, where

$$\tilde{j} : X \times C \cup A \times Y \hookrightarrow X \times Y$$

is the inclusion map. One has $\tilde{j}_! = j_! \boxtimes j'_!$. Hence, we have a natural exterior product map

$$H^i(X, A; \mathbb{Z}) \otimes H^j(Y, C; \mathbb{Z}) \xrightarrow{\times} H^{i+j}(X \times Y, X \times C \cup A \times Y; \mathbb{Z}).$$

This is related to the so-called *Künneth formula*:

Theorem 2.4.1 (Künneth formula for pairs)**.** *Let $A \subset X$ and $C \subset Y$ be closed subsets. The exterior product map induces a natural isomorphism*

$$\bigoplus_{i+j=n} H^i(X, A; \mathbb{Q}) \otimes H^j(Y, C; \mathbb{Q}) \xrightarrow{\cong} H^n(X \times Y, X \times C \cup A \times Y; \mathbb{Q}).$$

The same result holds with $\mathbb{Z}$-coefficients, provided all cohomology groups of (X, A) and (Y, C) in all degrees are free.

Proof. Using the sheaves of singular cochains, see the proof of theorem 2.2.5, one has fine resolutions $j_! \mathbb{Z} \to F^\bullet$ on X, and $j'_! \mathbb{Z} \to G^\bullet$ on Y. The tensor product $F^\bullet \boxtimes G^\bullet$ thus is a fine resolution of $\tilde{j}_! \mathbb{Z} = j_! \mathbb{Z} \boxtimes j'_! \mathbb{Z}$. Here one uses that the tensor product of fine sheaves is fine [Wa, pg. 193]. The cohomology of the tensor product complex $F^\bullet \otimes G^\bullet$ induces a short exact sequence

$$0 \to \bigoplus_{i+j=n} H^i(X, A; \mathbb{Z}) \otimes H^j(Y, C; \mathbb{Z}) \to H^n(X \times Y, X \times C \cup A \times Y; \mathbb{Z})$$

$$\to \bigoplus_{i+j=n+1} \mathrm{Tor}_1^{\mathbb{Z}}(H^i(X, A; \mathbb{Z}), H^j(Y, C; \mathbb{Z})) \to 0$$

by [God, thm. 5.5.1] or [We, thm. 3.6.3]. If all cohomology groups are free, the last term vanishes. □

Proposition 2.4.2. *The Künneth isomorphism of Theorem 2.4.1 is associative and graded commutative.*

Proof. This is a standard consequence of the definition of the Künneth isomorphism from complexes of groups. □

In later constructions, we will need a certain compatibility of the exterior product with coboundary maps. Assume that $X \supset A \supset B$ and $Y \supset C$ are closed subsets.

Proposition 2.4.3. *The diagram involving coboundary maps*

$$\begin{array}{ccc}
H^i(A, B; \mathbb{Z}) \otimes H^j(Y, C; \mathbb{Z}) & \longrightarrow & H^{i+j}(A \times Y, A \times C \cup B \times Y; \mathbb{Z}) \\
{\scriptstyle \delta \otimes \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \delta} \\
H^{i+1}(X, A; \mathbb{Z}) \otimes H^j(Y, C; \mathbb{Z}) & \longrightarrow & H^{i+j+1}(X \times Y, X \times C \cup A \times Y; \mathbb{Z})
\end{array}$$

commutes up to a sign $(-1)^j$*. The diagram*

$$\begin{array}{ccc} H^i(Y,C;\mathbb{Z}) \otimes H^j(A,B;\mathbb{Z}) & \longrightarrow & H^{i+j}(Y \times A, Y \times B \cup C \times A;\mathbb{Z}) \\ {\scriptstyle \mathrm{id}\otimes\delta}\downarrow & & \downarrow{\scriptstyle \delta} \\ H^i(Y,C;\mathbb{Z}) \otimes H^{j+1}(X,A;\mathbb{Z}) & \longrightarrow & H^{i+j+1}(Y \times X, Y \times A \cup C \times X;\mathbb{Z}) \end{array}$$

commutes (without a sign).

Proof. We indicate the argument, without going into full details. Let $F^\bullet$ be a complex computing $H^\bullet(Y,C;\mathbb{Z})$ Let $G_1^\bullet$ and $G_2^\bullet$ be complexes computing $H^\bullet(A,B;\mathbb{Z})$ and $H^\bullet(X,A;\mathbb{Z})$. Let $K_1^\bullet$ and $K_2^\bullet$ be the complexes computing cohomology of the corresponding product varieties. Cup product is induced from maps of complexes $F_i^\bullet \otimes G^\bullet \to K_i^\bullet$. In order to get compatibility with the boundary map, we have to consider the diagram of the form

$$\begin{array}{ccc} F_1 \otimes G & \longrightarrow & K_1 \\ \downarrow & & \downarrow \\ (F_2[1]) \otimes G & \longrightarrow & K_2[1] \end{array}$$

However, by Lemma 1.3.6, the complexes $(F_2[1]) \otimes G$ and $(F_2 \otimes G)[1]$ are not equal. We need to introduce the sign $(-1)^j$ in bidegree (i,j) to make the identification and get a commutative diagram.

The argument for the second type of boundary map is the same, but does not need the introduction of signs by Lemma 1.3.6. □

Assume now that $X = Y$ and $A = C$. Then, $j_!\mathbb{Z}$ has an algebra structure, and we obtain the *cup product* maps:

$$H^i(X,A;\mathbb{Z}) \otimes H^j(X,A;\mathbb{Z}) \longrightarrow H^{i+j}(X,A;\mathbb{Z})$$

via the multiplication maps

$$H^{i+j}(X \times X, \tilde{j}_!\mathbb{Z}) \to H^{i+j}(X, j_!\mathbb{Z}),$$

induced by

$$\tilde{j}_! = j_! \boxtimes j_! \to j_! \ .$$

In the case where $A = \emptyset$, the cup product induces Poincaré duality:

Proposition 2.4.4 (Poincaré Duality)**.** *Let X be a compact, orientable topological manifold of dimension m. Then the cup product pairing*

$$H^i(X,\mathbb{Q}) \times H^{m-i}(X,\mathbb{Q}) \longrightarrow H^m(X,\mathbb{Q}) \cong \mathbb{Q}$$

is non-degenerate in both factors. With $\mathbb{Z}$-coefficients, the same result holds for the two left groups modulo torsion.

Proof. We will give a proof of a slightly more general statement in the algebraic situation below. A proof of the stated theorem can be found in [GH, pg. 53]. There it is shown that $H^{2n}(X)$ is torsion-free of rank one, and the cup-product pairing is unimodular modulo torsion, using simplicial cohomology, and the relation between Poincaré duality and the dual cell decomposition. □

We will now prove a relative version in the algebraic case. It implies the version above in the case where $A = B = \emptyset$. By abuse of notation, we again do not distinguish between an algebraic variety over $\mathbb{C}$ and its underlying topological space.

Theorem 2.4.5 (Poincaré duality for algebraic pairs)**.** *Let X be a smooth and proper complex variety of dimension n over $\mathbb{C}$ and $A, B \subset X$ two normal crossing divisors, such that $A \cup B$ is also a normal crossing divisor. Then there is a non-degenerate duality pairing*

$$H^d(X\backslash A, B\backslash(A\cap B); \mathbb{Q}) \times H^{2n-d}(X\backslash B, A\backslash(A\cap B); \mathbb{Q}) \longrightarrow H^{2n}(X, \mathbb{Q}) \cong \mathbb{Q}(-n).$$

Again, with $\mathbb{Z}$-coefficients this is true modulo torsion by unimodularity of the cup-product pairing.

Proof. We give a sheaf theoretic proof using Verdier duality and some formulas and ideas of Beilinson (see [Be1]). Look at the commutative diagram:

$$\begin{tikzcd} U = X \setminus (A \cup B) \arrow[r, "\ell_U"] \arrow[d, "\kappa_U"'] & X \setminus A \arrow[d, "\kappa"] \\ X \setminus B \arrow[r, "\ell"] & X. \end{tikzcd}$$

Assuming $A \cup B$ is a normal crossing divisor, we want to show first that the natural map

$$\ell_! R\kappa_{U*}\mathbb{Q}_U \longrightarrow R\kappa_* \ell_{U!}\mathbb{Q}_U,$$

extending $\mathrm{id} : \mathbb{Q}_U \to \mathbb{Q}_U$, is an isomorphism. This is a local computation. We look without loss of generality at a neighborhood of an intersection point $x \in A \cap B$, since the computation at other points is even easier. If we work in the analytic topology, we may choose a polydisk neighborhood D in X around x such that D decomposes as

$$D = D_A \times D_B$$

and such that

$$A \cap D = A_0 \times D_B, \quad B \cap D = D_A \times B_0$$

for some suitable topological spaces A_0, B_0. Using the same symbols for the maps as in the above diagram, the situation looks locally like

$$\begin{array}{ccc} (D_A \setminus A_0) \times (D_B \setminus B_0) & \xrightarrow{\ell_U} & (D_A \setminus A_0) \times D_B \\ {\scriptstyle \kappa_U}\downarrow & & \downarrow{\scriptstyle \kappa} \\ D_A \times (D_B \setminus B_0) & \xrightarrow{\ell} & D = D_A \times D_B. \end{array}$$

Using the Künneth formula, one concludes that both sides $\ell_! R\kappa_{U*}\mathbb{Q}_U$ and $R\kappa_*\ell_{U!}\mathbb{Q}_U$ are isomorphic to

$$R\kappa_{U*}\mathbb{Q}_{D_A \setminus A_0} \otimes \ell_! \mathbb{Q}_{D_B \setminus B_0}$$

near the point x, and the natural map provides an isomorphism.

Now, one has

$$H^d(X \setminus A, B \setminus (A \cap B); \mathbb{Q}) = H^d(X, \ell_! \kappa_{U*}\mathbb{Q}_U),$$

(using that the maps involved are affine), and

$$H^{2n-d}(X \setminus B, A \setminus (A \cap B); \mathbb{Q}) = H^{2n-d}(X, \kappa_! \ell_{U*}\mathbb{Q}_U).$$

We have to show that there is a perfect pairing

$$H^d(X \setminus A, B \setminus (A \cap B); \mathbb{Q}) \times H^{2n-d}(X \setminus B, A \setminus (A \cap B); \mathbb{Q}) \to \mathbb{Q}(-n).$$

However, by Verdier duality, we have a perfect pairing

$$\begin{aligned} H^{2n-d}(X \setminus B, A \setminus (A \cap B); \mathbb{Q})^\vee &= H^{2n-d}(X, \kappa_! \ell_{U*}\mathbb{Q}_U)^\vee \\ &= H^{-d}(X, \kappa_! \ell_{U*}\mathbb{D}\mathbb{Q}_U)(-n) \\ &= H^{-d}(X, \mathbb{D}(\kappa_* \ell_{U!}\mathbb{Q}_U))(-n) \\ &= H^d(X, \kappa_* \ell_{U!}\mathbb{Q}_U)(-n) \\ &= H^d(X, \ell_! \kappa_{U*}\mathbb{Q}_U)(-n) \\ &= H^d(X \setminus A, B \setminus (A \cap B); \mathbb{Q}). \end{aligned}$$

□

Remark 2.4.6. The normal crossing condition is necessary, as one can see in the example of $X = \mathbb{P}^2$, where A consists of two distinct lines meeting in a point, and B a line going through the same point.

2.5 Basic Lemma

In this section we prove the basic lemma of Nori [N, N1, N2], a topological result, which was also known to Beilinson [Be1] and Vilonen (unpublished). Let $k \subset \mathbb{C}$ be a subfield. The proof of Beilinson works more generally in positive characteristics as we will see below.

Convention 2.5.1. We fix an embedding $k \hookrightarrow \mathbb{C}$. All sheaves and all cohomology groups in the following section are to be understood in the analytic topology on $X(\mathbb{C})$.

Theorem 2.5.2 (Basic Lemma I)**.** *Let $k \subset \mathbb{C}$. Let X be an affine variety over k of dimension n and $W \subset X$ be a Zariski closed subset with* $\dim(W) < n$*. Then there exists a Zariski closed subset $Z \supset W$ with* $\dim(Z) < n$ *and*

$$H^q(X, Z; \mathbb{Z}) = 0, \text{ for } q \neq n$$

and, moreover, the cohomology group $H^n(X, Z; \mathbb{Z})$ is a free $\mathbb{Z}$-module.

We formulate the Lemma for coefficients in $\mathbb{Z}$, but by the universal coefficient theorem [We, thm. 3.6.4] it will hold with other coefficients as well.

Example 2.5.3. There is an example where there is an easy way to obtain Z. Assume that X is of the form $\bar{X} \setminus H$ for some smooth projective $\bar{X}$ and a hyperplane H. Let $W = \emptyset$. Then take another hyperplane section H' meeting $\bar{X}$ and H transversally. Then $Z := H' \cap X$ will have the property that $H^q(X, Z; \mathbb{Z}) = 0$ for $q \neq n$ by the Lefschetz hyperplane theorem, see Thm. 2.3.9. This argument will be generalized in two of the proofs below.

An inductive application of this Basic Lemma in the case $Z = \emptyset$ yields a filtration of X by closed subsets

$$X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset$$

with $\dim(X_i) = i$ such that the complex of free $\mathbb{Z}$-modules

$$\cdots \xrightarrow{\delta_{i-1}} H^i(X_i, X_{i-1}) \xrightarrow{\delta_i} H^{i+1}(X_{i+1}, X_i) \xrightarrow{\delta_{i+1}} \cdots ,$$

where the maps $\delta_\bullet$ arise from the coboundary in the long exact sequence associated to the triples $X_{i+1} \supset X_i \supset X_{i-1}$, computes the cohomology of X.

Remark 2.5.4. This means that we can understand this filtration as algebraic analogue of the skeletal filtraton of simplicial complexes, see Corollary 2.3.6. Note that the filtration is not only algebraic, but even defined over the base field k.

The Basic Lemma is deduced from the following variant, which was also known to Beilinson [Be1]. To state it, we need the notion of a (weakly) constructible sheaf.

Definition 2.5.5. A sheaf of abelian groups on a variety X over k is *weakly constructible*, if there is a stratification of X into a disjoint union of finitely many Zariski locally closed subsets Y_i, and such that the restriction of F to Y_i is locally constant.

We will also need some basic facts about sheaf cohomology. If $j : U \hookrightarrow X$ is a Zariski open subset with closed complement $i : W \hookrightarrow X$ and F a sheaf of abelian groups on X, then there is an exact sequence of sheaves

$$0 \to j_! j^* F \to F \to i_* i^* F \to 0.$$

In addition, for F the constant sheaf $\mathbb{Z}$, one has $H^q(X, j_! j^* F) = H^q(X, W; \mathbb{Z})$ and $H^q(X, i_* i^* F) = H^q(W, \mathbb{Z})$, see section 2.1.

Theorem 2.5.6 (Basic Lemma II). *Let X be an affine variety over k of dimension n and F be a weakly constructible sheaf on X. Then there exists a Zariski open subset $j : U \hookrightarrow X$ such the following three properties hold:*

1. $\dim(X \setminus U) < n$.
2. $H^q(X, F') = 0$ *for* $q \neq n$, *where* $F' := j_! j^* F \subset F$.
3. *There exists a finite subset $E \subset U(\mathbb{C})$ such that $H^{\dim(X)}(X, F')$ is isomorphic to a direct sum $\oplus_x F_x$ of stalks of F at points of E.*

Version II of the Lemma implies version I. Let $V = X \setminus W$ with open immersion $h : V \hookrightarrow X$, and the sheaf $F = h_! h^* \mathbb{Z}$ on X. Version II for F gives an open subset $\ell : U \hookrightarrow X$ such that the sheaf $F' = \ell_! \ell^* F$ has non-vanishing cohomology only in degree n. Let $W' = X \setminus U$. Since F was zero on W, we have that F' is zero on $Z = W \cup W'$ and it is the constant sheaf on $X \setminus Z$, i.e., $F' = j_! j^* F$ for $j : X \setminus Z \hookrightarrow X$. In particular, F' computes the relative cohomology $H^q(X, Z; \mathbb{Z})$ and it vanishes for $q \neq n$. Freeness follows from property (3). $\square$

Now we will give two proofs of the Basic Lemma II. The first proof by Nori will prove all three assertions, the second proof of Beilinson we give below, proves (1) and (2).

2.5.1 Direct proof of Basic Lemma I

We start by giving a direct proof of Basic Lemma I. It was given by Nori in the unpublished notes [N1]. Close inspection shows that it is actually a variant of Beilinson's argument in this very special case.

Lemma 2.5.7. *Let X be affine, $W \subset X$ closed. Then there exist*

1. $\tilde{X}$ *smooth projective;*
2. $D_0, D_\infty \subset \tilde{X}$ *closed such that $D_0 \cup D_\infty$ is a simple normal crossings divisor and $\tilde{X} \setminus D_0$ is affine;*
3. $\pi : \tilde{X} \setminus D_\infty \to X$ *proper surjective, an isomorphism outside of D_0 such that $Y := \pi(D_0 \setminus D_\infty \cap D_0)$ contains W and $\pi^{-1}(Y) = D_0 \setminus D_\infty \cap D_0$.*

Proof. We may assume without loss of generality that $X \setminus W$ is smooth. Let $\bar{X}$ be a projective closure of X and $\bar{W}$ the closure of W in $\bar{X}$. By resolution of singularities, there is $\tilde{X} \to X$ proper surjective and an isomorphism above $X \setminus W$ such that $\tilde{X}$ is smooth. Let $D_\infty \subset \tilde{X}$ be the complement of the preimage of X. Let $\tilde{W}$ be the closure of the preimage of X. By resolution of singularities, we can also assume that $\tilde{W} \cup D_\infty$ is a divisor with normal crossings.

Note that $\bar{X}$ and hence also $\tilde{X}$ are projective. We choose a generic hyperplane $\tilde{H}$ such that $\bar{W} \cup D_\infty \cup \tilde{H}$ is a divisor with normal crossings. This is possible as the ground field k is infinite and the condition is satisfied in a Zariski open subset of the space of hyperplane sections. We put $D_0 = \tilde{H} \cup \bar{W}$. As $\tilde{H}$ is a hyperplane section, it is an ample divisor. Therefore, $D_0 = \tilde{H} \cup \bar{W}$ is the support of the ample divisor $\tilde{H} + m\bar{W}$ for m sufficiently large [Ha2, Exer. II 7.5(b)]. Hence $\tilde{X} \setminus D_0$ is affine, as the complement of an ample divisor in a projective variety is affine. $\square$

Proof of Basic Lemma I. We use the varieties constructed in the last lemma. We claim that Y has the right properties if the coefficients form an arbitrary field K. We have $Y \supset W$. From Artin vanishing, see Corollary 2.3.8, we immediately have vanishing of $H^i(X, Y; K)$ for $i > n$.

By excision (see Proposition 2.1.7)

$$H^i(X, Y; K) = H^i(\tilde{X} \setminus D_\infty, D_0 \setminus D_0 \cap D_\infty; K).$$

By Poincaré duality for pairs (see Theorem 2.4.5), it is dual to

$$H^{2n-i}(\tilde{X} \setminus D_0, D_\infty \setminus D_0 \cap D_\infty; K).$$

The variety $\tilde{X} \setminus D_0$ is affine. Hence by Artin vanishing, the cohomology group vanishes for all $i \neq n$ and any coefficient field K.

It remains to treat the case of integral coefficients. Let i be the smallest index such that $H^i(X, Y; \mathbb{Z})$ is non-zero. By Artin vanishing for $\mathbb{Z}$-coefficients 2.3.5, we have $i \leq n$.

If $i < n$, then the group $H^i(X, Y; \mathbb{Z})$ has to be torsion because the cohomology vanishes with $\mathbb{Q}$-coefficients. By the universal coefficient theorem [We, thm. 3.6.4]

$$H^{i-1}(X, Y; \mathbb{F}_p) = \mathrm{Tor}_1^{\mathbb{Z}}(H^i(X, Y; \mathbb{Z}), \mathbb{F}_p) ,$$

which implies that $H^{i-1}(X, Y; \mathbb{F}_p)$ is non-trivial for the occuring torsion primes. This is a contradiction to the vanishing for $K = \mathbb{F}_p$. Hence $i = n$. The same argument shows that $H^n(X, Y; \mathbb{Z})$ is torsion-free. $\square$

2.5.2 Nori's proof of Basic Lemma II

We now present the proof of the stronger Basic Lemma II published by Nori in [N2].

We start with a couple of lemmas on weakly constructible sheaves.

Lemma 2.5.8. *Let $0 \to F_1 \to F_2 \to F_3 \to 0$ be a short exact sequence of sheaves on X with F_1, F_3 weakly constructible. Then F_2 is weakly constructible.*

Proof. By assumption, there are stratifications of X such that F_1 and F_3 become locally constant, respectively. We take a common refinement. We replace X by one of the strata and are now in the situation that F_1 and F_3 are locally constant. Then F_3 is also locally constant. Indeed, by passing to a suitable open cover (in the analytic topology), F_1 and F_3 become even constant. If $V \subset U$ is an inclusion of open connected subsets, then the restrictions $F_1(U) \to F_1(V)$ and $F_3(U) \to F_3(U)$ are isomorphisms. This implies the same statement for F_2, because $H^1(U, F_1) = H^1(V, F_1) = 0$, as constant sheaves do not have higher cohomology. □

Lemma 2.5.9. *The notion of weak constructibility is stable under $j_!$ for j an open immersion and π_* for π finite.*

Proof. The assertion of $j_!$ is obvious, same as for i_* for closed immersions.

Now assume $\pi : X \to Y$ is finite and in addition étale. Let F be weakly constructible on X. Let $X_0, \dots, X_n \subset X$ be the stratification such that $F|_{X_i}$ is locally constant. Let Y_i be the image of X_i. These are locally closed subvarieties of Y because π is closed and open. We refine them into a stratification of Y. As π is finite étale, it is locally in the analytic topology of the form $I \times B$ with I finite and $B \subset Y(\mathbb{C})$ an open in the analytic topology. Obviously $\pi_* F|_B$ is locally constant on the strata we have defined.

Now let π be finite. As we have already discussed closed immersion, it suffices to assume π surjective. There is an open dense subscheme $U \subset Y$ such π is étale above U. Let $U' = \pi^{-1}(U)$, $Z = Y \setminus U$ and $Z' = X \setminus U'$. We consider the exact sequence on X

$$0 \to j_{U'!} j_{U'}^* F \to F \to i_{Z'*} i_{Z'}^* F \to 0.$$

As π is finite, the functor π_* is exact and hence

$$0 \to \pi_* j_{U'!} j_{U'}^* F \to \pi_* F \to \pi_* i_{Z'*} i_{Z'}^* F \to 0.$$

By Lemma 2.5.8, it suffices to consider the outer terms. We have

$$\pi_* j_{U'!} j_{U'}^* F = j_{U*} \pi|_{U'*} j_{U'}^* F,$$

and this is weakly constructible by the étale case and the assertion on open immersions. We also have

$$\pi_* i_{Z'*} i_{Z'}^* F = i_{Z*} \pi|_{Z'*} i_{Z'}^* F,$$

and this is weakly constructible by noetherian induction and the case of closed immersions. □

Nori's proof of Basic Lemma II. Let $n := \dim(X)$. In the first step, we reduce to $X = \mathbb{A}^n$. We use Noether normalization to obtain a finite morphism $\pi : X \to \mathbb{A}^n$. By Lemma 2.5.9, the sheaf $\pi_* F$ is weakly constructible.

Let then $v : V \hookrightarrow \mathbb{A}^n$ be a Zariski open set with the property that $F' := v_! v^* \pi_* F$ satisfies the Basic Lemma II on $\mathbb{A}^n$. Let $U := \pi^{-1}(V) \overset{j}{\hookrightarrow} X$ be the preimage in X. One has an equality of sheaves:

$$\pi_* j_! j^* F = v_! v^* \pi_* F.$$

Therefore, $H^q(X, j_! j^* F) = H^q(\mathbb{A}^n, v_! v^* \pi_* F)$ and the latter vanishes for $q \neq n$.

So let us now assume that F is weakly constructible on $X = \mathbb{A}^n$. We argue by induction on n and all F. The case $n = 0$ is trivial.

By replacing F by $j_! j^* F$ for an appropriate open $j : U \to \mathbb{A}^n$, we may assume that F is locally constant on U and that $\mathbb{A}^n \setminus U = V(f)$. By Noether normalization or its proof, there is a surjective projection map $p : \mathbb{A}^n \to \mathbb{A}^{n-1}$ such that $p|_{V(f)} : V(f) \to \mathbb{A}^{n-1}$ is surjective and finite.

We will see in Lemma 2.5.10 that $R^q p_* F = 0$ for $q \neq 1$ and $R^1 p_* F$ is weakly constructible. The Leray spectral sequence now gives that

$$H^q(\mathbb{A}^n, F) = H^{q-1}(\mathbb{A}^{n-1}, R^1 \pi_* F).$$

In the induction procedure we apply the Basic Lemma II to $R^1 p_* F$ on $\mathbb{A}^{n-1}$. By induction, there exists a Zariski open $h : V \hookrightarrow \mathbb{A}^{n-1}$ such that $h_! h^* R^1 \pi_* F$ has cohomology only in degree $n - 1$. Let $U := \pi^{-1}(V)$ and $j : U \hookrightarrow \mathbb{A}^n$ be the inclusion. Then $j_! j^* F$ has cohomology only in degree n. $\square$

Lemma 2.5.10. *Let p be as in the above proof. Then $R^q \pi_* F = 0$ for $q \neq 0$ and $R^1 \pi_* F$ is weakly constructible.*

Proof. This is a standard fact, but Nori gives a direct proof.

The stalk of $R^q p_* F$ at $y \in \mathbb{A}^{n-1}$ is given by $H^q(\{y\} \times \mathbb{A}^1, F_{\{y\} \times \mathbb{A}^1})$ by the variation of proper base change in Theorem 2.5.11 below.

Let, more generally, G be a sheaf on $\mathbb{A}^1$ which is locally constant outside a finite, non-empty set S. Let T be a tree in $\mathbb{A}^1(\mathbb{C})$ with vertex set S. Then the restriction map to the tree defines a retraction isomorphism $H^q(\mathbb{A}^1, G) \cong H^q(T, G_T)$ for all $q \geq 0$. Using Čech cohomology, we can compute that $H^q(T, G_T)$: For each vertex $v \in S$, let U_s be the star of half edges of length more than half the length of all outgoing edges at the vertex s. Then U_a and U_b only intersect if the vertices a and b have a common edge $e = e(a, b)$. The intersection $U_a \cap U_b$ is contractible and contains the center $t(e)$ of the edge e. There are no triple intersections. Therefore $H^q(T, G_T) = 0$ for $q \geq 2$. Since G is zero on S, U_s is simply connected, and G is locally constant, $G(U_s) = 0$ for all s. Therefore also $H^0(T, G_T) = 0$ and $H^1(T, G_T)$ is isomorphic to $\bigoplus_e G_{t(e)}$.

This implies already that $R^q p_* F = 0$ for $q \neq 1$.

To show that R^1p_*F is weakly constructible, means to show that it is locally constant on some stratification. We see that the stalks $(R^1p_*F)_y$ depend only on the set of points in $\{y\} \times \mathbb{A}^1 = p^{-1}(y)$ where $F_{\{y\}\times\mathbb{A}^1}$ vanishes. But the sets of points where the vanishing set has the same degree (cardinality) defines a suitable stratification. Note that the stratification only depends on the branching behaviour of $V(f) \to \mathbb{A}^{n-1}$, hence the stratification is algebraic and defined over k. $\square$

Theorem 2.5.11 (Variation of Proper Base Change)**.** *Let $p : X \to Y$ be a continuous map between locally compact, locally contractible topological spaces which is a fiber bundle and let G be a sheaf on X. Assume $W \subset X$ is closed and such that G is locally constant on $X \setminus W$ and p restricted to W is proper. Then $(R^qp_*G)_y \cong H^q(p^{-1}(y), G_{p^{-1}(y)})$ for all q and all $y \in Y$.*

Proof. The statement is local on Y, so we may assume that $X = T\times Y$ is a product with p the projection. Since Y is locally compact and locally contractible, we may assume that Y is compact by passing to a compact neighbourhood of y. As $W \to Y$ is proper, this implies that W is compact. By enlarging W, we may assume that $W = K \times Y$ is a product of compact sets for some compact subset $K \subset X$. Since Y is locally contractible, we replace Y be a contractible neighbourhood. (We may loose compactness, but this does not matter anymore.) Let $i : K \times Y \to X$ be the inclusion and and $j : (T \setminus K) \times Y \to X$ the complement.

Look at the exact sequence

$$0 \to j_!G_{(T\setminus K)\times Y} \to G \to i_*G_{K\times Y} \to 0.$$

The result holds for $G_{K\times Y}$ by the usual proper base change.

Since Y is contractible, we may assume that $G_{(T\setminus K)\times Y}$ is the pull-back of constant sheaf on $T \setminus K$. Now the result for $j_!G_{(T\setminus K)\times Y}$ follows from the Künneth formula. $\square$

2.5.3 Beilinson's proof of Basic Lemma II, 1. and 2.

We follow Beilinson [Be1] Proof 3.3.1. His proof is even more general, as he does not assume X to be affine. Note that Beilinson's proof is in the setting of étale sheaves, independent of the characteristic of the ground field. We have translated it to weakly constructible sheaves. The argument is intrinsically about perverse sheaves, even though we have downplayed their use as far as possible. For a complete development of the theory of perverse sheaves in the weakly constructible setting see Schürmann's monograph [Schü].

Let X be affine reduced of dimension n over a field $k \subset \mathbb{C}$. Let F be a weakly constructible sheaf on X. We choose a projective compactification $\kappa : X \hookrightarrow \bar{X}$ such that κ is an affine morphism. Let W be a divisor on X such that F is a locally constant sheaf on $h : X \setminus W \hookrightarrow X$ and $X \setminus W$ is smooth. Then define $M := h_!h^*F$.

Let $\bar{H} \subset \bar{X}$ be a generic hyperplane. (We will see in the proof of Lemma 2.5.12 below what the conditions on $\bar{H}$ are.) Let $H = X \cap \bar{H}$ be the hyperplane in X.

We denote by $V = \bar{X} \setminus \bar{H}$ the complement and by $\ell : V \hookrightarrow \bar{X}$ the open inclusion. Furthermore, let $\kappa_V : V \cap X \hookrightarrow V$ and $\ell_X : V \cap X \hookrightarrow X$ be the open inclusion maps, and $i : \bar{H} \hookrightarrow \bar{X}$ and $i_X : H \hookrightarrow X$ the closed immersions. We set $U := X \setminus (W \cup H)$ and consider the open inclusion $j : U \hookrightarrow X$ with complement $Z = W \cup (H \cap X)$. Let $M_{V\cap X}$ be the restriction of M to $V \cap X$. Summarizing, we have a commutative diagram

$$\begin{tikzcd}
 & U \arrow[d, "j"] & \\
V \cap X \arrow[r, "\ell_X"] \arrow[d, "\kappa_V"'] & X \arrow[d, "\kappa"] & H \arrow[l, "i_X"'] \arrow[d, "\tilde{\kappa}"] \\
V \arrow[r, "\ell"] & \bar{X} & \bar{H}. \arrow[l, "i"']
\end{tikzcd}$$

Lemma 2.5.12. *For generic $\bar{H}$ in the above set-up, there is an isomorphism*

$$\ell_! \ell^* R\kappa_* M \xrightarrow{\cong} R\kappa_* \ell_{X*} M_{V\cap X}$$

extending naturally $\mathrm{id} : M_{V\cap X} \to M_{V\cap X}$.

Proof. We consider the map of distinguished triangles

$$\begin{tikzcd}
\ell_! \ell^* R\kappa_* M \arrow[r] \arrow[d] & R\kappa_* M \arrow[r] \arrow[d, "\mathrm{id}"'] & i_* i^* R\kappa_* M \arrow[d] \\
R\kappa_* \ell_{X!} M_{V\cap X} \arrow[r] & R\kappa_* M \arrow[r] & i_* R\tilde{\kappa}_* i_X^* M
\end{tikzcd}$$

(the existence of the arrows follows from standard adjunctions together with proper base change in the simple form $\kappa^* \ell_! = \ell_{X!} \kappa_V^*$ and $\kappa^* i_* = i_{X*} \bar{\kappa}^*$, respectively).

Hence it is sufficient to prove that

$$i^* R\kappa_* M \xrightarrow{\cong} R\tilde{\kappa}_* i_X^* M. \tag{2.1}$$

To prove this, we make a base change to the universal hyperplane section. In detail: Let $\mathbb{P}$ be the space of hyperplanes in $\bar{X}$. Let

$$\overline{\mathcal{H}}_{\mathbb{P}} \to \mathbb{P}$$

be the universal family. It comes with a natural map

$$i_{\mathbb{P}} : \overline{\mathcal{H}} \to \bar{X}.$$

Let $\mathcal{H}$ be the preimage of X. By [Gro2, pg. 9] and [Jo, Thm. 6.10] there is a dense Zariski open subset $T \subset \mathbb{P}$ such that the induced map

$$i_T : \overline{\mathcal{H}}_T \hookrightarrow \bar{X} \times T \longrightarrow \bar{X}$$

is smooth.

We apply smooth base change in the square

$$\begin{array}{ccc} \mathcal{H}_T & \xrightarrow{i_{X,T}} & X \\ {\scriptstyle \bar\kappa_T}\downarrow & & \downarrow{\scriptstyle \kappa} \\ \overline{\mathcal{H}}_T & \xrightarrow{i_T} & \bar{X} \end{array}$$

and obtain a quasi-isomorphism

$$i_T^* R\kappa_* M \xrightarrow{\cong} R\tilde{\kappa}_{T*} i_{X,T}^* M$$

of complexes of sheaves on $\overline{\mathcal{H}}_T$.

We specialize to some $t \in T(k)$ and get a hyperplane $t : \bar{H} \subset \overline{\mathcal{H}}_T$ to which we restrict. The left hand side turns into $i^* R\kappa_* M$.

The right hand side turns into

$$t^* R\bar{\kappa}_{T*} i_{X,T}^* M = R\bar{\kappa}_* t_X^* i_{X,T}^* M = R\bar{\kappa} i_X^* M$$

by applying the generic base change theorem 2.5.14 to $\bar{\kappa}_T$ over the base T and $\mathcal{G} = i_{X,T}^* M$. This requires to shrink T further.

Putting these equations together, we have verified equation 2.1. □

Proof of Basic Lemma II. We keep the notation fixed in this section. By Artin vanishing for constructible sheaves (see Theorem 2.5.13), the group $H^i(X, j_! j^* F)$ vanishes for $i > n$. It remains to show that $H^i(X, j_! j^* F)$ vanishes for $i < n$. We obviously have $j_! j^* F = \ell_{X!} M_{V \cap X}$. Therefore,

$$\begin{aligned} H^i(X, j_! j^* F) &= H^i(X, \ell_{X!} M_{V \cap X}) \\ &= H^i(\bar{X}, R\kappa_* \ell_{X!} M_{V \cap X}) \\ &= H^i(\bar{X}, \ell_! \ell^* R\kappa_* M) \quad \text{by 2.5.12} \\ &= H^i_c(V, (R\kappa_* M)_V). \end{aligned}$$

The last group vanishes for $i < n$ by Artin's vanishing theorem 2.5.13 for compact supports once we have checked that $R\kappa_* M_V[n]$ is perverse. Recall that $M = h_! h^* F|_{X \setminus W}$ with $F|_{X \setminus W}$ locally constant sheaf on a smooth variety. Hence $F|_{X \setminus W}[n]$ is perverse. Both h and κ are affine, hence the same is true for $R\kappa_* h_! F|_{X \setminus W}$ by Theorem 2.5.13 3. □

We now formulate the version of Artin vanishing used in the above proof. If X is a topological space, and $j : X \hookrightarrow \bar{X}$ an arbitrary compactification, then cohomology with supports with coefficients in a weakly constructible sheaf $\mathcal{G}$ is defined by

$$H^i_c(X, \mathcal{G}) := H^i(\bar{X}, j_!\mathcal{G}).$$

It follows from proper base change that this is independent of the choice of compactification.

Theorem 2.5.13 (Artin vanishing for constructible sheaves)**.** *Let X be affine of dimension n.*

1. *Let $\mathcal{G}$ be weakly constructible on X. Then $H^q(X, \mathcal{G}) = 0$ for $q > n$;*
2. *Let $\mathcal{F}_\bullet$ be a perverse sheaf on X for the middle perversity. Then $H^q_c(X, \mathcal{F}_\bullet) = 0$ for $q < 0$.*
3. *Let $g : U \to X$ be an open immersion and $\mathcal{F}_\bullet$ a perverse sheaf on U. Then both $g_!\mathcal{F}_\bullet$ and $Rg_*\mathcal{F}_\bullet$ are perverse on X.*

Proof. The first two statements are [Schü, Corollary 6.0.4, p. 391]. Note that a weakly constructible sheaf lies in ${}^mD^{\leq n}(X)$ in the notation of loc.cit.

The last statement combines the vanishing results for affine morphisms [Schü, Theorem 6.0.4, p. 409] with the standard vanishing for all compactifiable morphisms [Schü, Corollary 6.0.5, p. 397] for a morphism of relative dimension 0. □

Theorem 2.5.14 (Generic base change)**.** *Let S be of finite type over k, $f : X \to Y$ a morphism of S-varieties. Let $\mathcal{F}$ be a weakly constructible sheaf on X. Then there is a dense open subset $U \subset S$ such that:*

1. *over U, the sheaves $R^if_*\mathcal{F}$ are weakly constructible and almost all vanish;*
2. *the formation of $R^if_*\mathcal{F}$ is compatible with any base change $S' \to U \subset S$.*

This is the analogue of [SGA 4 1/2, Théorème 1.9 in sect. Thm. finitude], which is for constructible étale sheaves in the étale setting.

Proof. The case $S = Y$ was treated by Arapura, see [Ara, Theorem 3.1.10]. We explain the reduction to this case, using the same arguments as in the étale case.

All schemes can be assumed reduced.

Using Nagata, we can factor f as a composition of an open immersion and a proper map. The assertion holds for the latter by the proper base change theorem, hence it suffices to consider open immersions.

As the question is local on Y, we may assume that it is affine over S. We can then cover X by affines. Using the hypercohomology spectral sequence for the covering, we may reduce to the case X affine. In this case (X and Y affine, f

an open immersion) we argue by induction on the dimension of the generic fibre of $X \to S$.

If $n = 0$, then, at least after shrinking S, we are in the situation where f is the inclusion of a connected component and the assertion is trivial.

We now assume the case $n-1$. We embed Y into $\mathbb{A}^m_S$ and consider the coordinate projections $p_i : Y \to \mathbb{A}^1_S$. We apply the inductive hypothesis to the map f over $\mathbb{A}^1_S$. Hence there is an open dense $U_i \subset \mathbb{A}^1_S$ such that the conclusion is valid over $p^{-1}U_i$. Hence the conclusion is valid over their union, i.e., outside a closed subvariety $Y_1 \subset Y$ finite over S. By shrinking S, we may assume that it is finite étale.

We fix the notation in the resulting diagram as follows:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "a"'] & Y \arrow[d, "b"] & Y_1 \arrow[l, "i"'] \arrow[dl, "b_1"] \\ & S & \end{tikzcd}$$

Let j be the open complement of i. We have checked that $j^*Rf_*\mathcal{G}$ is weakly constructible and compatible with any base change. We apply Rb_* to the triangle defined by the sequence

$$j_!j^*Rf_*\mathcal{G} \to Rf_*\mathcal{G} \to i_*i^*Rf_*\mathcal{G}$$

and obtain

$$Rb_*j_!j^*Rf_*\mathcal{G} \to Ra_*\mathcal{G} \to b_{1*}i^*Rf_*\mathcal{G}.$$

The first two terms are (after shrinking of S) constructible by the previous considerations and the case $S = Y$. We also obtain that they are compatible with any base change. Hence the same is true for the third term. As b_1 is finite étale this also implies that $i^*Rf_*\mathcal{G}$ is constructible and compatible with base change. (Indeed, this follows because a direct sum of sheaves is constant if and only if every summand is constant.) The same is true for $j_!j^*Rf_*\mathcal{G}$ by the previous considerations and base change for $j_!$. Hence the conclusion also holds for the middle term of the first triangle and we are done.

$\square$

2.6 Triangulation of Algebraic Varieties

If X is a variety defined over $\mathbb{Q}$, we may ask whether any singular homology class $\gamma \in H^{\mathrm{sing}}_\bullet(X^{\mathrm{an}};\mathbb{Q})$ can be represented by an object described by polynomials. This is indeed the case: for a precise statement we need several definitions. The result will be formulated in Proposition 2.6.8.

This section follows closely the Diploma thesis of Benjamin Friedrich, see [Fr]. The results are due to him.

We work over $k = \widetilde{\mathbb{Q}}$, i.e., the integral closure of $\mathbb{Q}$ in $\mathbb{R}$. Note that $\widetilde{\mathbb{Q}}$ is a field. In this section, we use X to denote a variety over $\widetilde{\mathbb{Q}}$, and X^{an} for the associated analytic space over $\mathbb{C}$ (cf. Subsection 1.2).

2.6.1 Semi-algebraic Sets

Definition 2.6.1 ([Hi2, Def. 1.1., p.166])**.** A subset of $\mathbb{R}^n$ is said to be $\widetilde{\mathbb{Q}}$-*semi-algebraic*, if it is of the form

$$\{\underline{x} \in \mathbb{R}^n | f(\underline{x}) \geq 0\}$$

for some polynomial $f \in \widetilde{\mathbb{Q}}[x_1, \ldots, x_n]$, or can be obtained from sets of this form in a finite number of steps, where each step consists of one of the following basic operations:

1. complementary set,
2. finite intersection,
3. finite union.

We need also a definition for maps:

Definition 2.6.2 ($\widetilde{\mathbb{Q}}$-semi-algebraic map [Hi2, p. 168])**.** A continuous map f between $\widetilde{\mathbb{Q}}$-semi-algebraic sets $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ is said to be $\widetilde{\mathbb{Q}}$-*semi-algebraic*, if its graph

$$\Gamma_f := \big\{\big(a, f(a)\big) \,|\, a \in A\big\} \subseteq A \times B \subseteq \mathbb{R}^{n+m}$$

is $\widetilde{\mathbb{Q}}$-semi-algebraic.

Example 2.6.3. Any polynomial map

$$\begin{aligned} f : A &\longrightarrow B \\ (a_1, \ldots, a_n) &\mapsto (f_1(a_1, \ldots, a_n), \ldots, f_m(a_1, \ldots, a_n)) \end{aligned}$$

between $\widetilde{\mathbb{Q}}$-semi-algebraic sets $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ with $f_i \in \widetilde{\mathbb{Q}}[x_1, \ldots, x_n]$ for $i = 1, \ldots, m$ is $\widetilde{\mathbb{Q}}$-semi-algebraic, since it is continuous and its graph $\Gamma_f \subseteq \mathbb{R}^{n+m}$ is cut out from $A \times B$ by the polynomials

$$y_i - f_i(x_1, \ldots, x_n) \in \widetilde{\mathbb{Q}}[x_1, \ldots, x_n, y_1, \ldots, y_m] \quad \text{for} \quad i = 1, \ldots, m. \tag{2.2}$$

We can even allow f to be a rational map with rational component functions

$$f_i \in \widetilde{\mathbb{Q}}(x_1, \ldots, x_n), \quad i = 1, \ldots, m$$

as long as none of the denominators of the f_i vanish at a point of A. The argument remains the same except that the expression (2.2) has to be multiplied by the denominator of f_i.

Fact 2.6.4 ([Hi2, Prop. II, p. 167], [Sb, Thm. 3, p. 370]).
By a result of Seidenberg-Tarski, the image (respectively preimage) of a $\widetilde{\mathbb{Q}}$-semi-algebraic set under a $\widetilde{\mathbb{Q}}$-semi-algebraic map is again $\widetilde{\mathbb{Q}}$-semi-algebraic.

As the name suggests, any algebraic set should be in particular $\widetilde{\mathbb{Q}}$-semi-algebraic.

Lemma 2.6.5. *Let X be a quasi-projective algebraic variety defined over $\widetilde{\mathbb{Q}}$. Then we can regard the complex analytic space X^{an} associated to the base change $X_{\mathbb{C}} = X \times_{\widetilde{\mathbb{Q}}} \mathbb{C}$ as a bounded $\widetilde{\mathbb{Q}}$-semi-algebraic subset*

$$X^{\mathrm{an}} \subseteq \mathbb{R}^N \tag{2.3}$$

for some N. Moreover, if $f : X \to Y$ is a morphism of varieties defined over $\widetilde{\mathbb{Q}}$, we can consider $f^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$ as a $\widetilde{\mathbb{Q}}$-semi-algebraic map.

Remark 2.6.6. We will mostly need the case when X is even *affine*. Then $X \subset \mathbb{C}^n$ is defined by polynomial equations with coefficients in $\widetilde{\mathbb{Q}}$. We identify $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and rewrite the equations for the real and imaginary part. Hence X is obviously $\widetilde{\mathbb{Q}}$-semialgebraic. In the lemma, we will show in addition that X can be embedded as a *bounded* $\widetilde{\mathbb{Q}}$-semialgebraic set.

Proof of Lemma 2.6.5. First step $X = \mathbb{P}^n_{\widetilde{\mathbb{Q}}}$: Consider

- $\mathbb{P}^n_{\mathbb{C}} = (\mathbb{P}^n_{\widetilde{\mathbb{Q}}} \times_{\widetilde{\mathbb{Q}}} \mathbb{C})^{\mathrm{an}}$ with homogenous coordinates $x_0, \ldots, x_n$, which we split as $x_m = a_m + ib_m$ with $a_m, b_m \in \mathbb{R}$ in real and imaginary part, and
- $\mathbb{R}^N$, $N = 2(n+1)^2$, with coordinates $\{y_{kl}, z_{kl}\}_{k,l=0,\ldots,n}$.

We define a map

$$\psi : \underset{[x_0:\ldots:x_n]}{\mathbb{P}^n_{\mathbb{C}}} \longrightarrow \underset{(y_{00},z_{00},\ldots,y_{nn},z_{nn})}{\mathbb{R}^N}$$

$$[x_0 : \ldots : x_n] \mapsto \Bigg(\ldots, \underbrace{\frac{\operatorname{Re} x_k \overline{x}_l}{\sum_{m=0}^n |x_m|^2}}_{y_{kl}}, \underbrace{\frac{\operatorname{Im} x_k \overline{x}_l}{\sum_{m=0}^n |x_m|^2}}_{z_{kl}}, \ldots \Bigg)$$

$$[a_0 + ib_0 : \ldots : a_n + ib_n] \mapsto \Bigg(\ldots, \underbrace{\frac{a_k a_l + b_k b_l}{\sum_{m=0}^n a_m^2 + b_m^2}}_{y_{kl}}, \underbrace{\frac{b_k a_l - a_k b_l}{\sum_{m=0}^n a_m^2 + b_m^2}}_{z_{kl}}, \ldots \Bigg).$$

Rewriting the last line (with the convention $0 \cdot \cos(\genfrac{}{}{0pt}{}{\text{indeterminate}}{\text{angle}}) = 0$) as

$$[r_0 e^{i\phi_0} : \ldots : r_n e^{i\phi_n}] \mapsto \left(\ldots, \frac{r_k r_l \cos(\phi_k - \phi_l)}{\sum_{m=0}^n r_m^2}, \frac{r_k r_l \sin(\phi_k - \phi_l)}{\sum_{m=0}^n r_m^2}, \ldots \right) \tag{2.4}$$

shows that ψ is injective: Assume

$$\psi\big([r_0e^{i\phi_0} : \ldots : r_ne^{i\phi_n}]\big) = (y_{00}, z_{00}, \ldots, y_{nn}, z_{nn})$$

where $r_k \neq 0$, or equivalently $y_{kk} \neq 0$, for a fixed k. We find

$$\frac{r_l}{r_k} = \frac{\sqrt{y_{kl}^2 + z_{kl}^2}}{y_{kk}}, \quad \text{and}$$
$$\phi_k - \phi_l = \begin{cases} \arctan(z_{kl}/y_{kl}) & \text{if } y_{kl} \neq 0, \\ \pi/2 & \text{if } y_{kl} = 0, z_{kl} > 0, \\ \text{indeterminate} & \text{if } y_{kl} = z_{kl} = 0, \\ -\pi/2 & \text{if } y_{kl} = 0, z_{kl} < 0; \end{cases}$$

that is, the preimage of $(y_{00}, z_{00}, \ldots, y_{nn}, z_{nn})$ is uniquely determined.

Therefore, we can consider $\mathbb{P}^n_{\mathbb{C}}$ via ψ as a subset of $\mathbb{R}^N$. It is bounded since it is contained in the unit sphere $S^{N-1} \subset \mathbb{R}^N$. We claim that $\psi(\mathbb{P}^n_{\mathbb{C}})$ is also $\widetilde{\mathbb{Q}}$-semi-algebraic. The composition of the projection

$$\begin{aligned} \pi : \mathbb{R}^{2(n+1)} \setminus \{(0, \ldots, 0)\} &\longrightarrow \mathbb{P}^n_{\mathbb{C}} \\ (a_0, b_0, \ldots, a_n, b_n) &\mapsto [a_0 + ib_0 : \ldots : a_n + ib_n] \end{aligned}$$

with the map ψ is a polynomial map, hence $\widetilde{\mathbb{Q}}$-semi-algebraic by Example 2.6.3. Thus

$$\operatorname{Im} \psi \circ \pi = \operatorname{Im} \psi \subseteq \mathbb{R}^N$$

is $\widetilde{\mathbb{Q}}$-semi-algebraic by Fact 2.6.4.

Second step (zero set of a polynomial): We use the notation

$$\begin{aligned} &V(g) := \{\underline{x} \in \mathbb{P}^n_{\mathbb{C}} \mid g(\underline{x}) = 0\} \quad \text{for} \quad g \in \mathbb{C}[x_0, \ldots, x_n] \text{ homogenous,} \quad \text{and} \\ &W(h) := \{\underline{t} \in \mathbb{R}^N \mid h(\underline{t}) = 0\} \quad \text{for} \quad h \in \mathbb{C}[y_{00}, \ldots, z_{nn}]. \end{aligned}$$

Let $X^{\text{an}} = V(g)$ for some homogenous $g \in \widetilde{\mathbb{Q}}[x_0, \ldots, x_n]$. Then $\psi(X^{\text{an}}) \subseteq \mathbb{R}^N$ is a $\widetilde{\mathbb{Q}}$-semi-algebraic subset, as a little calculation shows. Setting for $k = 0, \ldots, n$

$$\begin{aligned} g_k &:= \text{“}g(\underline{x}\,\overline{x}_k)\text{”} \\ &= g(x_0\overline{x}_k, \ldots, x_n\overline{x}_k) \\ &= g\big((a_0a_k + b_0b_k) + i(b_0a_k - a_0b_k), \ldots, (a_na_k + b_nb_k) + i(b_na_k - a_nb_k)\big), \end{aligned}$$

where $x_j = a_j + ib_j$ for $j = 0, \ldots, n$, and

$$h_k := g(y_{0k} + iz_{0k}, \ldots, y_{nk} + iz_{nk}),$$

we obtain

$$\begin{aligned}\psi(X^{\mathrm{an}}) &= \psi(V(g))\\ &= \bigcap_{k=0}^{n} \psi(V(g_k))\\ &= \bigcap_{k=0}^{n} \psi(\mathbb{P}^n_{\mathbb{C}}) \cap W(h_k)\\ &= \bigcap_{k=0}^{n} \psi(\mathbb{P}^n_{\mathbb{C}}) \cap W(\operatorname{Re} h_k) \cap W(\operatorname{Im} h_k).\end{aligned}$$

Final step: We can choose an embedding

$$X \subseteq \mathbb{P}^n_{\widetilde{\mathbb{Q}}},$$

thus getting

$$X^{\mathrm{an}} \subseteq \mathbb{P}^n_{\mathbb{C}}.$$

Since X is a locally closed subvariety of $\mathbb{P}^n_{\widetilde{\mathbb{Q}}}$, the space X^{an} can be expressed in terms of subvarieties of the form $V(g)$ with $g \in \widetilde{\mathbb{Q}}[x_0, \ldots, x_n]$, using only the following basic operations

1. complementary set,
2. finite intersection,
3. finite union.

Now $\widetilde{\mathbb{Q}}$-semi-algebraic sets are stable under these operations as well and the first assertion is proved.

Second assertion: The first part of the lemma provides us with $\widetilde{\mathbb{Q}}$-semi-algebraic inclusions

$$\psi : X^{\mathrm{an}} \subseteq \underset{\underline{x}=[x_0:\ldots:x_n]}{\mathbb{P}^n_{\mathbb{C}}} \subseteq \underset{(y_{00}, z_{00}, \ldots, y_{nn}, z_{nn})}{\mathbb{R}^N},$$

$$\phi : Y^{\mathrm{an}} \subseteq \underset{\underline{u}=[u_0:\ldots:u_m]}{\mathbb{P}^m_{\mathbb{C}}} \subseteq \underset{(v_{00}, w_{00}, \ldots, v_{mm}, w_{mm})}{\mathbb{R}^M},$$

and a choice of coordinates as indicated. We use the notation

$$\begin{aligned}V(g) &:= \{(\underline{x}, \underline{u}) \in \mathbb{P}^n_{\mathbb{C}} \times \mathbb{P}^m_{\mathbb{C}} \mid g(\underline{x}, \underline{u}) = 0\},\\ &\quad \text{for } g \in \mathbb{C}[x_0, \ldots, x_n, u_0, \ldots, u_m] \text{ homogenous in both } \underline{x} \text{ and } \underline{u}, \quad \text{and}\\ W(h) &:= \{\underline{t} \in \mathbb{R}^{N+M} \mid h(\underline{t}) = 0\}, \quad \text{for } h \in \mathbb{C}[y_{00}, \ldots, z_{nn}, v_{00}, \ldots, w_{mm}].\end{aligned}$$

Let $\{U_i\}$ be a finite open affine covering of X such that $f(U_i)$ satisfies

- $f(U_i)$ does not meet the hyperplane $\{u_j = 0\} \subset \mathbb{P}^m_{\widetilde{\mathbb{Q}}}$ for some j, and
- $f(U_i)$ is contained in an open affine subset V_i of Y.

This is always possible, since we can start with the open covering $Y \cap \{u_j \neq 0\}$ of Y, take a subordinated open affine covering $\{V_{i'}\}$, and then choose a finite open affine covering $\{U_i\}$ subordinated to $\{f^{-1}(V_{i'})\}$. Now each of the maps

$$f_i := f^{\mathrm{an}}{}_{|U_i} : U_i^{\mathrm{an}} \longrightarrow Y^{\mathrm{an}}$$

has image contained in V_i^{an} and does not meet the hyperplane $\{\underline{u} \in \mathbb{P}^m_{\mathbb{C}} \mid u_j = 0\}$ for an appropriate j

$$f_i : U_i^{\mathrm{an}} \longrightarrow V_i^{\mathrm{an}}.$$

Being associated to an algebraic map between affine varieties, this map is rational

$$f_i : \underline{x} \mapsto \left[\frac{g_0'(\underline{x})}{g_0''(\underline{x})} : \ldots : \frac{g_{j-1}'(\underline{x})}{g_{j-1}''(\underline{x})} : \underset{j}{1} : \frac{g_{j+1}'(\underline{x})}{g_{j+1}''(\underline{x})} : \ldots : \frac{g_m'(\underline{x})}{g_m''(\underline{x})} \right],$$

with $g_k', g_k'' \in \widetilde{\mathbb{Q}}[x_0, \ldots, x_n]$, $k = 0, \ldots, \widehat{j}, \ldots, m$. Since the graph $\Gamma_{f^{\mathrm{an}}}$ of f^{an} is the finite union of the graphs Γ_{f_i} of the f_i, it is sufficient to prove that $(\psi \times \phi)(\Gamma_{f_i})$ is a $\widetilde{\mathbb{Q}}$-semi-algebraic subset of $\mathbb{R}^{N+M}$. Now

$$\Gamma_{f_i} = (U_i^{\mathrm{an}} \times V_i^{\mathrm{an}}) \cap \bigcap_{\substack{k=0 \\ k \neq j}}^{n} V\left(\frac{y_k}{y_j} - \frac{g_k'(\underline{x})}{g_k''(\underline{x})} \right) = (U_i^{\mathrm{an}} \times V_i^{\mathrm{an}}) \cap \bigcap_{\substack{k=0 \\ k \neq j}}^{n} V(y_k g_k''(\underline{x}) - y_j g_k'(\underline{x})),$$

so all we have to deal with is

$$V(y_k g_k''(\underline{x}) - y_j g_k'(\underline{x})).$$

Again a little calculation is necessary. Setting

$$\begin{aligned}
g_{pq} &:= \text{``} u_k \overline{u}_q g_k''(\underline{x}\, \overline{x}_p) - u_j \overline{u}_q g_k'(\underline{x}\, \overline{x}_p) \text{''} \\
&= u_k \overline{u}_q g_k''(x_0 \overline{x}_p, \ldots, x_n \overline{x}_p) - u_j \overline{u}_q g_k'(x_0 \overline{x}_p, \ldots, x_n \overline{x}_p) \\
&= \big((c_k c_q + d_k d_q) + i(d_k c_q - c_k d_q) \big) \\
&\qquad g_k''\big((a_0 a_p + b_0 b_p) + i(b_0 a_p - a_0 b_p), \ldots, (a_n a_p + b_n b_p) + i(b_n a_p - a_n b_p) \big) \\
&\quad - \big((c_j c_q + d_j d_q) + i(d_j c_q - c_j d_q) \big) \\
&\qquad g_k'\big((a_0 a_p + b_0 b_p) + i(b_0 a_p - a_0 b_p), \ldots, (a_n a_p + b_n b_p) + i(b_n a_p - a_n b_p) \big),
\end{aligned}$$

where $x_l = a_l + ib_l$ for $l = 0, \ldots, n$, $u_l = c_l + id_l$ for $l = 0, \ldots, m$, and

$$h_{pq} := (v_{kq} + iw_{kq}) g_k''(y_{0p} + iz_{0p}, \ldots, y_{np} + iz_{np}) - (v_{jq} + iw_{jq}) g_k'(y_{0p} + iz_{0p}, \ldots, y_{np} + iz_{np}),$$

we obtain

$$
\begin{aligned}
(\psi \times \phi)\Big(V\big(y_k g_k''(\underline{x}) - y_j g_k'(\underline{x})\big)\Big) = & \\
&= \bigcap_{p=0}^{n} \bigcap_{q=0}^{m} (\psi \times \phi)(V(g_{pq})) \\
&= \bigcap_{p=0}^{n} \bigcap_{q=0}^{m} (\psi \times \phi)(U_i^{\mathrm{an}} \times V_j^{\mathrm{an}}) \cap W(h_{pq}) \\
&= \bigcap_{p=0}^{n} \bigcap_{q=0}^{m} (\psi \times \phi)(U_i^{\mathrm{an}} \times V_j^{\mathrm{an}}) \cap W(\operatorname{Re} h_{pq}) \cap W(\operatorname{Im} h_{pq}).
\end{aligned}
$$

□

2.6.2 Semi-algebraic singular chains

We need further prerequisites in order to state the promised Proposition 2.6.8.

Definition 2.6.7 ([Hi2, p. 168])**.** By an *open simplex* $\triangle^\circ$ we mean the interior of a simplex (= the convex hull of $r+1$ points in $\mathbb{R}^n$ which span an r-dimensional subspace). For convenience, a point is considered as an open simplex as well.

The notation $\triangle_d^{\mathrm{std}}$ will be reserved for the *closed standard simplex* spanned by the standard basis

$$\{e_i = (0, \dots, 0, \underset{i}{1}, 0, \dots, 0) \mid i = 1, \dots, d+1\}$$

of $\mathbb{R}^{d+1}$.

Consider the following data $(*)$:

- X a variety defined over $\widetilde{\mathbb{Q}}$,
- D a divisor in X with normal crossings,
- and finally $\gamma \in H_p^{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q})$, $p \in \mathbb{N}_0$.

As before, we have denoted by X^{an} (resp. D^{an}) the complex analytic space associated to the base change $X_{\mathbb{C}} = X \times_{\widetilde{\mathbb{Q}}} \mathbb{C}$ (resp. $D_{\mathbb{C}} = D \times_{\widetilde{\mathbb{Q}}} \mathbb{C}$).

By Lemma 2.6.5, we may consider both X^{an} and D^{an} as bounded $\widetilde{\mathbb{Q}}$-semi-algebraic subsets of $\mathbb{R}^N$.

We are now able to formulate our proposition.

Proposition 2.6.8. *With data $(*)$ as above, we can find a representative of γ that is a rational linear combination of singular simplices each of which is $\widetilde{\mathbb{Q}}$-semi-algebraic.*

The proof of this proposition relies on the following proposition due to Lojasiewicz which has been written down by Hironaka.

Proposition 2.6.9 ([Hi2, p. 170]). *For $\{X_i\}$ a finite system of bounded $\widetilde{\mathbb{Q}}$-semi-algebraic sets in $\mathbb{R}^n$, there exists a simplicial decomposition*

$$\mathbb{R}^n = \coprod_j \triangle^\circ{}_j$$

by open simplices $\triangle^\circ{}_j$ and a $\widetilde{\mathbb{Q}}$-semi-algebraic automorphism

$$\kappa : \mathbb{R}^n \to \mathbb{R}^n$$

such that each X_i is a finite union of some of the $\kappa(\triangle^\circ{}_j)$.

Note 2.6.10. Although Hironaka considers $\mathbb{R}$-semi-algebraic sets, we can safely replace $\mathbb{R}$ by $\widetilde{\mathbb{Q}}$ in this result (including the fact he cites from [Sb]). The only problem that could possibly arise concerns a "good direction lemma":

Lemma 2.6.11 (Good direction lemma for $\mathbb{R}$, [Hi2, p. 172], [KB, Thm. 5.I, p. 242]). *Let Z be a $\mathbb{R}$-semi-algebraic subset of $\mathbb{R}^n$, which is nowhere dense. A direction $v \in \mathbb{P}^{n-1}_{\mathbb{R}}(\mathbb{R})$ is called* good*, if any line l in $\mathbb{R}^n$ parallel to v meets Z in a discrete (maybe empty) set of points; otherwise v is called* bad. *Then the set $B(Z)$ of bad directions is a Baire category set in $\mathbb{P}^{n-1}_{\mathbb{R}}(\mathbb{R})$.*

This gives immediately good directions $v \in \mathbb{P}^{n-1}_{\mathbb{R}}(\mathbb{R}) \setminus B(Z)$, but not necessarily $v \in \mathbb{P}^{n-1}_{\widetilde{\mathbb{Q}}}(\widetilde{\mathbb{Q}}) \setminus B(Z)$. However, in Remark 2.1 of [Hi2], which follows directly after the lemma, the following statement is made: If Z is compact, then $B(Z)$ is closed in $\mathbb{P}^{n-1}_{\mathbb{R}}(\mathbb{R})$. In particular $\mathbb{P}^{n-1}_{\widetilde{\mathbb{Q}}}(\widetilde{\mathbb{Q}}) \setminus B(Z)$ will be non-empty. Since we only consider *bounded* $\widetilde{\mathbb{Q}}$-semi-algebraic sets Z', we may take $Z := \overline{Z'}$ (which is compact by Heine-Borel), and thus find a good direction $v \in \mathbb{P}^{n-1}_{\widetilde{\mathbb{Q}}}(\widetilde{\mathbb{Q}}) \setminus B(Z')$ using $B(Z') \subseteq B(Z)$. Hence:

Lemma 2.6.12 (Good direction lemma for $\widetilde{\mathbb{Q}}$). *Let Z' be a bounded $\widetilde{\mathbb{Q}}$-semi-algebraic subset of $\mathbb{R}^n$, which is nowhere dense. Then the set $\mathbb{P}^{n-1}_{\mathbb{R}}(\mathbb{R}) \setminus B(Z)$ of good directions is non-empty.*

Proof of Proposition 2.6.8. Applying Proposition 2.6.9 to the two-element system of $\widetilde{\mathbb{Q}}$-semi-algebraic sets $X^{\mathrm{an}}, D^{\mathrm{an}} \subseteq \mathbb{R}^N$, we obtain a $\widetilde{\mathbb{Q}}$-semi-algebraic decomposition

$$\mathbb{R}^N = \coprod_j \triangle^\circ{}_j$$

of $\mathbb{R}^N$ by open simplices $\triangle^\circ{}_j$ and a $\widetilde{\mathbb{Q}}$-semi-algebraic automorphism

$$\kappa : \mathbb{R}^N \to \mathbb{R}^N.$$

We write $\triangle_j$ for the closure of $\overset{\circ}{\triangle}_j$. The sets

$$K := \{\overset{\circ}{\triangle}_j \mid \kappa(\overset{\circ}{\triangle}_j) \subseteq X^{\mathrm{an}}\} \quad \text{and} \quad L := \{\overset{\circ}{\triangle}_j \mid \kappa(\overset{\circ}{\triangle}_j) \subseteq D^{\mathrm{an}}\}$$

can be thought of as finite simplicial complexes, but built out of open simplices instead of closed ones. We define their *geometric realizations*

$$|K| := \bigcup_{\overset{\circ}{\triangle}_j \in K} \overset{\circ}{\triangle}_j \quad \text{and} \quad |L| := \bigcup_{\overset{\circ}{\triangle}_j \in L} \overset{\circ}{\triangle}_j.$$

Then Proposition 2.6.9 states that κ maps the pair of topological spaces $(|K|, |L|)$ homeomorphically to $(X^{\mathrm{an}}, D^{\mathrm{an}})$.

Easy case: If X is complete, so is $X_{\mathbb{C}}$ (by [Ha2, Cor. II.4.8(c), p. 102]), hence X^{an} and D^{an} will be compact [Ha2, B.1, p. 439]. In this situation,

$$\overline{K} := \{\triangle_j \mid \kappa(\triangle_j) \subseteq X^{\mathrm{an}}\} \quad \text{and} \quad \overline{L} := \{\triangle_j \mid \kappa(\triangle_j) \subseteq D^{\mathrm{an}}\}$$

are (ordinary) simplicial complexes, whose geometric realizations coincide with those of K and L, respectively. In particular

$$\begin{aligned} H_\bullet^{\mathrm{simpl}}(\overline{K}, \overline{L}; \mathbb{Q}) &\cong H_\bullet^{\mathrm{sing}}(|\overline{K}|, |\overline{L}|; \mathbb{Q}) \\ &\cong H_\bullet^{\mathrm{sing}}(|K|, |L|; \mathbb{Q}) \\ &\cong H_\bullet^{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}; \mathbb{Q}). \end{aligned} \tag{2.5}$$

Here $H_\bullet^{\mathrm{simpl}}(\overline{K}, \overline{L}; \mathbb{Q})$ denotes simplicial homology of course.

We write $\gamma_{\mathrm{simpl}} \in H_p^{\mathrm{simpl}}(\overline{K}, \overline{L}; \mathbb{Q})$ and $\gamma_{\mathrm{sing}} \in H_p^{\mathrm{sing}}(|\overline{K}|, |\overline{L}|; \mathbb{Q})$ for the image of γ under this isomorphism. Any representative Γ_{simpl} of γ_{simpl} is a rational linear combination

$$\Gamma_{\mathrm{simpl}} = \textstyle\sum_j a_j \, \triangle_j, \quad a_j \in \mathbb{Q}$$

of oriented closed simplices $\triangle_j \in \overline{K}$. We can choose orientation-preserving affine-linear maps of the standard simplex $\triangle_p^{\mathrm{std}}$ to $\triangle_j$

$$\sigma_j : \triangle_p^{\mathrm{std}} \longrightarrow \triangle_j \quad \text{for} \quad \triangle_j \in \Gamma_{\mathrm{simpl}}.$$

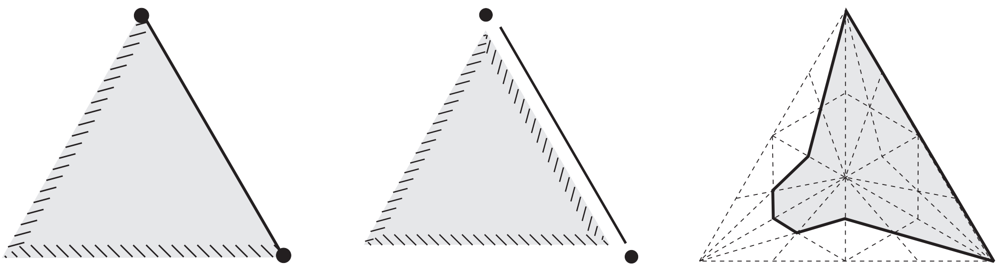
These maps yield a representative

$$\Gamma_{\mathrm{sing}} := \textstyle\sum_j a_j \, \sigma_j$$

of γ_{sing}. Composing with κ yields $\Gamma := \kappa_* \Gamma_{\mathrm{sing}} \in \gamma$, where Γ has the desired properties.

In the *general case*, we perform a barycentric subdivision $\mathcal{B}$ on K twice (once is not enough) and define $|K|$ and $|L|$ not as the "closure" of K and L, but as follows (see Figure 2.1)

$$\begin{aligned} \overline{K} &:= \{\triangle \mid \overset{\circ}{\triangle} \in \mathcal{B}^2(K) \text{ and } \triangle \subseteq |K|\}, \\ \overline{L} &:= \{\triangle \mid \overset{\circ}{\triangle} \in \mathcal{B}^2(K) \text{ and } \triangle \subseteq |L|\}. \end{aligned} \tag{2.6}$$



$\kappa^{-1}(X^{\mathrm{an}}) \cap \triangle_j$
Intersection of $\kappa^{-1}(X^{\mathrm{an}})$ with a closed 2-simplex $\triangle_j$, where we assume that part of the boundary $\partial\triangle_j$ does not belong to $\kappa^{-1}(X^{\mathrm{an}})$

$K \cap \triangle_j$
Open simplices of K contained in $\triangle_j$

$\left|\overline{K}\right| \cap \triangle_j$
Intersection of $\left|\overline{K}\right|$ with $\triangle_j$ (the dashed lines show the barycentric subdivision)

Figure 2.1: Definition of $\overline{K}$

The point is that the pair of topological spaces $(\left|\overline{K}\right|, \left|\overline{L}\right|)$ is a strong deformation retract of $(|K|, |L|)$. Assuming this, we see that in the general case with $\overline{K}$, $\overline{L}$ defined as in (2.6), the isomorphism (2.5) still holds and we can proceed as in the easy case to prove the proposition.

We define the retraction map

$$\rho : (|K| \times [0,1], |L| \times [0,1]) \to (\left|\overline{K}\right|, \left|\overline{L}\right|)$$

as follows: Let $\overset{\circ}{\triangle}_j \in K$ be an open simplex which is not contained in the boundary of any other simplex of K and set

$$\mathit{inner} := \triangle_j \cap \overline{K}, \qquad \mathit{outer} := \triangle_j \setminus \overline{K}.$$

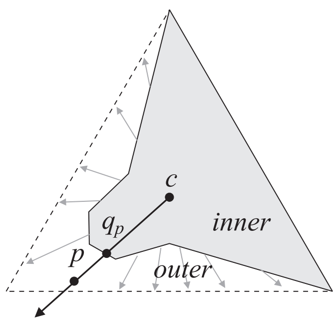


Figure 2.2: Definition of q_p

Note that *inner* is closed. For any point $p \in \mathit{outer}$ the ray $\overrightarrow{cp}$ from the center c of $\overset{\circ}{\triangle}_j$ through p "leaves" the set *inner* at a point q_p, i.e. $\overrightarrow{cp} \cap \mathit{inner}$ equals the line segment $c\,q_p$; see Figure 2.2. The map

$$\begin{aligned}\rho_j : \triangle_j \times [0,1] &\to \triangle_j \\ (p,t) &\mapsto \begin{cases} p & \text{if } p \in \mathit{inner}, \\ q_p + t \cdot (p - q_p) & \text{if} \quad p \in \mathit{outer} \end{cases}\end{aligned}$$

retracts $\triangle_j$ onto *inner*.

Now these maps ρ_j glue together to give the desired homotopy ρ. □

We want to state one of the intermediate results of this proof explicitly:

Corollary 2.6.13. *Let X and D be as above. Then the pair of topological spaces $(X^{\mathrm{an}}, D^{\mathrm{an}})$ is homotopy equivalent to a pair of (realizations of) simplicial complexes $(|X^{\mathrm{simpl}}|, |D^{\mathrm{simpl}}|)$.*

2.7 Singular cohomology via the h′-topology

In order to give a simple description of the period isomorphism for singular varieties, we are going to need a more sophisticated description of singular cohomology.

We work in the category of complex analytic spaces An.

Definition 2.7.1. Let X be a complex analytic space. The h′*-topology* on the category $(\mathrm{An}/X)_{\mathrm{h}'}$ of complex analytic spaces over X is the smallest Grothendieck topology such that the following are covering maps:

1. proper surjective morphisms;
2. open covers.

If $\mathcal{F}$ is a presheaf of An/X we denote $\mathcal{F}_{\mathrm{h}'}$ its sheafification in the h′-topology.

Remark 2.7.2. This definition is inspired by Voevodsky's h-topology on the category of schemes, see Section 3.2. We are not sure if it is the correct analogue in the analytic setting. However, it is good enough for our purposes.

Lemma 2.7.3. *For $Y \in \mathrm{An}$ let $\mathbb{C}_Y$ be the (ordinary) sheaf associated to the presheaf $\mathbb{C}$. Then*

$$Y \mapsto \mathbb{C}_Y(Y)$$

is an h′*-sheaf on* An.

Proof. We have to check the sheaf condition for the generators of the topology. By assumption it is satisfied for open covers. Let $\tilde{Y} \to Y$ be proper surjective.

Without loss of generality Y is connected. Let $\tilde{Y}_i$ for $i \in I$ be the collection of connected components of $\tilde{Y}$. Then

$$\tilde{Y} \times_Y \tilde{Y} = \bigcup_{i,j \in I} \tilde{Y}_i \times_Y \tilde{Y}_j$$

We have to compute the kernel of

$$\prod_{i \in I} \mathbb{C} \to \prod_{i,j} \mathbb{C}(\tilde{Y}_i \times_Y \tilde{Y}_j)$$

via the difference of the two natural restriction maps. Comparing a_i and a_j in $\mathbb{C}(\tilde{Y}_i \times_Y \tilde{Y}_j)$ we see that they agree. Hence the kernel is just one copy of $\mathbb{C} = \mathbb{C}_Y(Y)$. □

Proposition 2.7.4. *Let X be an analytic space and $i : Z \subset X$ a closed subspace. Then there is a morphism of sites $\rho : (\mathrm{An}/X)_{\mathrm{h}'} \to X$. It induces an isomorphism*

$$H^i_{\mathrm{sing}}(X, Z; \mathbb{C}) \to H^i_{\mathrm{h}'}((\mathrm{An}/X)_{\mathrm{h}'}, \mathrm{Ker}(\mathbb{C}_{\mathrm{h}'} \to i_*\mathbb{C}_{\mathrm{h}'}))$$

compatible with long exact sequences and products.

Remark 2.7.5. This statement and the following proof can be extended to more general sheaves $\mathcal{F}$.

The argument uses the notion of a hypercover, see Definition 1.5.8.

Proof. We first treat the absolute case with $Z = \emptyset$. We use the theory of cohomological descent as developed in [SGA4Vbis]. Singular cohomology satisfies cohomological descent for open covers and also for proper surjective maps (see Theorem 2.7.6). (The main ingredient for the second case is the proper base change theorem.) Hence it satisfies cohomological descent for h$'$-covers. This implies that singular cohomology can be computed as a direct limit

$$\lim_{\mathfrak{X}_\bullet} \mathbb{C}(\mathfrak{X}_\bullet),$$

where $\mathfrak{X}_\bullet$ runs through all h$'$-hypercovers. On the other hand, the same limit computes h$'$-cohomology, see Proposition 1.6.9 For the general case, recall that we have a short exact sequence

$$0 \to j_!\mathbb{C} \to \mathbb{C} \to i_*\mathbb{C} \to 0$$

of sheaves on X. Its pull-back to An$/X$ maps naturally to the short exact sequence

$$0 \to \mathrm{Ker}(\mathbb{C}_{\mathrm{h}'} \to i_*\mathbb{C}_{\mathrm{h}'})) \to \mathbb{C}_{\mathrm{h}'} \to i_*\mathbb{C}_{\mathrm{h}'} \to 0\ .$$

This reduces the comparison in the relative case to the absolute case once we have shown that $Ri_*\mathbb{C}_{\mathrm{h}'} = i_*\mathbb{C}_{\mathrm{h}'}$. The sheaf $R^n i_*\mathbb{C}_{\mathrm{h}'}$ is given by the h′-sheafification of the presheaf

$$X' \mapsto H^n_{\mathrm{h}'}(Z \times_X X', \mathbb{C}_{\mathrm{h}'}) = H^n_{\mathrm{sing}}(Z \times_X X', \mathbb{C})$$

for $X' \to X$ in An/X. By resolution of singularities for analytic spaces we may assume that X' is smooth and $Z' = X' \times_X Z$ a divisor with normal crossings. By passing to an open cover, we may assume that Z' an open ball in a union of coordinate hyperplanes, in particular contractible. Hence its singular cohomology is trivial. This implies that $R^n i_*\mathbb{C}_{\mathrm{h}'} = 0$ for $n \geq 1$. □

Theorem 2.7.6 (Descent for proper hypercoverings)**.** *Let $D \subset X$ be a closed subvariety and $D_\bullet \to D$ a proper hypercover(see Definition 1.5.8), such that there is a commutative diagram*

$$\begin{array}{ccc} D_\bullet & \longrightarrow & X_\bullet \\ \downarrow & & \downarrow \\ D & \longrightarrow & X \end{array}$$

Then one has cohomological descent for singular cohomology:

$$H^*(X, D; \mathbb{Z}) = H^*\left(\mathrm{Cone}(\mathrm{Tot}(X_\bullet) \to \mathrm{Tot}(D_\bullet))[-1]; \mathbb{Z}\right).$$

Here, $\mathrm{Tot}(-)$ *denotes the total complex in* $\mathbb{Z}[\mathrm{Var}]$ *associated to the corresponding simplicial variety, see Definition 1.5.11.*

Proof. The relative case follows from the absolute case. The essential ingredient is proper base change, which allows to reduce to the case where X is a point. The statement then becomes a completely combinatorial assertion on contractibility of simplicial sets. The results are summed up in [D5] (5.3.5). For a complete reference see [SGA4Vbis], in particular Corollaire 4.1.6. □

Chapter 3

Algebraic de Rham cohomology

Let k be a field of characteristic zero. We are going to define relative algebraic de Rham cohomology for general varieties over k, not necessarily smooth.

3.1 The smooth case

In this section, all varieties are smooth over k. In this case, de Rham cohomology is defined as hypercohomology of the complex of sheaves of differentials.

3.1.1 Definition

Definition 3.1.1. Let X be a smooth variety over k. Let Ω^1_X be the sheaf of k-differentials on X. For $p \geq 0$ let

$$\Omega^p_X = \Lambda^p \Omega^1_X$$

be the exterior power in the category of $\mathcal{O}_X$-modules. The universal k-derivation $d : \mathcal{O}_X \to \Omega^1_X$ induces

$$d^p : \Omega^p_X \to \Omega^{p+1}_X \ .$$

We call $(\Omega^\bullet_X, d)$ the *algebraic de Rham complex* of X.

If X is smooth of dimension n, the sheaf Ω^1_X is locally free of rank n. This allows to define exterior powers. Note that Ω^i_X vanishes for $i > n$. The differential is uniquely characterized by the properties:

1. $d^0 = d$ on $\mathcal{O}_X$;

2. $d^{p+1}d^p = 0$ for all $p \geq 0$;
3. $d(\omega \wedge \omega') = d\omega \wedge \omega' + (-1)^p \omega \wedge d\omega'$ for all local sections ω of Ω^p_X and ω' of $\Omega^{p'}_X$.

Indeed, if $t_1, \dots, t_n$ is a system of local parameters at $x \in X$, then local sections of Ω^p_X near x can be expressed as

$$\omega = \sum_{1 \leq i_1 < \dots < i_p \leq n} f_{i_1 \dots i_p} dt_{i_1} \wedge \dots \wedge dt_{i_p}$$

and we have

$$d^p \omega = \sum_{1 \leq i_1 < \dots < i_p \leq n} df_{i_1 \dots i_p} \wedge dt_{i_1} \wedge \dots \wedge dt_{i_p} \ .$$

Definition 3.1.2. Let X be smooth variety over a field k of characteristic 0. We define *algebraic de Rham cohomology* of X as the hypercohomology

$$H^i_{\mathrm{dR}}(X) = H^i(X, \Omega^\bullet_X) \ .$$

For background material on hypercohomology see Section 1.4.

If X is smooth and affine, this simplifies to

$$H^i_{\mathrm{dR}}(X) = H^i(\Omega^\bullet_X(X)) \ .$$

Example 3.1.3. 1. Consider the affine line $X = \mathbb{A}^1_k = \operatorname{Spec} k[t]$. Then

$$\Omega^\bullet_{\mathbb{A}^1}(\mathbb{A}^1) = \left[k[t] \xrightarrow{d} k[t]dt \right] \ .$$

We have

$$\operatorname{Ker}(d) = \{P \in k[t] | P' = 0\} = k \ , \quad \operatorname{Im}(d) = k[t]dt \ ,$$

because we have assumed characteristic zero. Hence

$$H^i_{\mathrm{dR}}(\mathbb{A}^1) = \begin{cases} k & i = 0, \\ 0 & i > 0. \end{cases}$$

2. Consider the multiplicative group $X = \mathbb{G}_m = \operatorname{Spec} k[t, t^{-1}]$. Then

$$\Omega^\bullet_{\mathbb{G}_m}(\mathbb{G}_m) = \left[k[t, t^{-1}] \xrightarrow{d} k[t, t^{-1}]dt \right] \ .$$

We have

$$\operatorname{Ker}(d) = \{P \in k[t] | P' = 0\} = k \ ,$$

$$\operatorname{Im}(d) = \{\sum_{i=n}^{N} a_i t^i dt | a_{-1} = 0\} \ ,$$

again because of characteristic zero. Hence

$$H^i_{\mathrm{dR}}(\mathbb{G}_m) = \begin{cases} k & i = 0, 1, \\ 0 & i > 1. \end{cases}$$

The isomorphism for $i = 1$ is induced by the residue for meromorphic differential forms.

3. Let X be a connected smooth projective curve of genus g. We use the stupid filtration on the de Rham complex

$$0 \to \Omega^1_X[-1] \to \Omega^\bullet_X \to \mathcal{O}_X[0] \to 0 \ .$$

The cohomological dimension of any variety X is the index i above which the cohomology $H^i(X, \mathcal{F})$ of any coherent sheaf $\mathcal{F}$ vanishes, see [Ha2, Chap. III, Section 4]. The cohomological dimension of a smooth, projective curve is 1, hence the long exact sequence reads

$$\begin{aligned} 0 \to H^{-1}(X, \Omega^1_X) \to H^0_{\mathrm{dR}}(X) \to H^0(X, \mathcal{O}_X) \\ \xrightarrow{\partial} H^0(X, \Omega^1_X) \to H^1_{\mathrm{dR}}(X) \to H^1(X, \mathcal{O}_X) \\ \xrightarrow{\partial} H^1(X, \Omega^1_X) \to H^2_{\mathrm{dR}}(X) \to 0 \end{aligned}$$

This is a special case of the Hodge spectral sequence. It is known to degenerate (e.g. [D4]). Hence the boundary maps ∂ vanish. By Serre duality, this yields

$$H^i_{\mathrm{dR}}(X) = \begin{cases} H^0(X, \mathcal{O}_X) = k & i = 0, \\ H^1(X, \Omega^1_X) = H^0(X, \mathcal{O}_X)^\vee = k & i = 2, \\ 0 & i > 2. \end{cases}$$

The most interesting group for $i = 1$ sits in an exact sequence

$$0 \to H^0(X, \Omega^1_X)^\vee \to H^1_{\mathrm{dR}}(X) \to H^0(X, \Omega^1_X) \to 0$$

and hence

$$\dim H^1_{\mathrm{dR}}(X) = 2g \ .$$

Remark 3.1.4. In these cases, the explicit computation shows that algebraic de Rham cohomology computes the standard Betti numbers of these varieties. We are going to show in chapter 5 that this is always true. In particular, it is always finite dimensional. A second algebraic proof of this fact will also be given in Corollary 3.1.17.

Lemma 3.1.5. *Let X be a smooth variety of dimension d. Then $H^i_{\mathrm{dR}}(X)$ vanishes for $i > 2d$. If in addition X is affine, it vanishes for $i > d$.*

Proof. We use the stupid filtration on the de Rham complex. This induces a system of long exact sequences relating the groups $H^i(X,\Omega^p_X)$ to algebraic de Rham cohomology.

Any variety of dimension d has cohomological dimension $\leq d$ for coherent sheaves [Ha2, ibid.]. All Ω^p_X are coherent, hence $H^i(X,\Omega^p_X)$ vanishes for $i>d$. The complex $\Omega^\bullet_X$ is concentrated in degrees at most d. This adds up to cohomological dimension $2d$ for algebraic de Rham cohomology. Affine varieties have cohomological dimension 0, hence $H^i(X,\Omega^p_X)$ vanishes already for $i>0$. □

3.1.2 Functoriality

Let $f:X\to Y$ be morphism of smooth varieties over k. We want to explain the functoriality

$$f^*:H^i_{\mathrm{dR}}(Y)\to H^i_{\mathrm{dR}}(X)\ .$$

We use the Godement resolution (see Definition 1.4.8) and put

$$R\Gamma_{\mathrm{dR}}(X)=\Gamma(X,Gd(\Omega^\bullet_X))\ .$$

The natural map $f^{-1}\mathcal{O}_X\to\mathcal{O}_X$ induces a unique multiplicative map

$$f^{-1}\Omega^\bullet_X\to\Omega^\bullet_Y\ .$$

By functoriality of the Godement resolution, we have

$$f^{-1}Gd_X(\Omega^\bullet_X)\to Gd_Y(f^{-1}\Omega^\bullet_X)\to Gd_Y(\Omega^\bullet_Y)\ .$$

Taking global sections, this yields

$$R\Gamma_{\mathrm{dR}}(Y)\to R\Gamma_{\mathrm{dR}}(X)\ .$$

We have shown:

Lemma 3.1.6. *De Rham cohomology $H^i_{\mathrm{dR}}(\cdot)$ is a contravariant functor on the category of smooth varieties over k with values in k-vector spaces. It is induced by a functor*

$$R\Gamma_{\mathrm{dR}}:\mathrm{Sm}\to C^+(k\mathrm{-Mod})\ .$$

Note that $\mathbb{Q}\subset k$, so the functor can be extended $\mathbb{Q}$-linearly to $\mathbb{Q}[\mathrm{Sm}]$. This allows to extend the definition of algebraic de Rham cohomology to complexes of smooth varieties in the next step. Explicitly: let $X^\bullet$ be an object of $C^-(\mathbb{Q}[\mathrm{Sm}])$. Then there is a double complex $K^{\bullet,\bullet}$ with

$$K^{n,m}=\Gamma(X^{-n},Gd^m(\Omega^\bullet))\ .$$

Definition 3.1.7. Let $X^\bullet$ be a object of $C^-(\mathbb{Z}[\mathrm{Sm}])$. We denote the total complex by

$$R\Gamma_{\mathrm{dR}}(X^\bullet)=\mathrm{Tot}(K^{\bullet,\bullet})$$

and set

$$H^i_{\mathrm{dR}}(X^\bullet)=H^i(R\Gamma_{\mathrm{dR}}(X^\bullet)\ .$$

We call this the *algebraic de Rham cohomology* of $X^\bullet$.

3.1.3 Cup product

Let X be a smooth variety over k. Wedge product of differential forms turns $\Omega^\bullet_X$ into a differential graded algebra:

$$\mathrm{Tot}(\Omega^\bullet_X \otimes_k \Omega^\bullet_X) \to \Omega^\bullet_X \ .$$

The compatibility with differentials was built into the definition of d in Definition 3.1.1.

Lemma 3.1.8. *$H^\bullet_{\mathrm{dR}}(X)$ carries a natural multiplication*

$$\cup : H^i_{\mathrm{dR}}(X) \otimes_k H^j_{\mathrm{dR}}(X) \to H^{i+j}_{\mathrm{dR}}(X)$$

induced from wedge product of differential forms.

Proof. We need to define

$$R\Gamma_{\mathrm{dR}}(X) \otimes_k R\Gamma_{\mathrm{dR}}(X) \to R\Gamma_{\mathrm{dR}}(X)$$

as a morphism in the derived category. We have quasi-isomorphisms

$$\Omega^\bullet_X \otimes \Omega^\bullet_X \to Gd(\Omega^\bullet_X) \otimes Gd(\Omega^\bullet_X)$$

and hence a quasi-isomorphism of flasque resolutions of $\Omega^\bullet_X \otimes \Omega^\bullet_X$

$$s : Gd(\Omega^\bullet_X \otimes \Omega^\bullet_X) \to Gd\left(Gd(\Omega^\bullet_X) \otimes Gd(\Omega^\bullet_X)\right)$$

In the derived category, this allows the composition

$$\begin{aligned} R\Gamma_{\mathrm{dR}}(X) \otimes_k R\Gamma_{\mathrm{dR}}(X) &= \Gamma(X, Gd(\Omega^\bullet_X)) \otimes_k \Gamma(X, Gd(\Omega^\bullet_X)) \\ &\to \Gamma(X, Gd(\Omega^\bullet) \otimes Gd(\Omega^\bullet_X)) \\ &\to \Gamma\left(X, Gd\left(Gd(\Omega^\bullet_X) \otimes Gd(\Omega^\bullet_X)\right)\right) \\ &\leftarrow s\Gamma(X, Gd(\Omega^\bullet_X \otimes \Omega^\bullet_X)) \\ &\to \Gamma(X, Gd(\Omega^\bullet_X)) = R\Gamma_{\mathrm{dR}}(X) \ . \end{aligned}$$

□

The same method also allows the construction of an exterior product.

Proposition 3.1.9 (Künneth formula)**.** *Let X, Y be smooth varieties. There is a natural multiplication induced from wedge product of differential forms*

$$H^i_{\mathrm{dR}}(X) \otimes_k H^j_{\mathrm{dR}}(Y) \to H^{i+j}_{\mathrm{dR}}(X \times Y) \ .$$

It induces an isomorphism

$$H^n_{\mathrm{dR}}(X \times Y) \cong \bigoplus_{i+j=n} H^i_{\mathrm{dR}}(X) \otimes_k H^j_{\mathrm{dR}}(Y) \ .$$

Proof. Let $p : X \times Y \to X$ and $q : X \times Y \to Y$ be the projection maps. The exterior multiplication is given by

$$H^i_{\mathrm{dR}}(X) \otimes H^j_{\mathrm{dR}}(Y) \xrightarrow{p^* \otimes q^*} H^i_{\mathrm{dR}}(X \times Y) \otimes H^j_{\mathrm{dR}}(X \times Y) \xrightarrow{\cup} H^{i+j}_{\mathrm{dR}}(X \times Y) \ .$$

The Künneth formula is most easily proved by comparison with singular cohomology. We postpone the proof to Lemma 5.3.2 in Chap. 5. □

Corollary 3.1.10 (Homotopy invariance)**.** *Let X be a smooth variety. Then the natural map*

$$H^n_{\mathrm{dR}}(X) \to H^n_{\mathrm{dR}}(X \times \mathbb{A}^1)$$

is an isomorphism.

Proof. We combine the Künneth formula with the compuation in the case of $\mathbb{A}^1$ in Example 3.1.3. □

3.1.4 Change of base field

Let K/k be an extension of fields of characteristic zero. We have the corresponding base change functor

$$X \mapsto X_K$$

from (smooth) varieties over k to (smooth) varieties over K. Let

$$\pi : X_K \to X$$

be the natural map of schemes. By standard calculus of differential forms,

$$\Omega^\bullet_{X_K/K} \cong \pi^*\Omega^\bullet_{X/k} = \pi^{-1}\Omega^\bullet_{X/k} \otimes_k K \ .$$

Lemma 3.1.11. *Let K/k be an extension of fields of characteristic zero. Let X be a smooth variety over k. Then there are natural isomorphisms*

$$H^i_{\mathrm{dR}}(X) \otimes_k K \to H^i_{\mathrm{dR}}(X_K) \ .$$

They are induced by a natural quasi-isomorphism

$$R\Gamma_{\mathrm{dR}}(X) \otimes_k K \to R\Gamma_{\mathrm{dR}}(X_K) \ .$$

Proof. By functoriality of the Godement resolution (see Lemma 1.4.10) and k-linarity, we get natural quasi-isomorphisms

$$\pi^{-1}Gd_X(\Omega^\bullet_{X/k}) \otimes_k K \to Gd_{X_K}(\pi^{-1}\Omega^\bullet_{X/k}) \to Gd_{X_K}(\Omega^\bullet_{X_K/K}) \ .$$

As K is flat over k, taking global sections induces a sequence of quasi-isomorphisms

$$\begin{aligned} R\Gamma_{\mathrm{dR}}(X) \otimes_k K &= \Gamma(X, Gd_X(\Omega^\bullet_{X/k})) \otimes_k K \\ &\cong \Gamma(X_K, \pi^{-1}Gd_X(\Omega^\bullet_{X/k}) \otimes_k K \\ &\cong \Gamma(X_K, \pi^{-1}Gd_X(\Omega^\bullet_{X/k}) \otimes_k K) \\ &\to \Gamma(X_K, Gd_{X_K}(\Omega^\bullet_{X_K/K}) \\ &= R\Gamma_{\mathrm{dR}}(X_K)\ . \end{aligned}$$

□

Remark 3.1.12. This immediately extends to algebraic de Rham cohomology of complexes of smooth varieties.

Conversely, we can also restrict scalars.

Lemma 3.1.13. *Let K/k be a finite field extension. Let Y be a smooth variety over K. Then there are a natural isomorphism*

$$H^i_{\mathrm{dR}}(Y/k) \to H^i_{\mathrm{dR}}(Y/K).$$

They are induced by a natural isomorphism

$$R\Gamma_{\mathrm{dR}}(Y/k) \to R\Gamma_{\mathrm{dR}}(Y/K).$$

Proof. We use the sequence of sheaves on Y ([Ha2] Proposition 8.11)

$$\pi^*\Omega^1_{K/k} \to \Omega^1_{Y/k} \to \Omega^1_{Y/K} \to 0$$

where $\pi : Y \to \mathrm{Spec} K$ is the structural map. As we are in characteristic 0, we have $\Omega^1_{K/k} = 0$. This implies that we actually have identical de Rham complexes

$$\Omega^\bullet_{Y/K} = \Omega^\bullet_{Y/k}$$

and identical Godement resolutions. □

3.1.5 Étale topology

In this section, we give an alternative interpretation of algebraic de Rham cohomology using the étale topology. The results are not used in our discussions of periods.

Let X_{et} be the small étale site on X, see section 1.6. The complex of differential forms $\Omega^\bullet_X$ can be viewed as a complex of sheaves on X_{et} (see [Mi], Chap. II, Example 1.2 and Proposition 1.3). We write $\Omega^\bullet_{X_{\mathrm{et}}}$ for distinction.

Lemma 3.1.14. *There is a natural isomorphism*

$$H^i_{\mathrm{dR}}(X) \to H^i(X_{\mathrm{et}}, \Omega^\bullet_{X_{\mathrm{et}}})\ .$$

Proof. The map of sites $s : X_{\mathrm{et}} \to X$ induces a map on cohomology

$$H^i(X, \Omega^\bullet_X) \to H^i(X_{\mathrm{et}}, \Omega^\bullet_{X_{\mathrm{et}}}) \ .$$

We filter $\Omega^\bullet_X$ by the stupid filtration $F^p\Omega^\bullet_X$

$$0 \to F^{p+1}\Omega^\bullet_X \to F^p\Omega^\bullet_X \to \Omega^p_X[-p] \to 0$$

and compare the induced long exact sequences in cohomology on X and X_{et}. As the Ω^p_X are coherent, the comparison maps

$$H^i(X, \Omega^p_X) \to H^i(X_{\mathrm{et}}, \Omega^p_{X_{\mathrm{et}}})$$

are isomorphisms by [Mi] Chap. III, Proposition 3.7. By descending induction on p, this implies that we have isomorphisms for all $F^p\Omega^\bullet_X$, in particular for $\Omega^\bullet_X$ itself. □

3.1.6 Differentials with log poles

We give an alternative description of algebraic de Rham cohomology using differentials with log poles as introduced by Deligne, see [D4], Chap. 3. We are not going to use this point of view in our study of periods.

Let X be a smooth variety and $j : X \to \bar{X}$ an open immersion into a smooth projective variety such that $D = \bar{X} \smallsetminus X$ is a simple divisor with normal crossings (see Definition 1.1.2).

Definition 3.1.15. Let

$$\Omega^1_{\bar{X}}\langle D\rangle \subset j_*\Omega^1_X$$

be the locally free $\mathcal{O}_{\bar{X}}$-module with the following basis: if $U \subset X$ is an affine open subvariety étale over $\mathbb{A}^n$ via coordinates $t_1, \dots, t_n$ and $D|_U$ given by the equation $t_1 \dots t_r = 0$, then $\Omega^1_{\bar{X}}\langle D\rangle|_U$ has $\mathcal{O}_{\bar{X}}$-basis

$$\frac{dt_1}{t_1}, \dots, \frac{dt_r}{t_r}, dt_{r+1}, \dots, dt_n \ .$$

For $p > 1$ let

$$\Omega^p_{\bar{X}}\langle D\rangle = \Lambda^p\Omega^1_{\bar{X}}\langle D\rangle \ .$$

We call the $\Omega^\bullet_{\bar{X}}\langle D\rangle$ the *complex of differentials with log poles along D.*

Note that the differential of $j_*\Omega^\bullet_X$ respects $\Omega^\bullet_{\bar{X}}\langle D\rangle$, so that this is indeed a complex.

Proposition 3.1.16. *The inclusion induces a natural isomorphism*

$$H^i(\bar{X}, \Omega^\bullet_{\bar{X}}\langle D\rangle) \to H^i(X, \Omega^\bullet_X) \ .$$

Proof. This is the algebraic version of [D4], Prop. 3.1.8. We indicate the argument. Note that $j : X \to \bar{X}$ is affine, hence j_* is exact and we have

$$H^i(X, \Omega^\bullet_X) \cong H^i(\bar{X}, j_*\Omega^\bullet_X) \ .$$

It remains to show that

$$\iota : \Omega^\bullet_{\bar{X}}\langle D\rangle \to j_*\Omega^\bullet_X$$

is a quasi-isomorphism, or, equivalently, that $\mathrm{Coker}(\iota)$ is exact. By Lemma 3.1.14 we can work in the étale topology. It suffices to check exactness in stalks in geometric points of $\bar{X}$ over closed points. As $\bar{X}$ is smooth and D a divisor with normal crossings, it suffices to consider the case $D = V(t_1 \dots t_r) \subset \mathbb{A}^n$ and the stalk in 0. As in the proof of the Poincaré lemma, it suffices to consider the case $n = 1$. If $r = 0$, then there is nothing to show.

In remains to consider the following situation: let $k = \bar{k}$, $\mathcal{O}$ be the henselization of $k[t]$ with respect to the ideal (t). We have to check that the complex

$$\mathcal{O}[t^{-1}]/\mathcal{O} \to \mathcal{O}[t^{-1}]/t^{-1}\mathcal{O}dt$$

is acyclic. The term in degree 0 has the $\mathcal{O}$-basis t^{-i} for $i > 0$. The term in degree 1 has the $\mathcal{O}$-basis $t^{-i}dt$ for $i > 1$. In this basis, the differential has the form

$$f\frac{dt}{t^i} \mapsto \begin{cases} f'\frac{dt}{t^i} - if\frac{dt}{t^{i+1}} & i > 1, \\ -f\frac{dt}{t^2} & i = 1. \end{cases}$$

It is injective because $\mathrm{char}(k) = 0$. By induction on i we also check that it is surjective. □

Corollary 3.1.17. *Let X be a smooth variety over k. Then the algebraic de Rham cohomology groups $H^i_{\mathrm{dR}}(X)$ are finite dimensional k-vector spaces.*

Proof. By resolution of singularities, we can embed X into a projective $\bar{X}$ such that D is a simple divisor with normal crossings. By the proposition

$$H^i_{\mathrm{dR}}(X) = H^i(\bar{X}, \Omega^\bullet_{\bar{X}}\langle D\rangle) \ .$$

Note that all $\Omega^\bullet_{\bar{X}}\langle D\rangle$ are coherent sheaves on a projective variety, hence the cohomology groups $H^p(\bar{X}, \Omega^q_{\bar{X}}\langle D\rangle)$ are finite dimensional over k. We use the stupid filtration on $\Omega^\bullet_{\bar{X}}\langle D\rangle$ and the induced long exact cohomology sequence. By induction, all $H^q(\bar{X}, F^p\Omega^\bullet_{\bar{X}}\langle D\rangle)$ are finite dimensional. □

Remark 3.1.18. The complex of differentials with log poles is studied intensively in the theory of mixed Hodge structures. Indeed, Deligne uses it in [D4] in order to define the Hodge and the weight filtration on cohomology of a smooth variety X. We are not going to use Hodge structures in the sequel though.

3.2 The general case: via the h-topology

We now want to extend the definition to the case of singular varieties and even to relative cohomology. The most simple minded idea – use Definition 3.1.2 – does not give the desired dimensions.

Example 3.2.1. Consider $X = \mathrm{Spec}A$ with $A = k[X,Y]/XY$, the union of two affine lines. This variety is homotopy equivalent to a point, so we expect its cohomology to be trivial. We compute the cohomology of the de Rham complex

$$A \to \langle dX, dY\rangle_A / \langle XdY + YdX\rangle_A \ .$$

Elements of A can be represented uniquely by polynomials of the form

$$P = \sum_{i=0}^{n} a_i X^i + \sum_{j=1}^{m} b_j Y^j$$

with

$$dP = \sum_{i=1}^{n} i a_i X^{i-1} dX + \sum_{j=1}^{m} b_j j Y^{j-1} dY \ .$$

P is in the kernel of d if it is constant. On the other hand d is not surjective because it misses differentials of the form $Y^i dX$.

There are different ways of adapting the definition in order to get a well-behaved theory.

The h-topology introduced by Voevodsky makes the handling of singular varieties straightforward. In this topology, any variety is locally smooth by resolution of singularities. The h-sheafification of the presheaf of Kähler differentials was studied in detail by Huber and Jörder in [HJ]. The weaker notion of eh-differential was already introduced by Geisser in [Ge].

We review a definition given by Voevodsky in [Voe].

Definition 3.2.2 ([Voe] Section 3.1)**.** A morphism of schemes $p : X \to Y$ is called *topological epimorphism* if Y has the quotient topology of X. It is a *universal topological epimorphism* if any base change of p is a topological epimorphism.

The h-topology on the category $(\mathrm{Sch}/X)_{\mathrm{h}}$ of separated schemes of finite type over X is the Grothendieck topology with coverings finite families $\{p_i : U_i \to Y\}$ such that $\bigcup_i U_i \to Y$ is a universal topological epimorphism.

By [Voe] the following are h-covers:

1. finite flat covers (in particular étale covers);
2. proper surjective morphisms;

3. quotients by finite groups actions.

The assignment $X \mapsto \Omega^p_{X/k}(X)$ is a presheaf on Sch. We denote by Ω^p_{h} (resp. $\Omega^p_{\mathrm{h}/X}$, if X needs to be specified) its sheafification in the h-topology, and by $\Omega^p_{\mathrm{h}}(X)$ its value as abelian group.

Definition 3.2.3. Let X be a separated k-scheme of finite type over k. We define

$$H^i_{\mathrm{dR}}(X_{\mathrm{h}}) = H^i((\mathrm{Sch}/X)_{\mathrm{h}}, \Omega^\bullet_{\mathrm{h}}) \ .$$

Proposition 3.2.4 ([HJ] Theorem 3.6, Proposition 7.4)**.** *Let X be smooth over k. Then*

$$\Omega^p_{\mathrm{h}}(X) = \Omega^p_{X/k}(X)$$

and

$$H^i_{\mathrm{dR}}(X_{\mathrm{h}}) = H^i_{\mathrm{dR}}(X) \ .$$

Proof. The statement on $\Omega^p_{\mathrm{h}}(X)$ is [HJ], Theorem 3.6. The statement on the de Rham cohomology is loc.cit., Proposition 7.4. together with the comparison of loc. cit., Lemma 7.22. □

Remark 3.2.5. The main ingredients of the proof are a normal form for h-covers established by Voevodsky in [Voe] Theorem 3.1.9, an explicit computation for the blow-up of a smooth variety in a smooth center and étale descent for the coherent sheaves $\Omega^p_{Y/k}$.

A particular useful h-cover are *abstract blow-ups*, covers of the form $(f : X' \to X, i : Z \to X)$ where Z is a closed immersion and f is proper and an isomorphism above $X - Z$.

Then, the above implies that there is a long exact blow-up sequence

$$\ldots \to H^i_{\mathrm{dR}}(X) \to H^i_{\mathrm{dR}}(X') \oplus H^i_{\mathrm{dR}}(Z) \to H^i_{\mathrm{dR}}(f^{-1}(Z)) \to \ldots$$

induced by the blow-up triangle

$$[f^{-1}(Z)] \to [X'] \oplus [Z] \to [X]$$

in SmCor.

Definition 3.2.6. Let $X \in \mathrm{Sch}$ and $i : Z \to X$ a closed subscheme. Put

$$\Omega^p_{\mathrm{h}/(X,Z)} = \mathrm{Ker}(\Omega^p_{\mathrm{h}/X} \to i_*\Omega^p_{\mathrm{h}/Z})$$

in the category of abelian sheaves on $(\mathrm{Sch}/X)_{\mathrm{h}}$.

We define *relative algebraic de Rham cohomology* as

$$H^p_{\mathrm{dR}}(X, Z) = H^p_{\mathrm{h}}(X, \Omega^\bullet_{\mathrm{h}/(X,Z)}) \ .$$

Lemma 3.2.7 ([HJ] Lemma 7.26)**.** *Let $i : Z \to X$ be a closed immersion.*

1. *Then*
$$Ri_*\Omega^p_{\mathrm{h}/Z} = i_*\Omega^p_{\mathrm{h}/Z}$$
and hence
$$H^q_{\mathrm{h}}(X, i_*\Omega^p_{\mathrm{h}/Z}) = H^q_{\mathrm{h}}(Z, \Omega^p_{\mathrm{h}}) \ .$$

2. *The natural map of sheaves of abelian groups on* $(\mathrm{Sch}/X)_{\mathrm{h}}$
$$\Omega^p_{\mathrm{h}/X} \to i_*\Omega^p_{\mathrm{h}/Z}$$
is surjective.

Remark 3.2.8. The main ingredient of the proof is resolution of singularities and the computation of $\Omega^p_{\mathrm{h}}(Z)$ for Z a divisor with normal crossings: it is given as Kähler differentials modulo torsion, see [HJ] Proposition 4.9.

Proposition 3.2.9 ((Long exact sequence) [HJ] Proposition 2.7)**.** *Let $Z \subset Y \subset X$ be closed immersions. Then there is a natural long exact sequence*

$$\cdots \to H^q_{\mathrm{dR}}(X, Y) \to H^q_{\mathrm{dR}}(X, Z) \to H^q_{\mathrm{dR}}(Y, Z) \to H^{q+1}_{\mathrm{dR}}(X, Y) \to \cdots$$

Remark 3.2.10. The sequence is the long exact cohomology sequence attached to the exact sequence of h-sheaves on X

$$0 \to \Omega^p_{\mathrm{h}/(X,Y)} \to \Omega^p_{\mathrm{h}/(X,Z)} \to i_{Y*}\Omega^p_{\mathrm{h}/(Y,Z)} \to 0$$

where $i_Y : Y \to X$ is the closed immersion.

Proposition 3.2.11 ((Excision) [HJ] Proposition 7.28)**.** *Let $\pi : \tilde{X} \to X$ be a proper surjective morphism, which is an isomorphism outside of $Z \subset X$. Let $\tilde{Z} = \pi^{-1}(Z)$. Then*
$$H^q_{\mathrm{dR}}(\tilde{X}, \tilde{Z}) \cong H^q_{\mathrm{dR}}(X, Z) \ .$$

Remark 3.2.12. This is an immediate consequence of the blow-up triangle.

Proposition 3.2.13 ((Künneth formula) [HJ] Proposition 7.29)**.** *Let $Z \subset X$ and $Z' \subset X'$ be closed immersions. Then there is a natural isomorphism*

$$H^n_{\mathrm{dR}}(X \times X', X \times Z' \cup Z \times X') = \bigoplus_{a+b=n} H^a_{\mathrm{dR}}(X, Z) \otimes_k H^b_{\mathrm{dR}}(X', Z') \ .$$

Proof. We explain the construction of the map. We work in the category of h-sheaves of k-vector spaces on $X \times X'$. Note that h-cohomology of an h-sheaf of k-vector spaces computed in the category of sheaves of abelian groups agrees with its h-cohomology computed in the category of sheaves of k-vector spaces because an injective sheaf of k-vector spaces is also injective as sheaf of abelian groups.

We abbreviate $T = X \times Z' \cup Z \times X'$. By h-sheafification of the product of Kähler differentials we have a natural multiplication

$$\mathrm{pr}_X^*\Omega^a_{\mathrm{h}/X} \otimes_k \mathrm{pr}_{X'}^*\Omega^b_{\mathrm{h}/X'} \to \Omega^{a+b}_{\mathrm{h}/X\times X'} \ .$$

It induces, with $i_Z : Z \to X$, $i_{Z'} : Z' \to X'$, and $i : T \to X \times X'$

$$\mathrm{pr}_X^*\mathrm{Ker}(\Omega^a_{\mathrm{h}/X} \to i_{Z*}\Omega^a_{\mathrm{h}/Z}) \otimes_k \mathrm{pr}_{X'}^*\mathrm{Ker}(\Omega^b_{\mathrm{h}/X'} \to i_{Z'*}\Omega^b_{\mathrm{h}/Z'}) \to \mathrm{Ker}(\Omega^{a+b}_{\mathrm{h}/X\times X'} \to i_*\Omega^{a+b}_{\mathrm{h}/T}) \ .$$

The resulting morphism

$$\mathrm{pr}_X^\bullet\Omega^*_{\mathrm{h}/(X,Z)} \otimes_k \mathrm{pr}_{X'}^\bullet\Omega^*_{\mathrm{h}/(X',Z')} \to \Omega^\bullet_{\mathrm{h}/(X\times X',T)} \ .$$

induces a natural Künneth morphism

$$\bigoplus_{a+b=n} H^a_{\mathrm{dR}}(X,Z) \otimes_k H^b_{\mathrm{dR}}(X',Z') \to H^n_{\mathrm{dR}}(X\times X',T) \ .$$

We refer to the proof of [HJ] Proposition 7.29 for the argument that this is an isomorphism. $\square$

Lemma 3.2.14. *Let K/k be an extension of fields of characteristic zero. Let X be a variety over k and $Z \subset X$ a subvariety. Then there are natural isomorphisms*

$$H^i_{\mathrm{dR}}(X,Z) \otimes_k K \to H^i_{\mathrm{dR}}(X_K, Z_K) \ .$$

They are induced by a natural quasi-isomorphism

$$R\Gamma_{\mathrm{dR}}(X) \otimes_k K \to R\Gamma_{\mathrm{dR}}(X_K) \ .$$

Proof. Via the long exact cohomology sequence for pairs, and the long exact sequence for a blow-up, it suffices to consider the case when X is a single smooth variety, where it follows from Lemma 3.1.11. $\square$

Lemma 3.2.15. *Let K/k be a finite extension of fields of characteristic* 0*. Let Y be variety over K and $W \subset Y$ a subvariety. We denote Y_k and W_k the same varieties when considered over k.*

Then there are natural isomorphisms

$$H^i_{\mathrm{dR}}(Y,W) \to H^i_{\mathrm{dR}}(Y_k, W_k) \ .$$

They are induced by a natural quasi-isomorphism

$$R\Gamma_{\mathrm{dR}}(Y) \to R\Gamma_{\mathrm{dR}}(Y_K) \ .$$

Proof. Note that if a variety is smooth over K, then it is also smooth when viewed over k.

The morphism on cohomology is induced by a morphism of sites from the category of k-varieties over Y to the category of K-varieties over k, both equipped with the h-topology. The pull-back of the de Rham complex over Y maps to the de Rham complex over Y_k. Via the long exact sequence for pairs and the blow-up sequence, it suffices to show the isomorphism for a single smooth Y. This was settled in Lemma 3.1.13. $\square$

3.3 The general case: alternative approaches

We are now going to present a number of earlier definitions in the literature. They all give the same results in the cases where they are defined.

3.3.1 Deligne's method

We present the approach of Deligne in [D5]. A singular variety is replaced by a suitable simplicial variety whose terms are smooth.

3.3.2 Hypercovers

See Section 1.5 for basics on simplicial objects. In particular, we have the notion of an S-hypercover for a class of covering maps of varieties.

We will need two cases:

1. S is the class of open covers, i.e., $X = \coprod_{i=1}^n U_i$ with $U_i \subset Y$ open and such that $\bigcup_{i=1}^n U_i = Y$.
2. S the class of proper surjective maps.

Lemma 3.3.1. *Let $X \to Y$ be in S. We put*

$$X_\bullet = \mathrm{cosq}_0^Y X \ .$$

In explicit terms,

$$X_p = X \times_Y \cdots \times_Y X \quad (p+1 \text{ factors})$$

where we number the factors from 0 to p. The face map ∂_i is the projection forgetting the factor number i. The degeneration s_i is induced by the diagonal from the factor i into the factors i and $i+1$.

Then $X_\bullet \to Y$ is an S-hypercover.

Proof. By [SGA4.2] Exposé V, Proposition 7.1.2, the morphism

$$\mathrm{cosq}_0 \to \mathrm{cosq}_{n-1}\mathrm{sq}_{n-1}\mathrm{cosq}_0$$

is an isomorphism of functors for $n \geq 1$. (This follows directly from the adjunction properties of the coskeleton functor.) Hence the condition on X_n is satisfied trivially for $n \geq 1$. In degree 0 we consider

$$X_0 = X \to (\mathrm{cosq}_{-1}^Y \mathrm{sq}_{-1} \mathrm{cosq}_0^Y)_0 = Y \ .$$

By assumption, it is in S. $\square$

It is worth spelling this out in complete detail.

Example 3.3.2. Let $X = \coprod_{i=1}^n U_i$ with $U_i \subset Y$ open. For $i_0, \ldots, i_p \in \{1, \ldots, n\}$ we abbreviate

$$U_{i_0,\ldots,i_p} = U_{i_0} \cap \cdots \cap U_{i_p} .$$

Then the open hypercover $X_\bullet$ is nothing but

$$X_p = \coprod_{i_0,\ldots,i_p=0^n} U_{i_0,\ldots,i_p}$$

with face and degeneracy maps given by the natural inclusions. Let $\mathcal{F}$ be a sheaf of abelian groups on X. Then the complex associated to the cosimplicial abelian group $\mathcal{F}(X_\bullet)$ is given by

$$\bigoplus_{i=1}^n \mathcal{F}(U_i) \to \bigoplus_{i_0,i_1=1}^n \mathcal{F}(U_{i_0,i_1}) \to \bigoplus_{i_0,i_1,i_2=1}^n \mathcal{F}(U_{i_0,i_1,i_2}) \to \ldots$$

with differential

$$\delta^p(\alpha)_{i_0,\ldots,i_p} = \sum_{j=0}^{p+1} \alpha_{i_0,\ldots,\hat{i}_j,\ldots,i_{p+1}}|_{U_{i_0,\ldots,i_{p+1}}} ,$$

i.e., the differential of the Čech complex. Indeed, the natural projection

$$\mathcal{F}(X_\bullet) \to C^\bullet(\mathfrak{U}, \mathcal{F})$$

to the Čech complex (see Definition 1.4.12) is a quasi-isomorphism.

Definition 3.3.3. We say that $X_\bullet \to Y_\bullet$ is a smooth proper hypercover if it is a proper hypercover with all X_n smooth.

Example 3.3.4. Let $Y = Y_1 \cup \ldots Y_n$ with $Y_i \subset Y$ closed. For $i_0, \ldots, i_p = 1, \ldots, n$ put

$$Y_{i_0,\ldots,i_p} = Y_{i_0} \cap \ldots Y_{i_p} .$$

Assume that all Y_i and all $Y_{i_0,\ldots,i_p}$ are smooth.

Then $X = \coprod_{i=1}^n Y_i \to Y$ is proper and surjective. The proper hypercover $X_\bullet$ is nothing but

$$X_n = \coprod_{i_0,\ldots,i_n=0^n} Y_{i_0} \cap \ldots Y_{i_n}$$

with face and degeneracy maps given by the natural inclusions. Hence $X_\bullet \to Y$ is a smooth proper hypercover. As in the open case, the projection to Čech complex of the closed cover $\mathfrak{Y} = \{V_i\}_{i=1}^n$ is a quasi-isomorphism.

Proposition 3.3.5. *Let $Y_\bullet$ be a simplicial variety. Then the system of all proper hypercovers of $Y_\bullet$ is filtered up to simplicial homotopy. It is functorial in $Y_\bullet$. The subsystem of smooth proper hypercovers is cofinal.*

Proof. The first statement is [SGA4.2], Exposé V, Théorème 7.3.2. For the second assertion, it suffices to construct a smooth proper hypercover for any $Y_\bullet$. Recall that by Hironaka's resolution of singularities [Hi1], or by de Jong's theorem on alterations [dJ], we have for any variety Y a proper surjective map $X \to Y$ with X smooth. By the technique of [SGA4.2], Exposé Vbis, Proposition 5.1.3 (see also [D5] 6.2.5), this allows to construct $X_\bullet$. $\square$

3.3.3 Definition of de Rham cohomology in the general case

Let again k be a field of characteristic 0.

Definition 3.3.6. Let X be a variety over k and $X_\bullet \to X$ a smooth proper hypercover. Let $C(X_\bullet) \in \mathbb{Z}\mathrm{Sm}$ be the associated complex We define *algebraic de Rham cohomology* of X by

$$H^i_{\mathrm{dR}}(X) = H^i\left(R\Gamma_{\mathrm{dR}}(X_\bullet)\right)$$

with $R\Gamma_{\mathrm{dR}}$ as in Definition 3.1.7. Let $D \subset X$ be a closed subvariety and $D_\bullet \to D$ a smooth proper hypercover such that there is a commutative diagram

$$\begin{array}{ccc} D_\bullet & \longrightarrow & X_\bullet \\ \downarrow & & \downarrow \\ D & \longrightarrow & X \end{array}$$

We define *relative algebraic de Rham cohomology* of the pair (X, D) by

$$H^i_{\mathrm{dR}}(X, D) = H^i\left(\mathrm{Cone}(R\Gamma(X_\bullet) \to R\Gamma(D_\bullet))[-1]\right) .$$

Proposition 3.3.7. *Algebraic de Rham cohomology is a well-defined functor, independent of the choice of hypercoverings of X and D.*

Remark 3.3.8. $R\Gamma_{\mathrm{dR}}$ defines a functor

$$\mathrm{Var} \to K^+(k\text{−Vect})$$

but *not* to $C^+(k\text{−Vect})$. Hence it does *not* extend directly to $C^b(\mathbb{Q}[\mathrm{Var}])$. We avoid addressing this point by the use of the h-topology instead.

Proof. This is a special case of descent for h-covers and hence a consequence of Proposition 3.2.4.

Alternatively, we can deduce if from the case of singular cohomology. Recall that algebraic de Rham cohomology is well-behaved with respect to extensions of the ground field. Without loss of generality, we may assume that k is finitely generated over $\mathbb{Q}$ and hence embeds into $\mathbb{C}$. Then we apply the period isomorphism of Definition 5.3.1. It remains to check the analogue for singular cohomology. This is Theorem 2.7.6. $\square$

Example 3.3.9. Let X be a smooth affine variety and D a simple divisor with normal crossings. Let $D_1, \ldots, D_n$ be the irreducible components. Let $X_\bullet$ be the constant simplicial variety X and $D_\bullet$ as in Example 3.3.4. Then algebraic de Rham cohomology D is computed by the total complex of the double complex ($D_{i_0,\ldots,i_p}$ being the $(p+1)$-fold intersection of components)

$$K^{p,q} = \bigoplus_{i_0<\cdots<i_p} \Omega^q_{D_{i_0,\ldots,i_p}}\left(D_{i_0,\ldots,i_p}\right)$$

with differential $d^{p,q} = \sum_{j=0}^p (-1)^j \partial_j^*$ the Čech differential and $\delta^{p,q}$ differentiation of differential forms.

Relative algebraic de Rham cohomology of (X, D) is computed by the total complex of the double complex

$$L^{p,q} = \begin{cases} K^{p-1,q} & p > 0, \\ \Omega^q_X(X) & p = 0. \end{cases}$$

Remark 3.3.10. Establishing the expected properties of relative algebraic de Rham cohomology is lengthy. Particularly complicated is the handling of the multiplicative structure which uses the the functor between complexes in $\mathbb{Z}[\mathrm{Sm}]$ and simplicial objects in $\mathbb{Z}[\mathrm{Sm}]$ and the product for simplicial objects. We do not go into the details but rely on the comparison with h-cohomology instead.

3.3.4 Hartshorne's method

We want to review Hartshorne's definition from [Ha1]. As before let k be a field of characteristic 0.

Definition 3.3.11. Let X be a smooth variety over k, $i : Y \subset X$ a closed subvariety. We define *algebraic de Rham cohomology* of Y as

$$H^i_{H-\mathrm{dR}}(Y) = H^i(\hat{X}, \hat{\Omega}^\bullet_X),$$

where $\hat{X}$ is the formal completion of X along Y and $\hat{\Omega}^\bullet_X$ the formal completion of the complex of algebraic differential forms on X.

Proposition 3.3.12 ([Ha1] Theorem (1.4))**.** *Let Y be as in Definition 3.3.11. Then $H^i_{H-\mathrm{dR}}(Y)$ is independent of the choice of X. In particular, if Y is smooth, the definition agrees with the one in Definition 3.1.2.*

Theorem 3.3.13. *The three definition of algebraic de Rham cohomology (Definition 3.3.6 via hypercovers, Definition 3.3.11 via embedding into smooth varieties, Definition 3.2.3 using the h-topology) agree.*

Proof. The comparison of $H^i_{H-\mathrm{dR}}(X)$ and $H^i_{\mathrm{dR}}(X_{\mathrm{eh}})$ is [Ge], Theorem 4.10. It agrees with $H^i_{\mathrm{dR}}(X_{\mathrm{h}})$ by [HJ], Proposition 6.1. By [HJ], Proposition 7.4 it agrees also with the definition via hypercovers. □

3.3.5 Using geometric motives

In Chapter 10 we are going to introduce the triangulated category of effective geometric motives $DM_{\mathrm{gm}}^{\mathrm{eff}}$ over k with coefficients in $\mathbb{Q}$. We only review the most important properties here and refer to Chapter 10 for more details. For technical reasons, it is easier to work with the affine version.

The objects in $DM_{\mathrm{gm}}^{\mathrm{eff}}$ are the same as the objects in $C^b(\mathrm{SmCor})$ where SmCor is the category of correspondences, see Section 1.1 and we denote SmCorAff the full subcategory with objects smooth affine varieties.

Lecomte and Wach in [LW] explain how to define an operation of correspondences on $\Omega^\bullet_X(X)$. We give a quick survey of their method.

For any normal variety Z let $\Omega_Z^{p,**}$ be the $\mathcal{O}_Z$-double dual of the sheaf of p-differentials. This is nothing but the sheaf of *reflexive differentials* on Z.

If $Z' \to Z$ is a finite morphism between normal varietes which is generically Galois with covering group G, then by [Kn]

$$\Omega_Z^{p,**}(Z) \cong \Omega_{Z'}^{p,**}(Z')^G \ .$$

Let X and Y be smooth affine varieties. Assume for simplicity that X and Y are connected. Let $\Gamma \in \mathrm{Cor}(X,Y)$ be a prime correspondence, i.e., $\Gamma \subset X \times Y$ an integral closed subvariety which is finite and dominant over X. Choose a finite $\tilde{\Gamma} \to \Gamma$ such that $\tilde{\Gamma}$ is normal and the covering $\tilde{\Gamma} \to X$ generically Galois with covering group G. In this case, $X = \tilde{\Gamma}/G$.

Definition 3.3.14. For a correspondence $\Gamma \in \mathrm{Cor}(X,Y)$ as above, we define

$$\Gamma^* : \Omega^\bullet_Y(Y) \to \Omega^\bullet_X(X)$$

as the composition

$$\Omega^\bullet_Y(Y) \to \Omega^\bullet_{\tilde{\Gamma}}(\tilde{\Gamma}) \to \Omega^{\bullet,**}_{\tilde{\Gamma}}(\tilde{\Gamma}) \xrightarrow{\frac{1}{|G|}\sum_{g\in G} g^*} \Omega^{\bullet,**}_{\tilde{\Gamma}}(\tilde{\Gamma})^G = \Omega^\bullet_X(X) \ .$$

This is well-defined and compatible with composition of correspondences. We can now define de Rham cohomology for complexes of correspondences.

Definition 3.3.15. Let $X^\bullet \in C^b(\mathrm{SmCorAff})$. We define

$$R\Gamma_{\mathrm{dR}}(X_\bullet) = \mathrm{Tot} R\Gamma_{\mathrm{dR}}(X_n)_{n\in\mathbb{Z}} \ .$$

and

$$H^i_{\mathrm{dR}}(X_\bullet) = H^i R\Gamma_{\mathrm{dR}}(X_\bullet) \ .$$

Note that there is a simple functor SmAff $\to$ SmCor. It assigns an object to itself and a morphism to its graph. This induces

$$i : C^b(\mathbb{Q}[\mathrm{SmAff}]) \to DM_{\mathrm{gm}}^{\mathrm{eff}} \ .$$

By construction,

$$f^* = \Gamma_f^* : \Omega_Y^\bullet(Y) \to \Omega_X^\bullet(X)$$

for any morphism $f : X \to Y$ between smooth affine varieties. Hence,

$$R\Gamma_{\mathrm{dR}}(X_\bullet) = R\Gamma_{\mathrm{dR}}(i(X_\bullet)),$$

where the left hand side was defined in Definition 3.1.7.

Proposition 3.3.16 (Voevodsky)**.** *The functor i extends naturally to a functor*

$$i : C^b(\mathbb{Q}[\mathrm{Var}]) \to DM_{\mathrm{gm}}^{\mathrm{eff}} \ .$$

Proof. The category of geometric motives constructed from affine varieties only agrees with the original $DM_{\mathrm{gm}}^{\mathrm{eff}}$. For details, see [Ha].

The extension to all varieties is a highly non-trivial result of Voevodsky. By [VSF], Chapter V, Corollary 4.1.4, there is functor

$$\mathrm{Var} \to DM_{\mathrm{gm}} \ .$$

Indeed, the functor

$$X \mapsto C_*L(X)$$

of loc. cit., Section 4.1, which assigns to every variety a homotopy invariant complex of Nisnevich sheaves, extends to $C^b(\mathbb{Z}[\mathrm{Var}])$ by taking total complexes. We consider it in the derived category of Nisnevich sheaves. Then the functor factors via the homotopy category $K^b(\mathbb{Z}[\mathrm{Var}])$.

By induction on the length of the complex, it follows from the result quoted above that $C_*L(\cdot)$ takes values in the full subcategory of geometric motives. $\square$

Definition 3.3.17. Let $D \subset X$ be a closed immersion of varieties. We define

$$H^i_{\mathrm{dR}}(X, D) = H^i R\Gamma_{\mathrm{dR}}(i([D \to X]) \ ,$$

where $[D \to X] \in C^b(\mathbb{Z}[\mathrm{Var}])$ is concentrated in degrees -1 and 0.

Proposition 3.3.18. *This definition agrees with the one given in Definition 3.3.6.*

Proof. The easiest way to formulate the proof is to invoke another variant of the category of geometric motives. It does not need transfers, but imposes h-descent instead. Scholbach [Sch1, Definition 3.10] defines the category $DM_{\mathrm{gm},h}^{\mathrm{eff}}$ as the localization of $K^-(\mathbb{Q}[\mathrm{Var}])$ with respect to the triangulated subcategory generated by complexes of the form $X \times \mathbb{A}^1 \to X$ and h-hypercovers $X_\bullet \to X$ and closed under certain infinite sums. By definition of $DM_{\mathrm{gm},h}^{\mathrm{eff}}$, any hypercovering $X_\bullet \to X$ induces an isomorphism of the associated complexes in $DM_{\mathrm{gm},h}^{\mathrm{eff}}$. By resolution of singularities, any object of $DM_{\mathrm{gm},h}^{\mathrm{eff}}$ is isomorphic to an object where all components are smooth. Hence we can replace $K^-(\mathbb{Q}[\mathrm{Var}])$ by

$K^-(\mathbb{Q}[\mathrm{Sm}])$ in the definition without any change. We have seen how algebraic de Rham cohomology is defined on $K^-(\mathbb{Q}[\mathrm{Sm}])$. By homotopy invariance (Corollary 3.1.10) and h-descent of the de Rham complex (Proposition 3.3.7), the definition of algebraic de Rham cohomology factors via $DM^{\mathrm{eff}}_{\mathrm{gm},h}$.

This gives a definition of algebraic de Rham cohomology for $K^-(\mathbb{Q}[\mathrm{Var}])$ which by construction agrees with the one in Definition 3.3.6. On the other hand, the main result of [Sch1] is that $DM^{\mathrm{eff}}_{\mathrm{gm}}$ can be viewed as full subcategory of $DM^{\mathrm{eff}}_{\mathrm{gm},h}$. This inclusion maps the motive of a (possibly singular) variety to the motive of a variety. As the two definitions of algebraic de Rham cohomology of motives agree on motives of smooth varieties, they agree on all motives. $\square$

3.3.6 The case of divisors with normal crossings

We are going to need the following technical result in order to give a simplified description of periods.

Proposition 3.3.19. *Let X be a smooth affine variety of dimension d and $D \subset X$ a simple divisor with normal crossings. Then every class in $H^d_{\mathrm{dR}}(X,D)$ is represented by some $\omega \in \Omega^d_X(X)$.*

The proof will be given at the end of this section.

Let $D = D_1 \cup \cdots \cup D_n$ be the decomposition into irreducible components. For $I \subset \{1,\dots,n\}$, let again

$$D_I = \bigcap_{i\in I} D_i \ .$$

Recall from Example 3.3.9 that the de Rham cohomology of (X,D) is computed by the total complex of

$$\Omega^\bullet_X(X) \to \bigoplus_{i=1}^n \Omega^\bullet_{D_i}(D_i) \to \bigoplus_{i<j} \Omega^\bullet_{D_{i,j}}(D_{i,j}) \to \cdots \to \Omega^\bullet_{D_{1,2,\dots,n}}(D_{1,2,\dots,n}) \ .$$

Note that D_I has dimension $d-|I|$, hence the double complex is concentrated in degrees $p,q\geq 0$, $p+q\leq d$. By definition, the classes in the top cohomology group $H^d_{\mathrm{dR}}(X,D)$ are presented by a tuple

$$(\omega_0,\omega_1,\dots,\omega_n) \quad \omega_0\in\Omega^d_X(X), \omega_i \in \bigoplus_{|I|=i}\Omega^{d-i}_{D_I}(D_I)\ , i>0\ .$$

All such tuples are cocycles for dimension reasons. We have to show that, modulo coboundaries, we can assume $\omega_i = 0$ for all $i>0$.

Lemma 3.3.20. *The maps*

$$\Omega^{d-1}_X(X) \to \bigoplus_{i=1}^n \Omega^{d-1}_{D_i}(D_i)$$

$$\bigoplus_{|I|=s} \Omega^{d-s-1}_{D_I}(D_I) \to \bigoplus_{|J|=s+1} \Omega^{d-s-1}_{D_J}(D_J)$$

are surjective.

Proof. X and all D_i are assumed affine, hence the global section functor is exact. It suffices to check the assertion for the corresponding sheaves on X and hence locally for the étale topology. By replacing X by an étale neighbourhood of a point, we can assume that there is a global system of regular paramters $t_1, \dots, t_d$ on X such that $D_i = \{t_i = 0\}$ for $i = 1, \dots, n$. First consider the case $s = 0$. The elements of $\Omega^{d-1}_{D_i}(D_i)$ are locally of the form $f_i dt_1 \wedge \dots \wedge \dot{dt_i} \wedge \dots \wedge t_d$ (omitting the factor at i). Again by replacing X by an open subvariety, we can assume they are globally of this shape. The forms can all be lifted to X.

$$\omega = \sum_{i=1}^{n} f_i dt_1 \wedge \dots \wedge \dot{dt_i} \wedge \dots \wedge t_d$$

is the preimage we were looking for.

For $s \geq 1$ we argue by induction on d and n. If $n = 1$, there is nothing to show. This settles the case $d = 1$. If $n > 0$, consider the decomposition

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\bigoplus_{|I|=s, I\subset\{1,\dots,n-1\}} \Omega^{d-s-1}_{D_I}(D_I) & \longrightarrow & \bigoplus_{|J|=s+1, J\subset\{1,\dots,n-1\}} \Omega^{d-s-1}_{D_J}(D_J) \\
\downarrow & & \downarrow \\
\bigoplus_{|I|=s, I\subset\{1,\dots,n\}} \Omega^{d-s-1}_{D_I}(D_I) & \longrightarrow & \bigoplus_{|J|=s+1, J\subset\{1,\dots,n\}} \Omega^{d-s-1}_{D_J}(D_J) \\
\downarrow & & \downarrow \\
\bigoplus_{|I|=s, I\subset\{1,\dots,n\}, n\in I} \Omega^{d-s-1}_{D_I}(D_I) & \longrightarrow & \bigoplus_{|J|=s+1, J\subset\{1,\dots,n\}, n\in J} \Omega^{d-s-1}_{D_J}(D_J) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

The arrow on the top is surjective by induction on n. The arrow on the bottom reproduces the assertion for X replaced by D_n and D replaced by $D_n \cap (D_1 \cup \dots \cup D_{n-1})$. By induction, it is surjective. Hence, the arrow in the middle is surjective. □

Proof of Proposition 3.3.19. Consider a cocycle $\omega = (\omega_0, \omega_1, \dots, \omega_n)$ as explained above. We argue by descending induction on the degree i. Consider $\omega_n \in \bigoplus_{|I|=n} \Omega^{d-n}_{D_I}(D_I)$. By the lemma, there is

$$\omega'_{n-1} \in \bigoplus_{|I|=n-1} \Omega^{d-n}_{D_I}(D_I)$$

such that $\partial \omega'_{n-1} = \omega_n$. We replace ω by $\omega \pm d\omega'_{n-1}$ (depending on the signs in the double complex). By construction, its component in degree n vanishes.

Hence, without loss of generality, we have $\omega_n = 0$. Next, consider ω_{n-1} etc. $\square$

Chapter 4

Holomorphic de Rham cohomology

We are going to define a natural comparison isomorphism between de Rham cohomology and singular cohomology of varieties over the complex numbers. The link is provided by holomorphic de Rham cohomology which we study in this chapter.

4.1 Holomorphic de Rham cohomology

Everything we did in the algebraic setting also works for complex manifolds, indeed this is the older notion.

We write $\mathcal{O}_X^{\mathrm{hol}}$ for the sheaf of holomorphic functions on a complex manifold X.

4.1.1 Definition

Definition 4.1.1. Let X be a complex manifold. Let Ω_X^1 be the sheaf of *holomorphic differentials* on X. For $p \geq 0$ let

$$\Omega_X^p = \Lambda^p \Omega_X^1$$

be the exterior power in the category of $\mathcal{O}_X^{\mathrm{hol}}$-modules and $(\Omega_X^\bullet, d)$ the *holomorphic de Rham complex*.

The differential is defined as in the algebraic case, see Definition 3.1.1.

Definition 4.1.2. Let X be a complex manifold. We define *holomorphic de Rham cohomology* of X as hypercohomology

$$H^i_{\mathrm{dR}^{\mathrm{an}}}(X) = H^i(X, \Omega_X^\bullet) \ .$$

As in the algebraic case, de Rham cohomology is a contravariant functor. The exterior products induces a cup-product.

Proposition 4.1.3 (Poincaré lemma). *Let X be a complex manifold. The natural map of sheaves $\mathbb{C} \to \mathcal{O}_X^{\mathrm{hol}}$ induces an isomorphism*

$$H^i_{\mathrm{sing}}(X, \mathbb{C}) \to H^i_{\mathrm{dR}^{\mathrm{an}}}(X) .$$

Proof. By Theorem 2.2.5, we can compute singular cohomology as sheaf cohomology on X. It remains to show that the complex

$$0 \to \mathbb{C} \to \mathcal{O}_X^{\mathrm{hol}} \to \Omega^1_X \to \Omega^2_X \to \dots$$

is exact. Let Δ be the unit ball in $\mathbb{C}$. The question is local, hence we may assume that $X = \Delta^d$. There is a natural isomorphism

$$\Omega^\bullet_{\Delta^d} \cong (\Omega^\bullet_\Delta)^{\otimes d}$$

Hence it suffices to treat the case $X = \Delta$. In this case we consider

$$0 \to \mathbb{C} \to \mathcal{O}^{\mathrm{hol}}(\Delta) \to \mathcal{O}^{\mathrm{hol}}(\Delta)dt \to 0 .$$

The elements of $\mathcal{O}^{\mathrm{hol}}(\Delta)$ are of the form $\sum_{i\geq 0} a_i t^i$ with radius of convergence 1. The differential has the form

$$\sum_{i\geq 0} a_i t^i \mapsto \sum_{i\geq 0} i a_i t^{i-1} dt .$$

The kernel is given by the constants. It is surjective because the antiderivative has the same radius of convergence as the original power series. □

Proposition 4.1.4 (Künneth formula). *Let X, Y be complex manifolds. There is a natural multiplication induced from wedge product of differential forms*

$$H^i_{\mathrm{dR}}(X) \otimes_k H^j_{\mathrm{dR}}(Y) \to H^{i+j}_{\mathrm{dR}}(X \times Y) .$$

It induces an isomorphism

$$H^n_{\mathrm{dR}}(X \times Y) \cong \bigoplus_{i+j=n} H^i_{\mathrm{dR}}(X) \otimes_k H^j_{\mathrm{dR}}(Y) .$$

Proof. The construction of the morphism is the same as in the algebraic case, see Proposition 3.1.9. The quasi-isomorphism $\mathbb{C} \to \Omega^\bullet$ is compatible with the exterior products. Hence the isomorphism reduces to the Künneth isomorphism for singular cohomology, see Proposition 2.4.1. □

4.1.2 Holomorphic differentials with log poles

Let $j : X \to \bar{X}$ be a an open immersion of complex manifolds. Assume that $D = \bar{X} \smallsetminus X$ is a divisor with normal crossings, i.e., locally on $\bar{X}$ there is a coordinate system $(t_1, \dots, t_n)$ such that D is given as the set of zeroes of $t_1 t_2 \dots t_r$ with $0 \leq r \leq n$.

Definition 4.1.5. Let
$$\Omega^1_{\bar{X}}\langle D\rangle \subset j_*\Omega^1_X$$
be the locally free $\mathcal{O}_{\bar{X}}$-module with the following basis: if $U \subset X$ is an open with coordinates $t_1, \dots, t_n$ and $D|_U$ given by the equation $t_1 \dots t_r = 0$, then $\Omega^1_{\bar{X}}\langle D\rangle|_U$ has $\mathcal{O}^{\mathrm{hol}}_{\bar{X}}$-basis
$$\frac{dt_1}{t_1}, \dots, \frac{dt_r}{t_r}, dt_{r+1}, \dots, dt_n \ .$$
For $p > 1$ let
$$\Omega^p_{\bar{X}}\langle D\rangle = \Lambda^p \Omega^1_{\bar{X}}\langle D\rangle \ .$$
We call the $\Omega^\bullet_{\bar{X}}\langle D\rangle$ the *complex of differentials with log poles along D.*

Note that the differential of $j_*\Omega^\bullet_X$ respects $\Omega^\bullet_{\bar{X}}\langle D\rangle$, so that this is indeed a complex.

Proposition 4.1.6. *The inclusion induces a natural isomorphism*
$$H^i(\bar{X}, \Omega^\bullet_{\bar{X}}\langle D\rangle) \to H^i(X, \Omega^\bullet_X) \ .$$

This is [D4] Proposition 3.1.8. The algebraic analogue was treated in Proposition 3.1.16.

Proof. Note that $j : X \to \bar{X}$ is Stein, hence j_* is exact and we have
$$H^i(X, \Omega^\bullet_X) \cong H^i(\bar{X}, j_*\Omega^\bullet_X) \ .$$
It remains to show that
$$\iota : \Omega^\bullet_{\bar{X}}\langle D\rangle \to j_*\Omega^\bullet_X$$
is a quasi-isomorphism, or, equivalently, that $\mathrm{Coker}(\iota)$ is exact. The statement is local, hence we may assume that $\bar{X}$ is a coordinate ball and $D = V(t_1 \dots t_r)$. We consider the stalk in 0. The complexes are tensor products of the complexes in the 1-dimensional situation. Hence it suffices to consider the case $n = 1$. If $r = 0$, then there is nothing to show.

In remains to consider the following situation: let $\mathcal{O}^{\mathrm{hol}}$ be ring of germs of holomorphic functions at $0 \in \mathbb{C}$ and $\mathcal{K}^{\mathrm{hol}}$ the ring of germs of holomorphic functions with an isolated singularity at 0. The ring $\mathcal{O}^{\mathrm{hol}}$ is given by power series with a positive radius of convergence. The field $\mathcal{K}^{\mathrm{hol}}$ is given by Laurent series converging on some punctured neighborhood $\{t \mid 0 < t < \epsilon\}$. We have to check that the complex

$$\mathcal{K}^{\mathrm{hol}}/\mathcal{O}^{\mathrm{hol}} \to (\mathcal{K}^{\mathrm{hol}}/t^{-1}\mathcal{O}^{\mathrm{hol}})dt$$

is acyclic.

We pass to the principal parts. The differential has the form

$$\sum_{i>0} a_i t^{-i} \mapsto \sum_{i>0} (-i) a_i t^{-i-1}$$

It is obviously injective. For surjectivity, note that the antiderivative

$$\int : \sum_{i>1} b_i t^{-i} \mapsto \sum_{i>1} \frac{b_i}{-i+1} t^{-i+1}$$

maps convergent Laurent series to convergent Laurent series. □

4.1.3 GAGA

We work over the field of complex numbers.

An affine variety $X \subset \mathbb{A}^n_{\mathbb{C}}$ is also a closed set in the analytic topology on $\mathbb{C}^n$. If X is smooth, the associated analytic space X^{an} in the sense of Section 1.2.1 is a complex submanifold. As in loc. cit., we denote by

$$\alpha : (X^{\mathrm{an}}, \mathcal{O}^{\mathrm{hol}}_{X^{\mathrm{an}}}) \to (X, \mathcal{O}_X)$$

the map of locally ringed spaces. Note that any algebraic differential form is holomorphic, hence there is a natural morphism of complexes

$$\alpha^{-1}\Omega^\bullet_X \to \Omega^\bullet_{X^{\mathrm{an}}} \ .$$

It induces

$$\alpha^* : H^i_{\mathrm{dR}}(X) \to H^i_{\mathrm{dR}^{\mathrm{an}}}(X^{\mathrm{an}}) \ .$$

Proposition 4.1.7 (GAGA for de Rham cohomology)**.** *Let X be a smooth variety over $\mathbb{C}$. Then the natural map*

$$\alpha^* : H^i_{\mathrm{dR}}(X) \to H^i_{\mathrm{dR}^{\mathrm{an}}}(X^{\mathrm{an}})$$

is an isomorphism.

If X is smooth and projective, this is a standard consequence of Serre's comparison result for cohomology of coherent sheaves (GAGA). We need to extend this to the open case.

Proof. Let $j : X \to \bar{X}$ be a compactification such that $D = \bar{X} \smallsetminus X$ is a simple divisor with normal crossings. The change of topology map α also induces

$$\alpha^{-1} j_* \Omega^\bullet_X \to j^{\mathrm{an}}_* \Omega^\bullet_{X^{\mathrm{an}}}$$

which respects differential with log-poles

$$\alpha^{-1} \Omega_{\bar{X}^\bullet} \langle D \rangle \to j^{\mathrm{an}}_* \Omega^\bullet_{\bar{X}^{\mathrm{an}}} \langle D^{\mathrm{an}} \rangle \ .$$

Hence we get a commutative diagram

$$\begin{array}{ccc} H^i_{\mathrm{dR}}(X) & \longrightarrow & H^i_{\mathrm{dR}^{\mathrm{an}}}(X^{\mathrm{an}}) \\ \uparrow & & \uparrow \\ H^i(\bar{X}, \Omega^\bullet_{\bar{X}} \langle D \rangle) & \longrightarrow & H^i(\bar{X}^{\mathrm{an}}, \Omega^\bullet_{\bar{X}^{\mathrm{an}}} \langle D^{\mathrm{an}} \rangle) \end{array}$$

By Proposition 3.1.16 in the algebraic, and Proposition 4.1.6 in the holomorphic case, the vertical maps are isomorphism. By considering the Hodge to de Rham spectral sequence (attached to the stupid filtration on $\Omega^\bullet_{\bar{X}} \langle D \rangle$), it suffices to show that

$$H^p(\bar{X}, \Omega^q_{\bar{X}} \langle D \rangle) \to H^p(\bar{X}^{\mathrm{an}}, \Omega^q_{\bar{X}^{\mathrm{an}}} \langle D^{\mathrm{an}} \rangle)$$

is an isomorphism for all p, q. Note that $\bar{X}$ is smooth, projective and $\Omega^q_{\bar{X}} \langle D \rangle$ is coherent. Its analytification $\alpha^{-1} \Omega^q_{\bar{X}} \langle D \rangle \otimes_{\alpha^{-1} \mathcal{O}_{\bar{X}}} \mathcal{O}^{\mathrm{hol}}_{\bar{X}^{\mathrm{an}}}$ is nothing but $\Omega^q_{\bar{X}^{\mathrm{an}}} \langle D^{\mathrm{an}} \rangle$. By GAGA [Se1], we have an isomorphism in cohomology. □

4.2 De Rham cohomology via the h′-topology

We address the singular case via the h′-topology on (An/X) introduced in Definition 2.7.1.

4.2.1 h′-differentials

Definition 4.2.1. Let $\Omega^p_{\mathrm{h}'}$ be the h′-sheafification of the presheaf

$$Y \mapsto \Omega^p_Y(Y)$$

on the category of complex analytic spaces An.

Theorem 4.2.2 (Jörder [Joe])**.** *Let X be a complex manifold. Then*

$$\Omega^p_X(X) = \Omega^p_{\mathrm{h}'}(X) \ .$$

Proof. Jörder defines in [Joe, Definition 1.4.1] what he calls h-differentials Ω^p_{h} as the presheaf pull-back of Ω^p from the category of manifolds to the category of complex analytic spaces. (There is no mention of a topology in loc.cit.) In

[Joe, Proposition 1.4.2 (4)] he establishes that $\Omega^p_{\mathrm{h}}(X) = \Omega^p_X(X)$ in the smooth case. It remains to show that $\Omega^p_{\mathrm{h}} = \Omega^p_{\mathrm{h}'}$. By resolution of singularities, every X is smooth locally for the h′-topology. Hence it suffices to show that Ω^p_{h} is an h′-sheaf. By [Joe, Lemma 1.4.5], the sheaf condition is satisfied for proper covers. The sheaf condition for open covers is satisfied because already Ω^p_X is a sheaf in the ordinary topology. □

Lemma 4.2.3 (Poincaré lemma)**.** *Let X be a complex analytic space. Then the complex*

$$\mathbb{C}_{\mathrm{h}'} \to \Omega^\bullet_{\mathrm{h}'}$$

of h′*-sheaves on* $(\mathrm{An}/X)_{\mathrm{h}'}$ *is exact.*

Proof. We may check this locally in the h′-topology. By resolution of singuarities it suffices to consider sections over some Y which is smooth and even an open ball in $\mathbb{C}^n$. By Theorem 4.2.2 the complex reads

$$\mathbb{C} \to \Omega^\bullet_Y(Y) \ .$$

By the ordinary holomorphic Poincaré Lemma 4.1.3, it is exact. □

Remark 4.2.4. The main topic of Jörder's thesis [Joe] is to treat the question of a Poincaré Lemma for h′-forms with respect to the usual topology. This is more subtle and fails in general.

4.2.2 De Rham cohomology

We now turn to de Rham cohomology.

Definition 4.2.5. Let X be a complex analytic space.

1. We define h′*-de Rham cohomology* as hypercohomology

$$H^i_{\mathrm{dR}^{\mathrm{an}}}(X_{\mathrm{h}'}) = H^i_{\mathrm{h}'}((\mathrm{Sch}/X)_{\mathrm{h}'}, \Omega^\bullet_{\mathrm{h}'}) \ .$$

2. Let $i : Z \to X$ a closed subspace. Put

$$\Omega^p_{\mathrm{h}/(X,Z)} = \mathrm{Ker}(\Omega^p_{\mathrm{h}/X} \to i_*\Omega^p_{\mathrm{h}/Z})$$

   in the category of abelian sheaves on $(\mathrm{An}/X)_{\mathrm{h}'}$.

   We define *relative* h′*-de Rham cohomology* as

$$H^p_{\mathrm{dR}^{\mathrm{an}}}(X_{\mathrm{h}'}, Z_{\mathrm{h}'}) = H^p_{\mathrm{h}'}((\mathrm{An}/X)_{\mathrm{h}'}, \Omega^*_{\mathrm{h}/(X,Z)}) \ .$$

Lemma 4.2.6. *The properties (long exact sequence, excision, Künneth formula) of relative algebraic H-de Rham cohomology (see Section 3.2) are also satisfied in relative* h′*-de Rham cohomology.*

Proof. The proofs are the same as Section 3.2, respectively in [HJ, Section 7.3]. The proof relies on the computation of $\Omega^p_{\mathrm{h}'}(D)$ when D is a normal crossings space. Indeed, the same argument as in the proof of [HJ, Proposition 4.9] shows that

$$\Omega^p_{\mathrm{h}'}(D) = \Omega^p_D(D)/\mathrm{torsion}\ .$$

□

As in the previous case, exterior multiplication of differential forms induces a product structure on h′-de Rham cohomology.

Corollary 4.2.7. *For all $X \in \mathrm{An}$ and closed immersions $i : Z \to X$ the inclusion of the Poincaré lemma induces a natural isomorphism*

$$H^i_{\mathrm{sing}}(X, Z, \mathbb{C}) \to H^i_{\mathrm{dR}^{\mathrm{an}}}(X_{\mathrm{h}'}, Z_{\mathrm{h}'})\ ,$$

compatible with long exact sequences and multiplication. Moreover, the natural map

$$H^i_{\mathrm{dR}^{\mathrm{an}}}(X_{\mathrm{h}'}) \to H^i_{\mathrm{dR}^{\mathrm{an}}}(X)$$

is an isomorphism if X is smooth.

Proof. By the Poincaré Lemma 4.2.3, we have a natural isomorphism

$$H^i_{\mathrm{h}'}(X_{\mathrm{h}'}, Z_{\mathrm{h}'}, \mathbb{C}_{\mathrm{h}'}) \to H^i_{\mathrm{dR}^{\mathrm{an}}}(X_{\mathrm{h}'}, Z_{\mathrm{h}'})\ .$$

We combine it with the comparison isomorphism with singular cohomology of Proposition 2.7.4.

The second statement holds because both compute singular cohomology by Prop. 2.7.4 and Prop. 4.1.3. □

4.2.3 GAGA

We work over the base field $\mathbb{C}$. As before we consider the analytification functor

$$X \mapsto X^{\mathrm{an}}$$

which takes a separated scheme of finite type over $\mathbb{C}$ to a complex analytic space. We recall the map of locally ringed spaces

$$\alpha : X^{\mathrm{an}} \to X\ .$$

We want to view it as a morphism of topoi

$$\alpha : (\mathrm{An}/X^{\mathrm{an}})_{\mathrm{h}'} \to (\mathrm{Sch}/X)_{\mathrm{h}}\ .$$

Definition 4.2.8. Let $X \in \mathrm{Sch}/\mathbb{C}$. We define the h′-topology on the category $(\mathrm{Sch}/X)_{\mathrm{h}'}$ to be the smallest Grothendieck topology such that the following are covering maps:

1. proper surjective morphisms;
2. open covers.

If $\mathcal{F}$ is a presheaf of An/X, we denote by $\mathcal{F}_{\mathrm{h}'}$ its sheafification in the h′-topology.

Lemma 4.2.9. *1. The morphism of sites* $(\mathrm{Sch}/X)_{\mathrm{h}} \to (\mathrm{Sch}/X)_{\mathrm{h}'}$ *induces an isomorphism on the categories of sheaves.*

2. The analytification functor induces a morphism of sites

$$(\mathrm{An}/X^{\mathrm{an}})_{\mathrm{h}'} \to (\mathrm{Sch}/X)_{\mathrm{h}'} \ .$$

Proof. By [Voe] Theorem 3.1.9 any h-cover can be refined by a cover in normal form which is a composition of open immersions followed by proper maps. This shows the first assertion. The second is clear by construction. □

By h′-sheafifiying, the natural morphism of complexes

$$\alpha^{-1}\Omega^\bullet_X \to \Omega^\bullet_{X^{\mathrm{an}}}$$

of Section 4.1.3, we also obtain

$$\alpha^{-1}\Omega^\bullet_{\mathrm{h}} \to \Omega^\bullet_{\mathrm{h}'}$$

on $(\mathrm{An}/X^{\mathrm{an}})_{\mathrm{h}'}$. It induces

$$\alpha^* : H^i_{\mathrm{dR}}(X_{\mathrm{h}}) \to H^i_{\mathrm{dR}^{\mathrm{an}}}(X^{\mathrm{an}}_{\mathrm{h}'}) \ .$$

Proposition 4.2.10 (GAGA for h′-de Rham cohomology)**.** *Let X be a variety over $\mathbb{C}$ and Z a closed subvariety. Then the natural map*

$$\alpha^* : H^i_{\mathrm{dR}}(X_{\mathrm{h}}, Z_{\mathrm{h}}) \to H^i_{\mathrm{dR}^{\mathrm{an}}}(X^{\mathrm{an}}_{\mathrm{h}'}, Z^{\mathrm{an}}_{\mathrm{h}'})$$

is an isomorphism. It is compatible with long exact sequences and products.

Proof. By naturality, the comparison morphism is compatible with long exact sequences. Hence it suffices to consider the absolute case.

Let $X_\bullet \to X$ be a smooth proper hypercover. This is a cover in h′-topology, hence we may replace X by $X_\bullet$ on both sides. As all components of $X_\bullet$ are smooth, we may replace h-cohomology by Zariski-cohomology in the algebraic setting (see Proposition 3.2.4). On the analytic side, we may replace h′-cohomology by ordinary sheaf cohomology (see Corollary 2.7.4). The statement then follows from the comparison in the smooth case, see Proposition 4.1.7. □

Chapter 5

The period isomorphism

The aim of this section is to define well-behaved isomorphisms between singular and de Rham cohomology of algebraic varieties.

5.1 The category $(k,\mathbb{Q})-\mathrm{Vect}$

We introduce a simple linear algebra category which will later allow to formalize the notion of periods. Throughout, let $k \subset \mathbb{C}$ be a subfield.

Definition 5.1.1. Let $(k,\mathbb{Q})-\mathrm{Vect}$ be the category of triples $(V_k, V_{\mathbb{Q}}, \phi_{\mathbb{C}})$ where V_k is a finite dimensional k-vector space, $V_{\mathbb{Q}}$ a finite dimensional $\mathbb{Q}$-vector space and

$$\phi_{\mathbb{C}} : V_k \otimes_k \mathbb{C} \to V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$$

a $\mathbb{C}$-linear isomorphism. Morphisms in $(k,\mathbb{Q})-\mathrm{Vect}$ are linear maps on V_k and $V_{\mathbb{Q}}$ compatible with comparison isomorphisms.

Note that $(k,\mathbb{Q})-\mathrm{Vect}$ is a $\mathbb{Q}$-linear additive tensor category with the obvious notion of tensor product. It is rigid, i.e., all objects have strong duals. It is even Tannakian with projection to the $\mathbb{Q}$-component as fibre functor.

For later use, we make the duality explicit:

Remark 5.1.2. Let $V = (V_k, V_{\mathbb{Q}}, \phi_{\mathbb{C}}) \in (k,\mathbb{Q})-\mathrm{Vect}$. The the dual $V^\vee$ is given by

$$V^\vee = (V_k^*, V_{\mathbb{Q}}^*, (\phi^*)^{-1}$$

where $\cdot^*$ denotes the vector space dual over k and $\mathbb{Q}$ or $\mathbb{C}$. Note that the inverse is needed in order to make the map go in the right direction.

Remark 5.1.3. The above is a simplification of the category of mixed Hodge structures introduced by Deligne, see [D4]. It does not take the weight and Hodge filtration into account. In other words: there is a faithful forgetful functor from mixed Hodge structures over k to $(k,\mathbb{Q})-\mathrm{Vect}$.

Example 5.1.4. The invertible objects are those where $\dim_k V_k = \dim_{\mathbb{Q}} V_{\mathbb{Q}} = 1$. Up to isomorphism they are of the form

$$L(\alpha) = (k, \mathbb{Q}, \alpha) \text{ with } \alpha \in \mathbb{C}^* \ .$$

5.2 A triangulated category

We introduce a triangulated category with a t-structure whose heart is $(k, \mathbb{Q})-\mathrm{Vect}$.

Definition 5.2.1. A *cohomological* $(k, \mathbb{Q})-\mathrm{Vect}$*-complex* consists of the following data:

- a bounded below complex $K_k^\bullet$ of k-vector spaces with finite dimensional cohomology;
- a bounded below complex $K_{\mathbb{Q}}^\bullet$ of $\mathbb{Q}$-vector spaces with finite dimensional cohomology;
- a bounded below complex $K_{\mathbb{C}}^\bullet$ of $\mathbb{C}$-vector spaces with finite dimensional cohomology;
- a quasi-isomorphism $\phi_{k,\mathbb{C}} : K_k^\bullet \otimes_k \mathbb{C} \to K_{\mathbb{C}}^\bullet$;
- a quasi-isomorphism $\phi_{\mathbb{Q},\mathbb{C}} : K_{\mathbb{Q}}^\bullet \otimes_{\mathbb{Q}} \mathbb{C} \to K_{\mathbb{C}}^\bullet$.

Morphisms of cohomological $(k, \mathbb{Q})-\mathrm{Vect}$-complexes are given by a pair of morphisms of complexes on the k-, $\mathbb{Q}$- and $\mathbb{C}$-component such that the obvious diagram commutes. We denote the category of cohomological $(k, \mathbb{Q})-\mathrm{Vect}$-complexes by $C^+_{(k,\mathbb{Q})}$.

Let K and L be objects of $C^+_{(k,\mathbb{Q})}$. A *homotopy* between K and L is a homotopy in the k-, $\mathbb{Q}$- and $\mathbb{C}$-component compatible under the comparison maps. Two morphisms in $C^+_{(k,\mathbb{Q})}$ are *homotopic* if they differ by a homotopy. We denote by $K^+_{(k,\mathbb{Q})}$ the *homotopy category of cohomological* $(k, \mathbb{Q})-\mathrm{Vect}$*-complexes.*

A morphism in $K^+_{(k,\mathbb{Q})}$ is called *quasi-isomorphism* if its k-, $\mathbb{Q}$-, and $\mathbb{C}$-components are quasi-isomorphisms. We denote by $D^+_{(k,\mathbb{Q})}$ the localization of $K^+_{(k,\mathbb{Q})}$ with respect to quasi-isomorphisms. It is called the *derived category of cohomological* $(k, \mathbb{Q})-\mathrm{Vect}$*-complexes.*

Remark 5.2.2. This is a simplification of the category of mixed Hodge complexes introduced by Beilinson [Be2]. A systematic study of this type of category can be found in [Hu1, §4]. In the language of loc.cit., it is the rigid glued category of the category of k-vector spaces and the category of $\mathbb{Q}$-vector spaces via the category of $\mathbb{C}$-vector spaces and the extension of scalars functors. Note that they are exact, hence the construction simplifies.

Lemma 5.2.3. *$D^+_{(k,\mathbb{Q})}$ is a triangulated category. It has a natural t-structure with*

$$H^i : D^+_{(k,\mathbb{Q})} \to (k,\mathbb{Q})-\mathrm{Vect}$$

defined componentwise. The heart of the t-structure is $(k,\mathbb{Q})-\mathrm{Vect}$.

Proof. This is straightforward. For more details see [Hu1, §4]. □

Remark 5.2.4. In [Hu1, 4.2, 4.3], explicit formulas are given for the morphisms in $D^+_{(k,\mathbb{Q})}$. The category has cohomological dimension 1. For $K, L \in (k,\mathbb{Q})-\mathrm{Vect}$, the group $\mathrm{Hom}_{D^+_{(k,\mathbb{Q})}}(K, L[1])$ is equal to the group of Yoneda extensions. As in [Be2], this implies that $D^+_{(k,\mathbb{Q})}$ is equivalent to the bounded derived category $D^+((k,\mathbb{Q})-\mathrm{Vect})$. We do not spell out the details because we are not going to need these properties.

There is an obvious definition of a tensor product on $C^+_{(k,\mathbb{Q})}$. Let $K^\bullet, L^\bullet \in C^+_{(k,\mathbb{Q})}$. We define $K^\bullet \otimes \mathrm{L}^\bullet$ with $k, \mathbb{Q}, \mathbb{C}$-component given by the tensor product of complexes of vector spaces over k, $\mathbb{Q}$, and $\mathbb{C}$, respectively (see Example 1.3.4). Tensor product of two quasi-isomorphisms defines the comparison isomorphism on the tensor product.

It is associative and commutative. Note that the

Lemma 5.2.5. *$C^+_{(k,\mathbb{Q})}$, $K^+_{(k,\mathbb{Q})}$ and $D^+_{(k,\mathbb{Q})}$ are associative and commutative tensor categories with the above tensor product. The cohomology functor H^* commutes with $\otimes$. For $K^\bullet, L^\bullet$ in $D^+_{(k,\mathbb{Q})}$, we have a natural isomorphism*

$$H^*(K^\bullet) \otimes H^*(L^\bullet) \to H^*(K^\bullet \otimes L^\bullet).$$

It is compatible with the associativity constraint. It is compatible with the commutativity constraint up to the sign $(-1)^{pq}$ on $H^p(K^\bullet) \otimes H^q(L^\bullet)$.

Proof. The case of $D^+_{(k,\mathbb{Q})}$ follows immediately from the case of complexes of vector spaces, where it is well-known. The signs come from the signs in the total complex of a bicomplex, in this case, tensor product of complexes, see Section 1.3.3. □

Remark 5.2.6. This is again simpler than the case treated in [Hu1, Chapter 13], because we do not need to control filtrations and because our tensor products are exact.

5.3 The period isomorphism in the smooth case

Let k be a subfield of $\mathbb{C}$. We consider smooth varieties over k and the complex manifold X^{an} associated to $X \times_k \mathbb{C}$.

Definition 5.3.1. Let X be a smooth variety over k. We define the *period isomorphism*

$$\mathrm{per}: H^\bullet_{\mathrm{dR}}(X)\otimes_k\mathbb{C}\to H^\bullet_{\mathrm{sing}}(X,\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C}$$

to be the isomorphism given by the composition of the isomorphisms

1. $H^\bullet_{\mathrm{dR}}(X)\otimes_k\mathbb{C}\to H^\bullet_{\mathrm{dR}}(X\times_k\mathbb{C})$ of Lemma 3.1.11,
2. $H^\bullet_{\mathrm{dR}}(X\times_k\mathbb{C})\to H^\bullet_{\mathrm{dR}^{\mathrm{an}}}(X^{\mathrm{an}})$ of Proposition 4.1.7,
3. the inverse of $H^\bullet_{\mathrm{dR}^{\mathrm{an}}}(X^{\mathrm{an}})\to H^\bullet_{\mathrm{sing}}(X^{\mathrm{an}},\mathbb{C})$ of Proposition 4.1.3,
4. the inverse of the change of coefficients isomorphism $H^\bullet_{\mathrm{sing}}(X^{\mathrm{an}},\mathbb{C})\to H^\bullet_{\mathrm{sing}}(X^{\mathrm{an}},\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C}$.

We define the *period pairing*

$$\mathrm{per}: H^\bullet_{\mathrm{dR}}(X)\times H_\bullet^{\mathrm{sing}}(X^{\mathrm{an}},\mathbb{Q})\to\mathbb{C}$$

to be the map

$$(\omega,\gamma)\mapsto\gamma(\mathrm{per}(\omega))$$

where we view classes in singular homology as linear forms on singular cohomology.

Recall the category $(k,\mathbb{Q})-\mathrm{Vect}$ introduced in Section 5.1.

Lemma 5.3.2. *The assignment*

$$X\mapsto(H^\bullet_{\mathrm{dR}}(X),H^\bullet_{\mathrm{sing}}(X),\mathrm{per})$$

defines a functor

$$\mathrm{H}:\mathrm{Sm}\to(k,\mathbb{Q})-\mathrm{Vect}\ .$$

For all $X,Y\in\mathrm{Sm}$*, the Künneth isomorphism induces an natural isomorphism*

$$\mathrm{H}(X)\otimes\mathrm{H}(Y)\to\mathrm{H}(X\times Y)\ .$$

The image of H *is closed under direct sums and tensor product.*

Proof. Functoriality holds by construction. The Künneth morphism is induced from the Künneth isomorphism in singular cohomology (Proposition 2.4.1) and algebraic de Rham cohomology (see Proposition 3.1.9). All identifications in Definition 5.3.1 are compatible with the product structure. Hence we have defined a Künneth morphism in H. It is an isomorphism because it is an isomorphism in singular cohomology.

The direct sum realized by the disjoint union. The tensor product is realized by the product. □

In Chapter 9, we are going to study systematically the periods of the objects in H(Sm).

The period isomorphism has an explicit description in terms of integration.

Theorem 5.3.3. *Let X be a smooth affine variety over k and $\omega \in \Omega^i(X)$ a closed differential form with de Rham class $[\omega]$. Let $c \in H_d^{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{Q})$ be a singular homology class. Let $\sum a_j\gamma_j$ with $a_j \in \mathbb{Q}$ and $\gamma_j : \Delta_i \to X^{\mathrm{an}}$ differentiable singular cycles as in Definition 2.2.2. Then*

$$\mathrm{per}([\omega], c) = \sum a_j \int_{\Delta_i} \gamma^*(\omega) \ .$$

Remark 5.3.4. We could use the above formula as a definition of the period pairing, at least in the affine case. By Stokes' theorem, the value only depends on the class of ω.

Proof. Let $A^i(X^{\mathrm{an}})$ be group of $\mathbb{C}$-valued C^∞-differential forms and $\mathcal{A}^i_{X^{\mathrm{an}}}$ the associated sheaf. By the Poincaré lemma and its C^∞-analogue the morphisms

$$\mathbb{C} \to \Omega^\bullet_{X^{\mathrm{an}}} \to \mathcal{A}^\bullet_{X^{\mathrm{an}}}$$

are quasi-isomorphism. It induces a quasi-isomorphism

$$\Omega^\bullet_{X^{\mathrm{an}}}(X^{\mathrm{an}}) \to A^\bullet(X^{\mathrm{an}})$$

because both compute singular cohomology in the affine case. Hence it suffices to view ω as a C^∞-differential form. By the Theorem of de Rham, see [Wa], Sections 5.34-5.36, the period isomorphism is realized by integration over simplices. □

Example 5.3.5. For $X = \mathbb{P}^n_k$, we have

$$\mathrm{H}^{2j}(\mathbb{P}^n_k) = L((2\pi i)^j)$$

with $L(\alpha)$ the invertible object of Example 5.1.4.

5.4 The general case (via the h$'$-topology)

We generalize the period isomorphism to relative cohomology of arbitrary varieties.

Let k be a subfield of $\mathbb{C}$. We consider varieties over k and the complex analytic space X^{an} associated to $X \times_k \mathbb{C}$.

Definition 5.4.1. Let X be a variety over k, and $Z \subset X$ a closed subvariety. We define the *period isomorphism*

$$\mathrm{per} : H^\bullet_{\mathrm{dR}}(X, Z) \otimes_k \mathbb{C} \to H^\bullet_{\mathrm{sing}}(X, Z, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$$

to be the isomorphism given by the composition of the isomorphisms

1. $H^\bullet_{\mathrm{dR}}(X,Z)\otimes_k \mathbb{C}\to H^\bullet_{\mathrm{dR}}(X\times_k\mathbb{C}, Z\times_k\mathbb{C})$ of Lemma 3.2.14,

2. $H^\bullet_{\mathrm{dR}}(X\times_k\mathbb{C}, Z\times_k\mathbb{C})\to H^\bullet_{\mathrm{dR}^{\mathrm{an}}}(X^{\mathrm{an}}_{\mathrm{h}'}, Z^{\mathrm{an}}_{\mathrm{h}'})$ of Proposition 4.2.10,

3. the inverse of $H^\bullet_{\mathrm{dR}^{\mathrm{an}}}(X^{\mathrm{an}}_{\mathrm{h}'}, Z^{\mathrm{an}}_{\mathrm{h}'})\to H^\bullet_{\mathrm{sing}}(X^{\mathrm{an}}, Z^{\mathrm{an}},\mathbb{C})$ of Corollary 4.2.7,

4. the inverse of the change of coefficients isomorphism $H^\bullet_{\mathrm{sing}}(X^{\mathrm{an}}, Z^{\mathrm{an}},\mathbb{C})\to H^\bullet_{\mathrm{sing}}(X^{\mathrm{an}}, Z^{\mathrm{an}},\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C}$.

We define the *period pairing*

$$\mathrm{per}: H^\bullet_{\mathrm{dR}}(X,Z)\times H^{\mathrm{sing}}_\bullet(X^{\mathrm{an}}, Z^{\mathrm{an}},\mathbb{Q})\to\mathbb{C}$$

to be the map

$$(\omega,\gamma)\mapsto \gamma(\mathrm{per}(\omega)),$$

where we view classes in singular homology as linear forms on singular cohomology.

Lemma 5.4.2. *The assignment*

$$(X,Z)\mapsto (H^\bullet_{\mathrm{dR}}(X,Z), H^\bullet_{\mathrm{sing}}(X,Z),\mathrm{per})$$

defines a functor denoted H *on the category of pairs $X\supset Z$ with values in $(k,\mathbb{Q})-\mathrm{Vect}$. For all $Z\subset Z$, $Z'\subset X'$, the Künneth isomorphism induces a natural isomorphism*

$$\mathrm{H}(X,Z)\otimes\mathrm{H}(X',Z')\to\mathrm{H}(X\times X', X\times Z'\cup Z\times X')\ .$$

The image of H *is closed under direct sums and tensor product.*

If $Z\subset Y\subset X$ is a triple, the there is a induced long exact sequence in $(k,\mathbb{Q})-\mathrm{Vect}$.

$$\dots\to H^i(X,Y)\to H^i(X,Z)\to H^i(Y,Z)\xrightarrow{\partial} H^{i+1}(X,Y)\to\dots\ .$$

Proof. Functoriality and compatibility with long exact sequences holds by construction. The Künneth morphism is induced from the Künneth isomorphism in singular cohomology (Proposition 2.4.1) and algebraic de Rham cohomology (see Proposition 3.1.9). All identifications in Definition 5.3.1 are compatible with the product structure. Hence we have defined a Künneth morphism in H. It is an isomorphism because it is an isomorphism in singular cohomology.

The direct sum realized by the disjoint union. The tensor product is realized by the product. □

5.5 The general case (Deligne's method)

We generalize the period isomorphism to relative cohomology of arbitrary varieties.

Let k be a subfield of $\mathbb{C}$.

Recall from Section 3.1.2 the functor

$$R\Gamma_{\mathrm{dR}} : \mathbb{Z}[\mathrm{Sm}] \to C^+(k\mathrm{-Mod})$$

which maps a smooth variety to a natural complex computing its de Rham cohomology. In the same way, we define using the Godement resolution (see Definition 1.4.8)

$$R\Gamma_{\mathrm{sing}}(X) = \Gamma(X^{\mathrm{an}}, Gd(\mathbb{Q})) \in C^+(\mathbb{Q}\mathrm{-Mod})$$

a complex computing singular cohomology of X^{an}. Moreover, let

$$R\Gamma_{\mathrm{dR^{an}}}(X) = \Gamma(X^{\mathrm{an}}, Gd(\Omega^\bullet_{X^{\mathrm{an}}}) \in C^+(\mathbb{C}\mathrm{-Mod})$$

be a complex computing holomorphic de Rham cohomology of X^{an}.

Lemma 5.5.1. *Let X be a smooth variety over k.*

1. *As before let $\alpha : X^{\mathrm{an}} \to X \times_k \mathbb{C}$ be the morphism of locally ringed spaces and $\beta : X \times_k \mathbb{C} \to X$ the natural map. The inclusion $\alpha^{-1}\beta^{-1}\Omega^\bullet_X \to \Omega^\bullet_{X^{\mathrm{an}}}$ induces a natural quasi-isomorphism of complexes*

$$\phi_{\mathrm{dR},\mathrm{dR^{an}}} : R\Gamma_{\mathrm{dR}}(X) \otimes_k \mathbb{C} \to R\Gamma_{\mathrm{dR^{an}}}(X) \ .$$

2. *The inclusion $\mathbb{Q} \to \Omega^\bullet_{X^{\mathrm{an}}}$ induces a natural quasi-isomorphism of complexes*

$$\phi_{\mathrm{sing},\mathrm{dR^{an}}} : R\Gamma_{\mathrm{sing}}(X) \otimes_{\mathbb{Q}} \mathbb{C} \to R\Gamma_{\mathrm{dR^{an}}}(X) \ .$$

3. *We have*

$$\mathrm{per} = H^\bullet(\phi_{\mathrm{sing},\mathrm{dR^{an}}})^{-1} \circ H^\bullet(\phi_{\mathrm{sing},\mathrm{dR^{an}}}) : H^\bullet_{\mathrm{dR}}(X) \otimes_k \mathbb{C}) \to H^\bullet_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{Q}) \ .$$

Proof. The first assertion follows from applying Lemma 1.4.10 to β and α. As before, we identify sheaves on $X \times_k \mathbb{C}$ with sheaves on the set of closed points of $X \times_k \mathbb{C}$. This yields a quasi-isomorphism

$$\alpha^{-1}\beta^{-1}Gd_X(\Omega^\bullet_X) \to Gd_{X^{\mathrm{an}}}(\alpha^{-1}\beta^{-1}\Omega^\bullet_X) \ .$$

We compose with

$$Gd_{X^{\mathrm{an}}}(\alpha^{-1}\beta^{-1}\Omega^\bullet_X) \to Gd_{X^{\mathrm{an}}}(\Omega^\bullet_{X^{\mathrm{an}}}) \ .$$

Taking global sections yields by definition a natural $\mathbb{Q}$-linear map of complexes

$$R\Gamma_{\mathrm{dR}}(X) \to R\Gamma_{\mathrm{dR}^{\mathrm{an}}}(X) .$$

By extension of scalars we get $\phi_{\mathrm{dR},\mathrm{dR}^{\mathrm{an}}}$. It is a quasi-isomorphism because on cohomology it defines the maps from Lemma 3.1.11 and Proposition 4.1.7.

The second assertion follows from ordinary functoriality of the Godement resolution. The last holds by construction. □

In other words:

Corollary 5.5.2. *The assignment*

$$X \mapsto (R\Gamma_{\mathrm{dR}}(X), R\Gamma_{\mathrm{sing}}(X), R\Gamma_{\mathrm{dR}^{\mathrm{an}}}(X), \phi_{\mathrm{dR},\mathrm{dR}^{\mathrm{an}}}, \phi_{\mathrm{sing},\mathrm{dR}^{\mathrm{an}}})$$

defines a functor

$$R\Gamma : \mathrm{Sm} \to C^+_{(k,\mathbb{Q})}$$

*where $C^+_{(k,\mathbb{Q})}$ is the category of cohomological $(k,\mathbb{Q})-$*Vect*-complexes introduced in Definition 5.2.1.*

Moreover,

$$H^\bullet(R\Gamma(X)) = \mathrm{H}(X) ,$$

where the functor H *is defined as above.*

Proof. Clear from the lemma. □

By naturality, these definitions extend to objects in $\mathbb{Z}[\mathrm{Sm}]$.

Definition 5.5.3. Let

$$R\Gamma : K^-(\mathbb{Z}\mathrm{Sm}) \to D^+_{(k,\mathbb{Q})}$$

be defined componentwise as the total complex complex of the complex in $C^+_{(k,\mathbb{Q})}$. For $X_\bullet \in C^-(\mathbb{Z}\mathrm{Sm})$ and $i \in \mathbb{Z}$ we put

$$\mathrm{H}^i(X_\bullet) = H^i R\Gamma(X_\bullet) .$$

Definition 5.5.4. Let k be a subfield of $\mathbb{C}$ and X a variety over k with a closed subvariety D. We define the *period isomorphism*

$$\mathrm{per} : H^\bullet_{\mathrm{dR}}(X, D) \otimes_k \mathbb{C} \to H^\bullet_{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}) \otimes_{\mathbb{Q}} \mathbb{C}$$

as follows: let $D_\bullet \to X_\bullet$ be smooth proper hypercovers of $D \to X$ as in Definition 3.3.6. Let

$$C_\bullet = \mathrm{Cone} C(D_\bullet) \to C(X_\bullet)) \in C^-(\mathbb{Z}[\mathrm{Sm}]) .$$

Then $H^\bullet(R\Gamma(C_\bullet))$ consists of

$$(H^\bullet_{\mathrm{dR}}(X,D), H^\bullet_{\mathrm{sing}}(X,D), \mathrm{per}) \ .$$

In detail: per is given by the composition of the isomorphisms

$$H^\bullet_{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}, \mathbb{Q})) \otimes_{\mathbb{Q}} \mathbb{C} \to H^\bullet(R\Gamma_{\mathrm{sing}}(C_\bullet)$$

with

$$H^\bullet\phi_{\mathrm{sing},\mathrm{dR}^{\mathrm{an}}}(C_\bullet)^{-1} \circ H^\bullet\phi_{\mathrm{dR},\mathrm{dR}^{\mathrm{an}}}(C_\bullet) \ .$$

We define the *period pairing*

$$\mathrm{per} : H^\bullet_{\mathrm{dR}}(X,D) \times H_\bullet^{\mathrm{sing}}(X^{\mathrm{an}}, D^{\mathrm{an}}) \to \mathbb{C}$$

to be the map

$$(\omega,\gamma) \mapsto \gamma(\mathrm{per}(\omega))$$

where we view classes in relative singular homology as linear forms on relative singular cohomology.

Lemma 5.5.5. per *is well-defined, compatible with products and long exact sequences for relative cohomology.*

Proof. By definition of relative algebraic de Rham cohomology (see Definition 3.3.6), the morphism takes values in $H^\bullet_{\mathrm{dR}}(X,D)\otimes_k \mathbb{C}$. The first map is an isomorphism by proper descent in singular cohomology, see Theorem 2.7.6.

Compatibility with long exact sequences and multiplication comes from the definition. □

We make this explicit in the case of a divisor with normal crossings. Recall the description of relative de Rham cohomology in this case in Proposition 3.3.19.

Theorem 5.5.6. *Let X be a smooth affine variety of dimension d and $D \subset X$ a simple divisor with normal crossings. Let $\omega \in \Omega^d_X(X)$ with associated cohomology class $[\omega] \in H^d_{\mathrm{dR}}(X,D)$. Let $\sum a_j\gamma_j$ with $a_j \in \mathbb{Q}$ and $\gamma_j : \Delta_i \to X^{\mathrm{an}}$ be a differentiable singular cchain as in Definition 2.2.2 with boundary in D^{an}. Then*

$$\mathrm{per}([\omega], c) = \sum a_j \int_{\Delta_i} \gamma^*(\omega) \ .$$

Proof. Let $D_\bullet$ as in Section 3.3.6. We apply the considerations of the proof of Theorem 5.3.3 to X and the components of $D_\bullet$. Note that $\omega|_{D_I} = 0$ for dimension reasons. □